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Research Article

Composition Operators and Multiplication Operators on Weighted Spaces of Analytic Functions

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Let V be an arbitrary system of weights on an open connected subset G of $\mathbb{C}^N$ $(N \geq 1)$ and let $B(E)$ be the Banach algebra of all bounded linear operators on a Banach space E. Let $HV_b\ (G,E)$ and $HV_0\ (G,E)$ be the weighted locally convex spaces of vector-valued analytic functions. In this survey, we present a development of the theory of multiplication operators and composition operators from classical spaces of analytic functions $H(G)$ to the weighted spaces of analytic functions $HV_b\ (G,E)$ and $HV_0\ (G,E)$.

1. Introduction

Multiplication operators (also known as multipliers) and composition operators, on different spaces of analytic functions, have been actively appearing in different areas of mathematical sciences like dynamical systems, theory of semigroups, isometries, and, in turn, the theory of weighted composition operators besides their role in the theory of operator algebras and operator spaces. Evard and Jafari [1] and Siskakis [2, 3] have employed these operators to make a study of weighted composition semigroups and dynamical systems on Hardy Spaces. De Leeuw et al. [4] and Nagasawa [5] have described isometries of Hardy spaces $H^1(\mathbb{D})$ and $H^\infty(\mathbb{D})$ as a product of multiplication operators and composition operators. Isometries on H^p-spaces and Bergman spaces are very much related with multiplication operators and composition operators, and for details on these isometries, we refer to Forelli [6], Cambern and Jarosz [7], Kolaski [8], Mazur [9], and Lin [10]. In [11], Arveson has recently obtained Toeplitz C^*-algebras and operator spaces associated with these multiplication operators on Hardy Spaces.

In recent years, many authors like Attele [12], Axler [13–16], Bercovici [17], Eschmeier [18], Luecking [19], Vukotić [20], and Zhu [21] have made a study of multiplication operators on Bergman spaces, whereas Campbell and Leach [22], Feldman [23], Lin [10], and Ohno and Takagi [24] have obtained a study of these operators on Hardy spaces. On Bloch spaces, these operators are studied by Arazy [25], Axler [15] and Brown and Shields [26]. Also, Axler and Shields [16] and Stegenga [27] have explored multiplication operators on Dirichlet spaces. On BMOA, these operators are studied by Ortega and Fabregá [28]. Further, on Nevanlinna classes of analytic functions, these operators are studied by Jarchow et al. [29] and Yanagihara [30]. Besides these well-known analytic function spaces, a study of these operators on some other Banach spaces of analytic functions has also been pursued by Bonet et al. [31–34], Contreras and Hernández-Díaz [35], Ohno and Takagi [24], and Shields and Williams [36, 37].

In [38], Contreras and Hernández-Díaz have made a study of weighted composition operators on Hardy spaces, whereas Mirzakarimi and Siddighi [39] have considered these operators on Bergman and Dirichlet spaces. On Bloch and Block-type spaces, these operators are studied by MacCluer and Zhao [40], Ohno [41], Ohno and Zhao [42], and Ohno et al. [43]. In [24], Ohno and Takagi have obtained some properties of these operators on the disc algebra and the Hardy space $H^\infty(\mathbb{D})$. Also, recently, Montes-Rodríguez [44] and Contreras and Hernández-Díaz [35] have studied the behaviour of these operators on weighted Banach spaces of analytic functions. The applications of these operators can be found in the theory of semigroups and dynamical systems (see [2, 3, 45]). For more information on composition operators on spaces of analytic functions, we refer to three monographs (see Cowen and MacCluer [46], Shapiro [47], and Singh and Manhas [48]).

In the present survey, we report on a recent study of composition operators and multiplication operators on the weighted spaces of analytic functions.

2. Weighted spaces of analytic functions

Let G be an open connected subset of $\mathbb{C}^N$ $(N \geq 1)$ and let $H(G,E)$ be the space of all vector-valued analytic functions from G into the Banach space E. Let V be a set of nonnegative upper semicontinuous functions on G. Then V is said to be *directed upward*, if for every pair $u_1, u_2 \in V$ and $\lambda > 0$, there exists $v \in V$ such that $\lambda u_i \leq v$ (pointwise on G), for $i = 1,2$. If V is directed upward and for each $z \in G$, there exists $v \in V$ such that $v(z) > 0$, then we call V as an *arbitrary system* of weights on G. If U and V are two arbitrary systems of weights on G such that for each $u \in U$, there exists $v \in V$ for which $u \leq v$, then we write $U \leq V$. If $U \leq V$ and $V \leq U$, then we write $U \cong V$. Let V be an arbitrary system of weights on G. Then we define

$$\begin{aligned} HV_b(G,E) &= \{f \in H(G,E) : vf(G) \text{ is bounded in } E, \text{ for each } v \in V\}, \\ HV_0(G,E) &= \{f \in H(G,E) : vf \text{ vanishes at infinity on } G, \text{ for each } v \in V\}. \end{aligned} \tag{2.1}$$

For $v \in V$ and $f \in H(G,E)$, we define

$$\|f\|_{v,E} = \sup\{v(z)\|f(z)\| : z \in G\}. \tag{2.2}$$

Clearly, the family $\{\|\cdot\|_{v,E} : v \in V\}$ of seminorms defines a Hausdorff locally convex topology on each of theses spaces: $HV_b(G,E)$ and $HV_0(G,E)$. With this topology, the spaces $HV_b(G,E)$ and $HV_0(G,E)$ are called the weighted locally convex spaces of vector-valued analytic functions. These spaces have a basis of closed absolutely convex neighbourhoods of the form

$$B_{v,E} = \{f \in HV_b(G,E) \text{ (resp., } HV_0(G,E)) : \|f\|_{v,E} \leq 1\}. \tag{2.3}$$

If $E = \mathbb{C}$, then we write $HV_b(G,E) = HV_b(G)$, $HV_0(G,E) = HV_0(G)$ and

$$B_v = \{f \in HV_b(G) \text{ (resp., } HV_0(G)) : \|f\|_v \leq 1\}. \tag{2.4}$$

Throughout the paper, we assume for each $z \in G$, there exists $f_z \in HV_0(G)$ such that $f_z(z) \neq 0$.

If $v : \mathbb{D} \to \mathbb{R}^+$ is a continuous weight and $E = \mathbb{C}$, then the corresponding weighted Banach spaces of analytic functions are defined as follows:

$$\begin{aligned} H_v^\infty(\mathbb{D}) &= \{f \in H(\mathbb{D}) : vf(\mathbb{D}) \text{ is bounded}\}, \\ H_v^0(\mathbb{D}) &= \left\{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\right\}. \end{aligned} \tag{2.5}$$

Now, using the definitions of weights given in [32, 49–51], we give definitions of some systems of weights which are required for characterizing some results in the remaining sections.

Let V be an arbitrary system of weights on G and let $v \in V$. Then define $\widetilde{w} : G \to \mathbb{R}^+$ as

$$\begin{aligned} \widetilde{w}(z) &= \sup\{|f(z)| : \|f\|_v \leq 1\} \leq \frac{1}{v(z)}, \\ \widetilde{v}(z) &= \frac{1}{\widetilde{w}(z)}, \quad \text{for every } z \in G. \end{aligned} \tag{2.6}$$

In case $\widetilde{w}(z) \neq 0$, $\widetilde{v}$ is an upper semicontinuous, and we call it an associated weight of v. Let $\widetilde{V}$ denote the system of all associated weights of V. Then an arbitrary system of weights V is called a *reasonable system* as it satisfies the following properties:

$$\text{for each } v \in V, \text{ there exists } \widetilde{v} \in \widetilde{V} \text{ such that } v \leq \widetilde{v}; \tag{2.7a}$$

$$\begin{aligned} &\text{for each } v \in V,\ \|f\|_v \leq 1 \text{ iff } \|f\|_{\widetilde{v}} \leq 1, \\ &\text{for every } f \in HV_b(G); \end{aligned} \tag{2.7b}$$

$$\begin{aligned} &\text{if } v \in V, \text{ then for every } z \in G, \text{ there exists} \\ &f_z \in B_v \text{ such that } |f_z(z)| = \frac{1}{\widetilde{v}(z)}. \end{aligned} \tag{2.7c}$$

Let $v \in V$. Then v is called *essential* if there exists a constant $\lambda > 0$ such that $v(z) \leq \widetilde{v}(z) \leq \lambda v(z)$, for each $z \in G$. A reasonable system of weights V is called an *essential system* if each $v \in V$ is an essential weight. If V is an essential system of weights, then we have

$V \cong \widetilde{V}$. For example, let $G = \mathbb{D}$, the open unit disc, and let $f \in H(\mathbb{D})$ be nonzero. Then define $v_f(z) = [\sup\{|f(z)| : |z| = r\}]^{-1}$, for every $z \in \mathbb{D}$. Clearly, each v_f is a weight satisfying $\widetilde{v}_f = v_f$, and the family $V = \{v_f : f \in H(\mathbb{D}),\ f \text{ is nonzero}\}$ is an essential system of weights on $\mathbb{D}$. For more details on these weights, we refer to [50]. Let G be any balanced (i.e., $\lambda z \in G$, whenever $z \in G$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$) open subset of $\mathbb{C}^N$ $(N \geq 1)$. Then a weight $v \in V$ is called radial and typical if $v(z) = v(\lambda z)$ for all $z \in G$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and vanishes at the boundary ∂G. In particular, a weight v on $\mathbb{D}$ is radial and typical if $v(z) = v(|z|)$ and $\lim_{|z|\to 1} v(z) = 0$. For instance, in [35], it is shown that $v_p(z) = (1 - |z|^2)^p$, $(0 < p < \infty)$, for every $z \in \mathbb{D}$, are essential typical weights. For more details on the weighted Banach spaces of analytic functions and the weighted locally convex spaces of analytic functions associated with these weights, we refer to [32, 49–54]. For basic definitions and facts in complex analysis and functional analysis, we refer to [55–58].

Let $F(G,E)$ be a topological vector space of vector-valued analytic functions from G into E, and let $L(G,E)$ be the vector space of all vector-valued functions from G into E. Let $B(E)$ be the Banach algebra of all bounded linear operators on E. Then for an operator-valued map $\Psi : G \to B(E)$ and self-map $\phi : G \to G$, we define the linear map $W_{\Psi,\phi} : F(G,E) \to L(G,E)$ as $W_{\Psi,\phi}(f) = \Psi \cdot f \circ \phi$ for every $f \in F(G,E)$, where the product $\Psi \cdot f \circ \phi$ is defined pointwise on G as $(\Psi \cdot f \circ \phi)(z) = \Psi_z(f(\phi(z)))$ for every $z \in G$. In case $W_{\Psi,\phi}$ takes $F(G,E)$ into itself and is continuous, we call $W_{\Psi,\phi}$, the *weighted composition operator* on $F(G,E)$ induced by the symbols Ψ and ϕ. If $\Psi(z) = I$, the indentity operator on E for every $z \in G$, then $W_{\Psi,\phi}$ is called the *composition operator* induced by ϕ and we denote it by C_ϕ. In case $\phi(z) = z$ for every $z \in G$, $W_{\Psi,\phi}$ is called the *multiplication operator* induced by Ψ, and we denote it by M_Ψ.

3. Characterizations of multiplication operators

In this section, we give characterizations of multiplication operators on the weighted spaces of analytic functions. We begin with the following straightforward observations obtained by [31] on the weighted Banach spaces of scalar-valued analytic functions.

Proposition 3.1. *Let $\Psi : \mathbb{D} \to \mathbb{C}$ be analytic function. Then $M_\Psi : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded if and only if $\Psi \in H^\infty(\mathbb{D})$. If $H_v^0(\mathbb{D}) \neq \{0\}$, then the previous statement is equivalent to that $M_\Psi : H_v^0(\mathbb{D}) \to H_v^0(\mathbb{D})$ is bounded. Also, $\|M_\Psi\| = \|\Psi\|_\infty$.*

Proposition 3.2. *If $M_\Psi : H_v^0(\mathbb{D}) \to H_w^0(\mathbb{D})$ is bounded and both v and w are radial weights vanishing on the boundary, then $M_\Psi'' = M_\Psi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$.*

Proof. It is shown by [51, 59] that $(H_v^0(\mathbb{D}))'' = H_v^\infty(\mathbb{D})$ and $(H_w^0(\mathbb{D}))'' = H_w^\infty(\mathbb{D})$. In [32], it is observed that the evaluation functional $\delta_z : H_v^0(\mathbb{D}) \to \mathbb{C}$ defined as $\delta_z(f) = f(z)$ is also acting as the evaluation functional on $H_v^\infty(\mathbb{D})$. It is obvious that $M_\Psi'(\delta_z) = \Psi(z)\ \delta_z$. Now, for $f \in H_v^\infty(\mathbb{D})$, we have $\langle M_\Psi'' f, \delta_z\rangle = \langle f, \Psi(z)\ \delta_z\rangle = f(z)\Psi(z)$. □

Now, we present the generalizations of the above characterizations to the weighted spaces of vector-valued analytic functions for general systems of weights, which was obtained by Manhas in [60].

Proposition 3.3. *Let U and V be arbitrary systems of weights on G, and let $\Psi : G \to B(E)$ be an analytic map. Then $M_\Psi : HU_b(G,E) \to HV_b(G,E)$ is a multiplication operator, if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi(z)\| \leq u(z)$, for every $z \in G$.*

Remark 3.4. Proposition 3.3 makes it clear that every bounded analytic function $\Psi : G \to B(E)$ induces the multiplication operator M_Ψ on $HV_b(G,E)$, for any system of weights V on G. Also, if $V = \{\lambda\chi_K : \lambda \geq 0,\ K \subseteq G,\ K \text{ compact set}\}$, then every operator-valued analytic map $\Psi : G \to B(E)$ induces a multiplication operator M_Ψ on $HV_b(G,E)$. This makes it clear that even unbounded analytic operator-valued mappings generate multiplication operators on some of weighted locally convex spaces $HV_b(G,E)$, whereas it is not true for other spaces of analytic functions. For instance, Arveson [11] and Axler [13] have shown that only bounded analytic functions give rise to multiplication operators on Hardy spaces and Bergman spaces, respectively. Also, the same behaviour has been observed on the weighted Banach spaces of analytic functions $H_v^\infty(D)$ defined by a single continuous weights (see Proposition 3.1). Thus the behaviour of the multiplication operators on the weighted locally convex spaces of analytic functions is very much influenced by different systems of weights V on G.

Theorem 3.5. *Let V be an arbitrary system of weights and U a reasonable system of weights on G. Let $\Psi : G \to B(E)$ be an analytic map. Then $M_\Psi : H\widetilde{U}_b(G,E) \to HV_b(G,E)$ is a multiplication operator if and only if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi(z)\| \leq \widetilde{u}(z)$, for every $z \in G$.*

Proof. The sufficient part follows from Proposition 3.3. Conversely, suppose $M_\Psi : H\widetilde{U}_b(G, E) \to HV_b(G,E)$ is a multiplication operator. Let $v \in V$. Then by the continuity of M_Ψ at the origin, there exists $\widetilde{u} \in \widetilde{U}$ with $u \in U$ such that $u \leq \widetilde{u}$ and $M_\Psi(B_{\widetilde{u},E}) \subseteq B_{v,E}$. To establish the inequality $v(z)\|\Psi(z)\| \leq \widetilde{u}(z)$ for every $z \in G$, it is enough to prove that $v(z)\|\Psi_z(y)\| \leq \widetilde{u}(z)\|y\|$, for every $z \in G$ and $y \in E$. Fix $z_0 \in G$ and $y_0 \in E$. Then by (2.7c), there exists $f_{z_0} \in B_u$ such that $\|f_{z_0}(z_0)\| = 1/\widetilde{u}(z_0)$. Let $g_0 : G \to E$ be defined as

$$g_0(z) = \frac{1}{\|y_0\|} f_{z_0}(z)y_0, \qquad \text{for every } z \in G. \tag{3.1}$$

Clearly, $g_0 \in B_{u,E}$ and $\|g_0(z_0)\| = 1/\widetilde{u}(z_0)$. Also, according to (2.7b), $f_{z_0} \in B_{\widetilde{u}}$ and therefore $g_0 \in B_{\widetilde{u},E}$. Thus it follows that $M_\Psi(g_0) \in B_{v,E}$. That is, $v(z)\|\Psi_z(g_0(z))\| \leq 1$, for every $z \in G$. In particular, for $z = z_0$, we have $v(z_0)\|\Psi_{z_0}(y_0)\| \leq \widetilde{u}(z_0)\|y_0\|$. This completes the proof of the theorem. □

Corollary 3.6. *Let V be an arbitrary system of weights and let U be an essential system of weights on G. Let $\Psi : G \to B(E)$ be an operator-valued analytic map. Then $M_\Psi : HU_b(G,E) \to HV_b(G,E)$ is a multiplication operator if and only if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi(z)\| \leq u(z)$, for every $z \in G$.*

Proof. It follows from Theorem 3.5 and from the relation that $\widetilde{U} \cong U$. □

Remark 3.7. All the results proved above are also hold for the spaces $HV_0(G,E)$ and $HU_0(G,E)$.

4. Invertible multiplication operators

In this section, we present characterizations of invertible multiplication operators on the weighted spaces of analytic functions. We begin with the following characterization of invertible multiplication operators on the weighted Banach spaces of scalar-valued analytic functions [31].

Proposition 4.1. *Let $\Psi \in H^\infty$. Then $M_\Psi : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is invertible if and only if $1/\Psi \in H^\infty$ (or equivalently, there exists $\epsilon > 0$ such that $|\Psi(z)| \geq \epsilon$, for all $z \in \mathbb{D}$). The same is true for $H_v^0(\mathbb{D})$ if $H_v^0(\mathbb{D}) \neq \{0\}$.*

Remark 4.2. Since $\lambda - M_\Psi = M_{\lambda-\Psi}$, the above result shows that the spectrum of M_Ψ satisfies $\sigma(M_\Psi) = \overline{\Psi(\mathbb{D})}$. This shows that the multiplication operator M_Ψ is not compact.

In [13], Axler has characterized the Fredholm multiplication operators on Bergman spaces. Then on the spaces $H_v^\infty(\mathbb{D})$, the Fredholm multiplication operators and closed range multiplication operators are characterized by [31]. Further, in [61], Cichon and Seip have proved the following theorem related to closed range multiplication operators, which was conjectured by [31].

Theorem 4.3. *Let $v \in C^2(\mathbb{D})$ be a radial weight such that $-(1-|z^2|)^2 \Delta \log v(z) \to +\infty$ as $\|z\| \to 1^-$, where Δ denotes the Laplacian. Then $M_\Psi : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ has closed range if and only if $\Psi = hb$, where h is invertible in H^∞ and b is finite Blaschke product.*

Further, Manhas has extended in [60] the characterizations of invertible multiplication operators to the weighted spaces of vector-valued analytic functions. We begin with stating an invertibility criterion on a Hausdorff topological vector space [62], which we have used for characterizing invertible multiplication operators on the spaces $HV_b(G,E)$.

Theorem 4.4. *Let E be a complete Hausdorff topological vector space and let $T : E \to E$ be a continuous linear operator. Then T is invertible if and only if T is bounded below and has dense range. Or, let E be a Hausdorff topological vector space and let $T : E \to E$ be a continuous linear operator. Then T is invertible if and only if T is bounded below and onto.*

In the above invertible criterion, a generalized definition of bounded below operators on Hausdorff topological vector spaces is used. Now, we give this definition as it is needed for proving some of the results of this section. A continuous linear operator T on a Hausdorff topological vector space E is said to be *bounded below* if for every neighbourhood N of the origin in E, there exists a neighbourhood M of the origin in E such that $T(N^c) \subseteq M^c$, where the symbol c stands for the complement of the neighborhood in E. We begin with the following proposition.

Proposition 4.5. *Let V be an arbitrary system of weights on G and let $\Psi : G \to B(E)$ be an analytic map such that M_Ψ is a multiplication operator on $HV_b(G,E)$. Then M_Ψ is invertible if*

(i) *for each $z \in G$, $\Psi(z) : E \to E$ is onto;*

(ii) *for each $v \in V$, there exists $u \in V$ such that $v(z)\|y\| \leq u(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$.*

Proof. Fix $z_0 \in G$. Let $v \in V$ such that $v(z_0) > 0$. Then by condition (ii), there exists $u \in V$ such that $v(z_0)\|y\| \leq u(z_0)\|\Psi_{z_0}(y)\|$, for every $y \in E$. That is, $\|\Psi_{z_0}(y)\| \geq \lambda_0\|y\|$, for every $y \in E$, where $\lambda_0 = v(z_0)/u(z_0) > 0$. This proves that $\Psi(z_0)$ is bounded below on E and hence by condition (i), $\Psi(z_0)$ is invertible in $B(E)$. We denote the inverse of $\Psi(z_0)$ by $\Psi_{z_0}^{-1}$. Now, we define $\Psi^{-1} : G \to B(E)$ as $\Psi^{-1}(z) = \Psi_z^{-1}$, for every $z \in G$. Clearly, Ψ^{-1} is an analytic map. Again, by condition (ii), it follows that

$$v(z)\left\|\Psi_z^{-1}(y)\right\| \leq u(z)\|y\|, \quad \text{for every } z \in G,\ y \in E. \tag{4.1}$$

That is, $v(z)\|\Psi^{-1}(z)\| \leq u(z)$, for every $z \in G$. Thus according to Proposition 3.3, Ψ^{-1} induces the multiplication operator $M_{\Psi^{-1}}$ on $HV_b(G,E)$ such that $M_\Psi M_{\Psi^{-1}} = M_{\Psi^{-1}} M_\Psi = I$, the identity operator. Hence M_Ψ is invertible. □

Corollary 4.6. *Let V be an arbitrary system of weights on G and let $\Psi : G \to B(E)$ be a bounded analytic map. Then M_Ψ is an invertible multiplication operator on $HV_b(G,E)$ if*

(i) *for each $z \in G$, $\Psi(z) : E \to E$ is onto;*

(ii) *Ψ is bounded away from zero.*

Proof. Since $\Psi : G \to B(E)$ is a bounded analytic function, Proposition 3.3 implies that M_Ψ is a multiplication operator on $HV_b(G,E)$, for any arbitrary system of weights V on G. By condition (ii), there exists $\lambda > 0$ such that $\|\Psi_z(y)\| \geq \lambda\|y\|$, for every $z \in G$ and $y \in E$. Let $v \in V$. Then $v(z)\|y\| \leq (1/\lambda)v(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$. Further, it implies that there exists $u \in V$ such that $v(z)\|y\| \leq u(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$. Thus according to Proposition 4.5, it follows that M_Ψ is invertible on $HV_b(G,E)$. □

The converse of the above Corollary 4.6 may not be true. That is, if an analytic map $\Psi : G \to B(E)$ is not bounded away from zero, even then M_Ψ is invertible on some of the weighted spaces $HV_b(G,E)$. This can be easily seen from the following corollary.

Corollary 4.7. *Let $V = \{\lambda\chi_K : \lambda \geq 0$ and $K \subseteq G$, K compact set$\}$. Then every analytic map $\Psi : G \to B(E)$ induces an invertible multiplication operator M_Ψ on $HV_b(G,E)$ if $\Psi(z)$ is invertible for every $z \in G$.*

Remark 4.8. Let $G = \{z \in \mathbb{C} : z = x + iy \text{ and } x > 0\}$ and let $V = \{\lambda\chi_K : \lambda \geq 0,\ K \subseteq G,$ K compact set$\}$. Let $E = H^\infty(G)$ be the Banach space of bounded analytic functions on G. We define an analytic map $\Psi : G \to B(E)$ as $\Psi(z) = M_z$, for $z \in G$, where the bounded operator $M_z : E \to E$ is defined as $M_z f = zf$, for every $f \in E$. Clearly, each $\Psi(z)$ is invertible in $B(E)$ and hence by Corollary 4.7, M_Ψ is an invertible multiplication operator on $HV_b(G,E)$. But $\Psi : G \to B(E)$ is not bounded away from zero. Also, we note that invertible multiplication operators on Bergman spaces of analytic functions [13] and weighted Banach spaces of analytic functions (see Proposition 4.1) are generated only by the functions which are bounded away from zero. Thus in general, the invertible behaviour is very much controlled by different systems of weights V on G.

Theorem 4.9. *Let V be a reasonable system of weights on G and let $\Psi : G \to B(E)$ be an operator-valued analytic map such that each $\Psi(z)$ is one-to-one and M_Ψ is a multiplication* *operator on* $H\tilde{V}_b(G,E)$*. Then* M_Ψ *is invertible if and only if*

(i) *for each* $z \in G$, $\Psi(z) : E \to E$ *is onto;*

(ii) *for each* $v \in V$*, there exists* $u \in V$ *such that* $\tilde{v}(z)\|y\| \leq \tilde{u}(z)\|\Psi_z(y)\|$*, for every* $z \in G$ *and* $y \in E$.

Proof. If conditions (i) and (ii) hold, then from Proposition 4.5, it clearly follows that M_Ψ is invertible.

Conversely, suppose that M_Ψ is invertible on $H\tilde{V}_b(G,E)$. To establish condition (i), let $z_0 \in G$ and let $f_{z_0} \in H\tilde{V}_b(G)$ such that $f_{z_0}(z_0) = 1$. Fix $0 \neq y_0 \in E$. Define an analytic map $g_{z_0} : G \to E$ as $g_{z_0}(z) = f_{z_0}(z)y_0$, for every $z \in G$. Clearly, $g_{z_0} \in H\tilde{V}_b(G,E)$. Since M_Ψ is onto, there exists $f_0 \in H\tilde{V}_b(G,E)$ such that $M_\Psi(f_0) = g_{z_0}$. That is, $\Psi_{z_0}(f_0(z_0)) = y_0$. This shows that each $\Psi(z)$ is onto. Now, to prove condition (ii), we fix $v \in V$. Then there exists $\tilde{v} \in \tilde{V}$ such that $v \leq \tilde{v}$. In view of Theorem 4.4, we conclude that M_Ψ is bounded below and onto. Further, it implies that there exists $\tilde{u} \in \tilde{V}$ with $u \in V$ such that $M_\Psi(B^c_{\tilde{v},E}) \subseteq B^c_{\tilde{u},E}$. Now, we claim that $\tilde{v}(z)\|\Psi_z^{-1}(y)\| \leq \tilde{u}(z)\|y\|$, for every $z \in G$ and $y \in E$. We fix $z_0 \in G$ and $y_0 \in E$. According to (2.7c), there exists $f_{z_0} \in B_u$ such that $|f_{z_0}(z_0)| = 1/\tilde{u}(z_0)$. Let $g_0 : G \to E$ be defined as $g_0(z) = (1/\|y_0\|)f_{z_0}(z)y_0$, for every $z \in G$. Clearly, $g_0 \in B_{u,E}$ and $\|g_0(z_0)\| = 1/\tilde{u}(z_0)$. Also, (2.7b) implies that $f_{z_0} \in B_{\tilde{u}}$ and hence $g_0 \in B_{\tilde{u},E}$. Since M_Ψ is onto, there exists $h_0 \in H\tilde{V}_b(G,E)$ such that $M_\Psi(h_0) = g_0$. That is, $\Psi_{z_0}(h_0(z_0)) = g_0(z_0)$. Since each Ψ_{z_0} is invertible, we have $h_0(z_0) = \Psi_{z_0}^{-1}(y_0)f_{z_0}(z_0)1/\|y_0\|$. Again, since $\|g_0\|_{\tilde{u},E} \leq 1$, we conclude that $M_\Psi(h_0) \notin B^c_{\tilde{u},E}$. Further, it implies that $h_0 \notin B^c_{\tilde{v},E}$. That is, $\tilde{v}(z)\|h_0(z)\| \leq 1$, for every $z \in G$. In particular, for $z = z_0$, we have $\tilde{v}(z_0)\|h_0(z_0)\| \leq 1$. That is, $\tilde{v}(z_0)\|\Psi_{z_0}^{-1}(y_0)\| \leq \tilde{u}(z_0)\|y_0\|$. This proves our claim. Since each $\Psi(z)$ is invertible, we have $\tilde{v}(z)\|y\| \leq \tilde{u}(z)\|\Psi_z(y)\|$, for every $z \in G$ and $y \in E$. This proves condition (ii). With this, the proof of the theorem is complete. □

Corollary 4.10. *Let* V *be an essential system of weights on* G *and let* $\Psi : G \to B(E)$ *be an analytic map such that each* $\Psi(z)$ *is one-to-one and* M_Ψ *is a multiplication operator on* $HV_b(G,E)$*. Then* M_Ψ *is invertible if and only if*

(i) *for each* $z \in G$, $\Psi(z) : E \to E$ *is onto;*

(ii) *for each* $v \in V$*, there exists* $u \in V$ *such that* $v(z)\|y\| \leq u(z)\|\Psi_z(y)\|$*, for every* $z \in G$ *and* $y \in E$.

Proof. Follows from Theorem 4.9 since $V \cong \tilde{V}$. □

Theorem 4.11. *Let* $V = \{\lambda\chi_K : \lambda \geq 0,\ K \subseteq G,\ K \text{ compact set}\}$ *and let* $\Psi \in H(G)$ *be nonzero. Then* M_Ψ *is not a compact multiplication operator on* $HV_b(G)$.

Proof. Suppose that M_Ψ is a compact multiplication operator on $HV_b(G)$. Since Corollary 4.7 implies that M_Ψ is invertible on $HV_b(G)$ if and only if $\Psi(z) \neq 0$, for every $z \in G$, we conclude that the spectrum of M_Ψ satisfies $\sigma(M_\Psi) = \overline{\Psi(G)}$. Now, from [57, Theorem 4], it follows that each point in the spectrum of a compact operator on a locally convex Hausdorff space is an isolated point which is a contradiction to the fact that $\sigma(M_\Psi) = \overline{\Psi(G)}$ is a connected set. Thus there is no nonzero compact multiplication operator on $HV_b(G)$. □

Now, we will characterize quasicompact multiplication operators on weighted Banach spaces of analytic functions. For this, we need to give some definitions: a continuous linear operator S on a Banach space E is said to be *quasicompact* [63] if there exists an integer n and a compact operator K on E such that $\|S^n - K\| < 1$. The *essential norm* of a continuous linear operator S on a Banach space E is defined by $\|S\|_e = \inf\{\|S-K\| : K$ compact on $E\}$. Clearly, S is compact if and only if $\|S\|_e = 0$. The *nth approximation number* of S is defined as $a_n(S) = \inf\{\|S-T_n\| : T_n$ is bounded on E, rank $T_n \le n\}$. Now, it readily follows that $\|S\|_e \le a_n(S) \le \|S\|$. For more details on the properties of the approximation numbers, we refer to [63].

Theorem 4.12. *Let v be a continuous weight on $\mathbb{D}$ and let $\Psi \in H^\infty(\mathbb{D})$. Then the multiplication operator M_Ψ on $Hv_b(\mathbb{D})$ is quasicompact if and only if $\|\Psi\|_\infty < 1$.*

Proof. If $\|\Psi\|_\infty < 1$, then by choosing K as the zero operator on $Hv_b(\mathbb{D})$, it follows that $\|M_\Psi^n - 0\| = \|M_{\Psi^n}\| = \|\Psi\|_\infty^n < 1$. Thus M_Ψ is quasicompact. Conversely, suppose that M_Ψ is a quasicompact multiplication operator on $Hv_b(\mathbb{D})$. Then there exists an integer n and a compact operator K on $Hv_b(\mathbb{D})$ such that $\|M_\Psi^n - K\| < 1$. Now, from [31, Corollary 2.5], it follows that $\|M_\Psi\|_e = \|M_\Psi\| = \|\Psi\|_\infty = a_n(M_\Psi)$, for all n. Further, it implies that $1 > \|M_\Psi^n - K\| \ge \|M_{\Psi^n}\|_e = \|\Psi\|_\infty^n$. Thus $\|\Psi\|_\infty < 1$. □

5. Dynamical systems and multiplication operators

Let $g \in H^\infty(G,B(E))$ and let $\|g\|_\infty = \sup\{\|g(z)\| : z \in G\}$. Then for each $t \in \mathbb{R}$, we define $\Psi_t : G \to B(E)$ as $\Psi_t(z) = e^{tg(z)}$, for every $z \in G$. Clearly, Ψ_t is an operator-valued bounded analytic map and hence by Proposition 3.3, M_{Ψ_t} is a multiplication operator on $HV_b(G,E)$, for any arbitrary system of weights V on G.

Theorem 5.1. *Let V be an arbitrary system of weights on G and let $\Pi : \mathbb{R}\times HV_b(G,E) \to H(G,E)$ be defined as $\Pi(t,f) = M_{\Psi_t} f$, for every $t \in \mathbb{R}$ and $f \in HV_b(G,E)$. Then Π is a linear dynamical system on $HV_b(G,E)$. Moreover, if V is a system of weights on G such that $HV_b(G,E)$ is completely metrizable, then the family $\mathcal{M} = \{M_{\Psi_t} : t \in \mathbb{R}\}$ is locally equicontinuous C_0-group of multiplication operators in $B(HV_b(G,E))$.*

Proof. We have already observed that M_{Ψ_t} is a multiplication operator on $HV_b(G,E)$, for every $t \in \mathbb{R}$. Thus it follows that $\Pi(t,f) \in HV_b(G,E)$, for every $t \in \mathbb{R}$ and $f \in HV_b(G,E)$. Clearly, Π is linear and $\Pi(0,f) = f$, for every $f \in HV_b(G,E)$. Again, it is easy to see that $\Pi(t+s,f) = \Pi(t,\Pi(s,f))$, for every $t,s \in \mathbb{R}$ and $f \in HV_b(G,E)$. To show that Π is a dynamical system, it is sufficient to prove that Π is jointly continuous. Let $\{(t_\alpha, f_\alpha)\}$ be a net in $\mathbb{R}\times HV_b(G,E)$ such that $(t_\alpha, f_\alpha) \to (t,f)$ in $\mathbb{R}\times HV_b(G,E)$. Let $v \in V$.

Then

$$\begin{aligned}
&||\Pi(t_\alpha, f_\alpha) - \Pi(t,f)||_{v,E} \\
&\quad = ||\Psi_{t_\alpha} f_\alpha - \Psi_t f||_{v,E} = \sup\{v(z)||\Psi_{t_\alpha}(z) f_\alpha(z) - \Psi_t(z) f(z)|| : z \in G\} \\
&\quad \le \sup\{v(z)||(\Psi_{t_\alpha}(z) - \Psi_t(z))(f_\alpha(z)) : z \in G||\} \\
&\qquad + \sup\{v(z)||\Psi_t(z)(f_\alpha(z) - f(z))|| : z \in G\}
\end{aligned}$$

$$\begin{aligned}
&\le \sup\left\{\left|\Psi_{t_\alpha}(z)-\Psi_t(z)\right|v(z)\left\|f_\alpha(z)\right\|: z\in G\right\}\\
&\quad+\sup\left\{\left|\Psi_t(z)\right|v(z)\left\|f_\alpha(z)-f(z)\right\|: z\in G\right\}\\
&\le \sup\left\{e^{|t|\|g\|_\infty}\left(e^{|t_\alpha-t|\|g\|_\infty}-1\right)v(z)\left\|f_\alpha(z)\right\|: z\in G\right\}\\
&\quad+\sup\left\{e^{|t|\|g\|_\infty}v(z)\left\|f_\alpha(z)-f(z)\right\|: z\in G\right\}\\
&= e^{|t|\|g\|_\infty}\left(e^{|t_\alpha-t|\|g\|_\infty}-1\right)\left\|f_\alpha\right\|_{v,E}\\
&\quad+e^{|t|\|g\|_\infty}\left\|f_\alpha-f\right\|_{v,E}\longrightarrow 0\quad\text{as } |t_\alpha-t|\longrightarrow 0,\ \left\|f_\alpha-f\right\|_{v,E}\longrightarrow 0.
\end{aligned}\tag{5.1}$$

This shows that Π is jointly continuous and hence Π is a (linear) dynamical system on $HV_b(G,E)$. Further, it implies that the family $\mathcal{M}$ is a C_o-group of multiplication operators on the weighted spaces $HV_b(G,E)$. Now, we will show that the family $\mathcal{M}$ is locally equicontinuous in $B(HV_b(G,E))$. For this, it is enough to see that for any fixed $s\in\mathbb{R}$, the subfamily $\mathcal{M}_s=\{M_{\Psi_t}: -s\le t\le s\}$ is equicontinuous on $HV_b(G,E)$. Now, it is easy to see that the subfamily $\mathcal{M}_s$ is a bounded set in $B(HV_b(G,E))$ because the map $t\to M_{\Psi_t}$ is continuous in the strong operator topology. Also, for each $f\in HV_b(G,E)$, the set $\mathcal{M}_s(f)=\{M_{\Psi_t}f: -s\le t\le s\}$ is bounded in $HV_b(G,E)$. Thus according to a corollary of the Banach-Steinhaus theorem [58], it follows that the family $\mathcal{M}$ is locally equicontinuous. □

6. Characterizations of composition operators

Every self-analytic map $\phi:\mathbb{D}\to\mathbb{D}$ induces a composition operator on the Hardy space $H^\infty(\mathbb{D})$. But these maps do not necessarily induce composition operators on the weighted space $H_v^\infty(\mathbb{D})$, for general weights v (see [32]). For example, consider the weight $v(z)=e^{-(1-|z|)^{-1}}$, for $z\in\mathbb{D}$. Then $v=\tilde{v}$. Let $\phi:\mathbb{D}\to\mathbb{D}$ be defined as $\phi(z)=(z+1)/2$, for every $z\in\mathbb{D}$. Then for $z=r\in\mathbb{R}$, we have $v(z)/v(\phi(z))=v(r)/v(\phi(r))=re^{1/(1-r)}$, for $0<r<1$. Then as $r\to 1$, $v(r)/v(\phi(r))\to\infty$, so C_ϕ is not bounded on $H_v^\infty(\mathbb{D})$.

In this section, we give characterizations of composition operators on the weighted spaces of analytic functions. We begin with a characterization of composition operators obtained in [32] on the weighted Banach spaces of scalar-valued analytic functions.

Theorem 6.1. *Let v and w be continuous bounded weights. Then the following are equivalent:*

(i) $C_\phi: H_v^\infty(\mathbb{D})\to H_w^\infty(\mathbb{D})$ *is bounded;*
(ii) $\sup_{z\in\mathbb{D}}(w(z)/\tilde{v}(\phi(z)))<\infty$;
(iii) $\sup_{z\in\mathbb{D}}(\tilde{w}(z)/\tilde{v}(\phi(z)))<\infty$.

If v and w are typical weights, then the above conditions are equivalent to

(iv) $C_\phi: H_v^0(\mathbb{D})\to H_w^0(\mathbb{D})$ *is bounded.*

Further, Garcia et al. [53] have generalized the above characterization to the weighted Banach spaces of scalar-valued analytic functions defined on the open unit ball of a Banach space. For presenting this generalization, we need to fix some definitions and notations.

Let X be a complex Banach space and B_X its open unit ball. Then clearly, the space $H_v^\infty(B_X)$ (defined in the same way as $H_v^\infty(\mathbb{D})$) is a Banach space. By $\mathfrak{B}_v$, we denote the closed unit ball of $H_v^\infty(B_X)$. It is well known that in $H_v^\infty(B_X)$, the τ_v (norm) topology is finer than the τ_0 (compact-open) topology and that $\mathfrak{B}_v$ is τ_0-compact [64]. A weight v satisfies *Condition-I* if $\inf_{x\in rB_X} v(x) > 0$, for every $0 < r < 1$ [65]. If v satisfies *Condition-I,* then $H_v^\infty(B_X) \subseteq H^\infty(B_X)$ [65]. If X is finite dimensional, then all weights on B_X satisfy *Condition-I.*

Now, we can present the extended version of the above theorem [53].

Theorem 6.2. *Let B_X and B_Y be open unit balls of the Banach spaces X and Y, respectively. Let w and v be weights on B_X and B_Y, respectively, satisfying Condition-I. Let $\phi : B_X \to B_Y$ be a holomorphic map. Then the following are equivalent:*

(i) $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ *is bounded;*

(ii) $\sup_{x\in B_X}(w(x)/\widetilde{v}(\phi(x))) < \infty$*;*

(iii) $\sup_{x\in B_X}(\widetilde{w}(x)/\widetilde{v}(\phi(x))) < \infty$*;*

(iv) $\sup_{\|\phi(x)\|>r_0}(w(x)/\widetilde{v}(\phi(x))) < \infty$, *for some* $0 < r_0 < 1$.

Proof. (iii)⇒(ii) is obvious because $w \le \widetilde{w}$.

(ii)⇒(i) let $f \in H_v^\infty(B_Y)$. Then we have $w(x)|f(\phi(x))| = (w(x)/\widetilde{v}(\phi(x)))\widetilde{v}(\phi(x))\times |f(\phi(x))| \le M\|f\|_{\widetilde{v}} = M\|f\|_v$, for all x. Hence C_ϕ is bounded.

(i)⇒(iii) if (iii) does not hold, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq B_X$ such that $\lim_{n\to\infty} \widetilde{w}(x_n)/\widetilde{v}(\phi(x_n)) = \infty$. Fix $n \in \mathbb{N}$. Let $f_n \in \mathfrak{B}_v$ be such that $|f_n(\phi(x_n))| = \widetilde{u}(\phi(x_n)) = 1/\widetilde{v}(\phi(x_n))$. Hence $|f_n(\phi(x_n))|\widetilde{w}(x_n) = \widetilde{w}(x_n)/\widetilde{v}(\phi(x_n))$, which is a contradiction to the fact that C_ϕ is bounded.

(ii)⇒(iv) is straightforward.

(iv)⇒(i) let $M = \sup_{\|\phi(x)\|>r_0}(w(x)/\widetilde{v}(\phi(x)))$. Let $x \in X$. If $\|\phi(x)\| > r_0$, then we have $w(x)|f(\phi(x))| = (w(x)/\widetilde{v}(\phi(x)))\widetilde{v}(\phi(x))|f(\phi(x))| \le M\|f\|_v$. If $\|\phi(x)\| \le r_0$, then we have $w(x)|f(\phi(x))| \le (\sup_{x\in B_X} w(x))(\sup_{x\in\overline{r_0 B_Y}} |f(y)|)$ because f is bounded in $\overline{r_0 B_Y}$. Thus we have $\sup_{x\in B_X} w(x)|f(\phi(x))| < \infty$ and $C_\phi(f) \in H_w^\infty(B_X)$, for all $f \in H_v^\infty(B_Y)$. Hence C_ϕ is bounded. □

Remark 6.3. In the above theorem, the first three conditions are equivalent even if *Condition-I* does not holds. On the other hand, in [53], Garcia et al. have given an example, which shows that *Condition-I* is necessary to prove that (iv) implies (i).

Also, Manhas [66] has further generalized Theorem 6.1 and related results of [32] to the general weighted spaces of analytic functions, which are given below.

Theorem 6.4. *Let U and V be arbitrary systems of weights on G. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator if $V \le U \circ \phi$.*

Remark 6.5. The condition $V \le U \circ \phi$ in the above theorem is not a sufficient condition for C_ϕ to be a composition operator from $HU_0(G) \to HV_0(G)$. For instance, let $G = \{z \in \mathbb{C} : z = x + iy,\ x > 0\}$ be the right half plane. Let $U = V$ be the system of constant weights on G. Let $\phi : G \to G$ be defined as $\phi(z) = z_0$, for every $z \in G$, where $z_0 \in G$ is fixed. Then, clearly, the inequality $V \le U \circ \phi$ is true. But $C_\phi : HU_0(G) \to HV_0(G)$ is not even an into map. For instance, if we take $f(z) = 1/z$, for every $z \in G$, then $f \in HU_0(G)$ but $C_\phi(f) \notin HV_0(G)$. So, in order to show that $C_\phi : HU_0(G) \to HV_0(G)$ is a composition operator, we need an additional condition on ϕ. Let $v \in V$ and $\varepsilon > 0$. Then consider the set $F(v,\varepsilon) = \{z \in G;\ v(z) \geq \varepsilon\}$. Clearly $F(v,\varepsilon)$ is a closed subset of G. In the next theorem, we have obtained a sufficient condition for C_ϕ to be a composition operator from $HU_0(G)$ into $HV_0(G)$.

Theorem 6.6. *Let U and V be arbitrary systems of weights on G. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : HU_0(G) \to HV_0(G)$ is a composition operator if*

(i) *$V \leq U \circ \phi$;*

(ii) *for every $v \in V$, $\varepsilon > 0$, and compact set $K \subseteq G$, the set $\phi^{-1}(K) \cap F(v,\varepsilon)$ is compact.*

Proof. In view of Theorem 6.4, condition (i) implies that $C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator. To show that $C_\phi : HU_0(G) \to HV_0(G)$ is a composition operator, it is enough to prove that C_ϕ is an into map. Let $f \in HU_0(G)$. Let $v \in V$ and $\varepsilon > 0$. Then we consider the set $K = \{z \in G : v(z)|f(\phi(z))| \geq \varepsilon\}$. We will show that K is a compact subset of G. By condition (i), there exists $u \in U$ such that $v(z) \leq u(\phi(z))$, for every $z \in G$. Let $S = \{z \in G : u(z)|f(z)| \geq \varepsilon\}$. Then clearly, S is a compact subset of G and $\phi(K) \subseteq S$. Let $M = \sup\{|f(z)| : z \in S\}$. Then $M > 0$ and $S \subseteq F(u,\varepsilon/M)$. By condition (ii), the set $\phi^{-1}(S) \cap F(v,\varepsilon/M)$ is compact. Since K is a closed subset of the set $\phi^{-1}(S) \cap F(v,\varepsilon/M)$, it follows that K is compact. Thus $C_\phi(f) \in HV_0(G)$. This completes the proof. □

Corollary 6.7. *Let U and V be arbitrary systems of weights on G. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then*

(i) *$C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator if $V \leq U \circ \phi$;*

(ii) *$C_\phi : HU_0(G) \to HV_0(G)$ is a composition operator if $V \leq U \circ \phi$ and ϕ is a conformal mapping of G onto itself.*

The converse of the above corollary may not be true. That is, if C_ϕ is a composition operator on $HV_b(G)$ and $HV_0(G)$, then $\phi \in H(G)$ may not be conformal mapping of G onto itself. For example, let $V = \{\lambda\chi_K : \lambda \geq 0,\ K \subseteq G,\ K \text{ is compact}\}$, then it can be easily seen that C_ϕ is a composition operator on $HV_0(G)$ if and only if $\phi : G \to G$ is an analytic map.

In the next theorem, Manhas [66] has obtained a necessary and sufficient condition for C_ϕ to be a composition operator on $HV_b(G)$ in terms of the inducing map ϕ and the system of weights V.

Theorem 6.8. *Let V be an arbitrary system of weights on G and let U be a reasonable system of weights on G. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator if and only if $V \leq \widetilde{U} \circ \phi$.*

Proof. Suppose that $C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator. Let $v \in V$. Then by the continuity of C_ϕ at the origin, there exists $u \in U$ and a neighbourhood B_u of the origin in $HU_b(G)$ such that $C_\phi(B_u) \subseteq B_v$. Let $\widetilde{u}$ be the associated weight of u. Then $\widetilde{u} \in \widetilde{U}$. Now, we claim that $v \leq \widetilde{u} \circ \phi$. Fix $z_0 \in G$. Then by (2.7c), there exists $f_0 \in B_u$ such that $|f_0(\phi(z_0))| = 1/\widetilde{u}(\phi(z_0))$. Further, it implies that $C_\phi(f_0) \in B_v$. That is, $v(z)|f_0(\phi(z))| \leq 1$, for every $z \in G$. In particular, for $z = z_0$, we have $v(z_0) \leq \widetilde{u}(\phi(z_0))$. This proves our claim and hence $V \leq \widetilde{U} \circ \phi$.

Conversely, suppose that the condition is true. To show that $C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator, it is sufficient to show that C_ϕ is continuous at the origin. Let $v \in V$ and B_v be a neighbourhood of the origin in $HV_b(G)$. Then by the given condition, there exists $\widetilde{u} \in \widetilde{U}$ with $u \in U$ such that $v \leq \widetilde{u} \circ \phi$. That is, $v(z) \leq \widetilde{u}(\phi(z))$, for every $z \in G$. Now, we claim that $C_\phi(B_u) \subseteq B_v$. Let $f \in B_u$. Then by (2.7b), $\|f\|_u \leq 1$ if and only if $\|f\|_{\widetilde{u}} \leq 1$. Now,

$$\begin{aligned} \|C_\phi f\|_v &= \sup\{v(z)\,|f(\phi(z))|: z \in G\} \\ &\leq \sup\{\widetilde{u}(\phi(z))\,|f(\phi(z))|: z \in G\} \\ &\leq \sup\{\widetilde{u}(z)\,|f(z)|: z \in G\} = \|f\|_{\widetilde{u}} \leq 1. \end{aligned} \tag{6.1}$$

This proves that $C_\phi f \in B_v$ and hence C_ϕ is a composition operator. This completes the proof of the theorem. □

Corollary 6.9. *Let U and V be reasonable systems of weights on G. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then the following statements are equivalent:*

(i) *$C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator;*
(ii) *$V \leq \widetilde{U} \circ \phi$;*
(iii) *$\widetilde{V} \leq \widetilde{U} \circ \phi$.*

Corollary 6.10. *Let V be an arbitrary system of weights on G and let U be an essential system of weights on G. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : HU_b(G) \to HV_b(G)$ is a composition operator if and only if $V \leq U \circ \phi$.*

Theorem 6.11. *Let V be an arbitrary system of weights on G and let U be an essential system of weights on G such that each weight of V and U vanishes at infinity. Let $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then $C_\phi : HU_0(G) \to HV_0(G)$ is a composition operator if and only if $V \leq U \circ \phi$.*

Example 6.12. Let $G = \mathbb{D}$, the open unit disc, and let v be a weight defined as $v(z) = 1 - |z|^2$, for every $z \in G$. Let $V = \{\lambda v : \lambda > 0\}$. Then clearly, V is an essential system of weights on G. Let $\phi : G \to G$ be an analytic map defined by $\phi(z) = (z+1)/2$, for every $z \in G$. Now, by the Pick-Schwarz lemma, it follows that

$$(1 - |z|^2)\,|\phi'(z)| \leq 1 - |\phi(z)|^2, \quad \text{for every } z \in G. \tag{6.2}$$

That is, $v(z) \leq 2v(\phi(z))$, for ever $z \in G$. Hence by Theorem 6.4, C_ϕ is a composition operator on $HV_b(G)$.

Remark 6.13. If $G = \mathbb{D}$ and U and V consist of single continuous weights only, then Corollaries 6.9, 6.10 and Theorem 6.11 reduce to the results of [32, Proposition 2.1 and Corollary 2.2].

7. Compact and weakly compact composition operators

In [67], Aron et al. have characterized the compact composition operators on the Banach algebra of bounded analytic functions. This is recorded in the following theorem.

Theorem 7.1. *Let $C_\phi : H^\infty(B_Y) \to H^\infty(B_X)$ be a composition operator. Then the following statements are equivalent:*

(i) *C_ϕ is compact;*

(ii) *C_ϕ is weakly compact and $\phi(B_X)$ is relatively compact in Y;*

(iii) *$\phi(B_X)$ lies strictly inside B_Y and $\phi(B_X)$ is relatively compact in Y.*

Further, Galindo et al. [68] have obtained a characterization of weakly compact composition operators in terms of the inducing map $\phi : B_X \to B_Y$, which is stated below.

Theorem 7.2. *The Composition operator $C_\phi : H^\infty(B_Y) \to H^\infty(B_X)$ is weakly compact if (i) $\phi(B_X) \subset rB_Y$, for some $0 < r < 1$ and (ii) $\phi(B_X)$ is relatively compact in $(Y,\sigma(Y,P(Y)))$. The converse holds, if moreover, Y has the approximation property. (By $(Y,\sigma(Y,P(Y)))$, we mean the space Y endowed with the weakest topology, making all $p \in P(Y)$ continuous, where $P(Y)$ denote the algebra of all continuous polynomials on Y.)*

Recently, Garcia et al. [53] has obtained characterizations of compact composition operators on weighted Banach spaces of analytic functions, which generalizes the above Theorem 7.1 and [32, Theorem 3.3]. This generalization is presented in the following theorem.

Theorem 7.3. *Let v and w be weights on B_Y and B_X, respectively, with $\lim_{\|x\|\to 1^-} w(x) = 0$. Let $\phi : B_X \to B_Y$ be a holomorphic map. Then $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ is compact if and only if*

(i) $\lim_{\|x\|\to 1^-}(w(x)/\tilde{v}(\phi(x))) = 0$;

(ii) *$\phi(rB_X)$ is relatively compact, for every $0 < r < 1$.*

It has been observed in [32, 50] that many weights do not satisfy this condition on the limit. In [32], Bonet et al. have characterized compact composition operators for general weights when $X = Y = \mathbb{C}$. This characterization is given in terms of an analytic condition (see (A) below in part (c)). Then by using a topological condition, Garcia et al. [53] have obtained a characterization of compact composition operators for general Banach spaces X and Y. This is presented in the following theorem.

Theorem 7.4. *Let v and w be weights on B_Y and B_X, respectively, with Condition-I. Let $\phi : B_X \to B_Y$ be a holomorphic map. Then the following hold.*

(a) *If $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ is compact, then $\phi(rB_X)$ is relatively compact, for every $0 < r < 1$.*

(b) *Suppose that $\|\phi\|_\infty < 1$. If $\phi(B_X)$ is relatively compact, then $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ is compact.*

(c) *Suppose that $\|\phi\|_\infty = 1$.*

(i) *If $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ is compact, then*

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}(\phi(x))} = 0. \tag{A}$$

(ii) *If* $\phi(B_X) \cap rB_Y$ *is relatively compact, for every* $0 < r < 1$, *and*

$$\lim_{r \to 1^-} \sup_{\|\phi(x)\| > r} \frac{w(x)}{\widetilde{v}(\phi(x))} = 0, \tag{7.1}$$

then $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ *is compact.*

In part (b) of the above theorem, if the space $H_w^\infty(B_X)$ is replaced by $H^\infty(B_X)$, then the improved result is as follows.

Proposition 7.5. *Let* v *be a weight on* Y *and Let* $\phi : B_X \to B_Y$ *be a holomorphic map. Then* $C_\phi : H_v^\infty(B_Y) \to H^\infty(B_X)$ *is compact if and only if* $\phi(B_X)$ *is relatively compact and* $\|\phi\|_\infty < 1$.

Further, if w is taken as a norm-radial weight (i.e., $w(x) = w(y)$, for every x, y, such that $\|x\| = \|y\|$), then the compactness of C_ϕ is better and given in the following corollary.

Corollary 7.6. *Let* v *and* w *be weights on* B_Y *and* B_X, *respectively, such that* w *is norm-radial. Let* $\phi : B_X \to B_Y$ *be a holomorphic map. Then we have the following:*

(a) *If* $w(x) \to 0$ *as* $\|x\| \to 1^-$, *then* $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ *is compact if and only if* $\phi(rB_X)$ *is relatively compact, for every* $0 < r < 1$ *and* $\lim_{\|x\| \to 1^-}(w(x)/\widetilde{v}(\phi(x))) = 0$.

(b) *If* $w(x) \not\to 0$ *as* $\|x\| \to 1^-$, *then* $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ *is compact if and only if* $\phi(B_X)$ *is relatively compact and* $\|\phi\|_\infty < 1$.

If Y is finite dimensional, then $\phi(B_X)$ is always relatively compact and in this case, the following corollary reduces to [32, Theorem 3.3], whenever $X = Y = \mathbb{C}$.

Corollary 7.7. *Let* Y *be a finite dimensional Banach space and* X *a complex Banach space. Let* v, w *be weights and let* $\phi : B_X \to B_Y$ *be a holomorphic map. Then we have the following.*

(a) *If* $\|\phi\|_\infty < 1$, *then* $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ *is compact.*

(b) *If* $\|\phi\|_\infty = 1$, *then* $C_\phi : H_v^\infty(B_Y) \to H_w^\infty(B_X)$ *is compact if and only if*

$$\lim_{r \to 1^-} \sup_{\|\phi(x)\| > r} \frac{w(x)}{\widetilde{v}(\phi(x))} = 0. \tag{7.2}$$

Remark 7.8. Garcia et al. [53] have given examples to show that the converse of (a), (b), and (c)(i) or (c)(ii) in Theorem 7.4 does not hold in general.

In [34], Bonet et al. have further generalized Theorem 7.2 to the Weighted Banach spaces of vector-valued analytic functions besides characterizing weakly compact composition operators on vector-valued Hardy spaces, Bergman spaces, and Bloch spaces. Here, we present a vector-valued version of Theorem 7.2.

Theorem 7.9. *Let* v *be an essential weight. Then* $C_\phi : H_v^\infty(\mathbb{D}, E) \to H_w^\infty(\mathbb{D}, E)$ *is weakly compact if and only if the Banach space* E *is reflexive and*

$$\lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{v(z)}{v(\phi(z))} = 0 \quad \text{or} \quad \|\phi\|_\infty < 1. \tag{7.3}$$

Remark 7.10. Recently in 2002, Bonet and Friz [69] have extended the above characterization of weakly compact composition operators to the weighted locally convex spaces

$HV(\mathbb{D},E)$ of vector-valued analytic functions, where $V=\{v_n\}$ is an increasing sequence of strictly positive, radial, continuous, bounded weights and E is a complete, barralled locally convex space.

8. Composition operators and homomorphisms

In this section, we present a few results which relate homomorphisms with composition operators [66]. We will begin with a characterization of all continuous linear operators on $HV_b(G)$, which are composition operators and this parallels a standard result for functional Hilbert spaces.

For each $z\in G$, the point evaluation δ_z defines a continuous linear functional on $HV_b(G)$. If we put $\Delta(G)=\{\delta_z : z\in G\}$, then $\Delta(G)$ is a subset of the continuous dual $HV_b(G)^*$.

Theorem 8.1. *Let $\Phi : HV_b(G)\to HV_b(G)$ be a linear transformation. Then there exists $\phi : G\to G$ such that $\Phi=C_\phi$ if and only if the transpose mapping Φ' from $HV_b(G)^*$ into the algebraic dual $HV_b(G)'$ leaves $\Delta(G)$ invariant. In case V is a reasonable system of bounded weights on G and $\Phi'(\Delta(G))\subset\Delta(G)$, ϕ is necessarily analytic and $\Phi=C_\phi$ is continuous if and only if $V\leq\tilde{V}o\phi$.*

Proof. Suppose that $\Phi=C_\phi$, for some $\phi : G\to G$. Let $z\in G$ and $f\in HV_b(G)$. Then

$$(\Phi'\delta_z)(f)=(\delta_z o\Phi)(f)=\delta_z(\Phi(f))=\delta_z(C_\phi f)=f(\phi(z))=\delta_{\phi(z)}(f). \tag{8.1}$$

This implies that $\Phi'\delta_z=\delta_{\phi(z)}$. Conversely, let us suppose that $\Phi'(\Delta(G))\subset\Delta(G)$. For $z\in G$ if we define $\phi(z)$ to be the unique element of G such that $\Phi'\delta_z=\delta_{\phi(z)}$. Let $f\in HV_b(G)$. Then

$$\Phi(f)(z)=\delta_z(\Phi(f))=(\delta_z o\Phi)(f)=(\Phi'\delta_z)(f)=\delta_{\phi(z)}(f)=f(\phi(z))=C_\phi(f)(z). \tag{8.2}$$

Thus $\Phi=C_\phi$. Also, since the identity function $f(z)=z$ belongs to $HV_b(G)$ and the range of C_ϕ is contained in $H(G)$, ϕ is necessarily an analytic map. Also, in view of Corollary 6.9, $\Phi=C_\phi$ is continuous when $V\leq\tilde{V}o\phi$. □

Theorem 8.2. *Let G be an open connected bounded subset of $\mathbb{C}$ and let V be a system of bounded weights on G such that $V\leq V^2$. Let $\Phi : HV_b(G)\to\mathbb{C}$ be a nonzero multiplicative linear functional. Then there exists $z_0\in G$ such that $\Phi=\delta_{z_0}$.*

Proof. Let $\lambda\in\mathbb{C}$ and let K_λ denote the constant function $K_\lambda(z)=\lambda$, for every $z\in G$. Since each weight $v\in V$ is bounded, it follows that each constant function $K_\lambda\in HV_b(G)$. Let $\Phi : HV_b(G)\to\mathbb{C}$ be a nonzero multiplicative linear functional. Then we have $\Phi(K_1)=\Phi(K_1\cdot K_1)=\Phi(K_1)\Phi(K_1)$. That is, $\Phi(K_1)$ is equal to zero or one. In case $\Phi(K_1)=0$, it follows that $\Phi(f)=\Phi(f\cdot K_1)=\Phi(f)\Phi(K_1)=0$, for every $f\in HV_b(G)$. Thus $\Phi=0$, a contradiction. This shows that $\Phi(K_1)=1$. Further, it implies that $\Phi(K_\lambda)=\Phi(K_\lambda\cdot K_1)=\Phi(\lambda\cdot K_1)=\lambda\Phi(K_1)=\lambda$. Let $f : G\to\mathbb{C}$ be defined as $f(z)=z$, for every $z\in G$. Then clearly, $f \in HV_b(G)$. Now, we fix $z_0 = \Phi(f)$. We will show that $z_0 \in G$. Suppose that $z_0 \notin G$. Then we define the function $h_{z_0} : G \to \mathbb{C}$ as $h_{z_0}(z) = 1/(z - z_0)$, for every $z \in G$. Again, since each weight $v \in V$ is bounded and G is a bounded domain, it follows that $h_{z_0} \in HV_b(G)$. Also, from the definition of h_{z_0}, we have $(z - z_0)h_{z_0}(z) = 1$, for every $z \in G$. That is, $(f(z) - K_{z_0}(z))h_{z_0}(z) = 1$, for every $z \in G$. Thus $(f - K_{z_0})h_{z_0} = K_1$. Further, it implies that $\Phi(K_1) = \Phi(f - K_{z_0})\Phi(h_{z_0}) = (\Phi(f) - \Phi(K_{z_0}))\Phi(h_{z_0}) = (z_0 - z_0)\Phi(h_{z_0}) = 0$, which is a contradiction because $\Phi(K_1) = 1$. This proves that $z_0 \in G$. Now, let $g \in HV_b(G)$. Then we define the function $h : G \to \mathbb{C}$ as

$$h(z) = h_{z_0}(z)(g(z) - g(z_0)), \quad \text{for } z \neq z_0, \qquad h(z) = g'(z_0), \quad \text{for } z = z_0. \tag{8.3}$$

It can be easily seen that $h \in HV_b(G)$. Now, it readily follows that $(f - K_{z_0})h = g - K_{g(z_0)}$. Further, we have $\Phi((f - K_{z_0})h) = \Phi(g - K_{g(z_0)})$. That is, $0 = \Phi(g) - \Phi(K_{g(z_0)})$. Thus it follows that $\Phi(g) = g(z_0) = \delta_{z_0}(g)$. This proves that $\Phi = \delta_{z_0}$. With this, the proof of the theorem is complete. □

Theorem 8.3. *Let G_1 and G_2 be open connected bounded subsets of $\mathbb{C}$. Let V and U be sytems of bounded weights on G_1 and G_2, respectively, such that $V \leq V^2$ and $U \leq U^2$. Let $\Phi : HV_b(G_1) \to HV_b(G_2)$ be a nonzero algebra homomorphism. Then there exists a holomorphic map $\phi : G_2 \to G_1$ such that $\Phi = C_\phi$.*

Proof. Since $HV_b(G_1)$ and $HV_b(G_2)$ contains constant functions, it follows that $K_1 \in HV_b(G_1)$ and $\Phi(K_1) = \Phi(K_1) \cdot \Phi(K_1)$. Then using connectedness of G_2, we can conclude that $\Phi(K_1) = K_1$. Further, it implies that $\Phi(K_\lambda) = K_\lambda$, for every $\lambda \in \mathbb{C}$. Now, let $z_0 \in G_2$. Then define $\delta_{z_0} : HV_b(G_1) \to \mathbb{C}$ as $\delta_{z_0}(f) = (\Phi f)(z_0)$. Clearly, δ_{z_0} is a multiplicative linear functional on $HV_b(G_1)$. Hence by Theorem 8.2, there exists $\alpha \in G_1$ such that $\delta_{z_0}(f) = \delta_\alpha(f) = f(\alpha)$, for every $f \in HV_b(G_1)$. Let $g : G_1 \to \mathbb{C}$ be defined as $g(z) = z$, for every $z \in G_1$. Then clearly, $g \in HV_b(G_1)$ and $\delta_{z_0}(g) = g(\alpha)$. Thus it follows that $(\Phi g)(z_0) = \alpha$. Let us define $\phi = \Phi(g)$. Thus $\phi : G_2 \to G_1$ is an analytic map such that $(\Phi f)(z_0) = f(\alpha) = f(\Phi g)(z_0) = (f o \phi)(z_0)$, $z_0 \in G_2$. This shows that $\Phi(f) = C_\phi(f)$, for every $f \in HV_b(G_1)$. Hence $\Phi = C_\phi$. With this, the proof of the theorem is complete. □

Theorem 8.4. *Let G be an open connected bounded subset of $\mathbb{C}$ and let V be a system of bounded weights on G such that $V \leq V^2$. Then a composition transformation C_ϕ on $HV_b(G)$ is invertible if and only if $\phi : G \to G$ is a conformal mapping.*

Proof. If ϕ is a conformal mapping, then obviously, C_ϕ is invertible on $HV_b(G)$. On the other hand, suppose A is the inverse of C_ϕ. Then we have $AC_\phi = C_\phi A = I$. For f and g in $HV_b(G)$, we have $C_\phi A(fg) = fg$. Further, it implies that $A(fg)o\phi = fg = (C_\phi Af) \cdot (C_\phi Ag) = (Af)o\phi \cdot (Ag)o\phi = (Af \cdot Ag)o\phi$. That is, $(A(fg) - Af \cdot Ag)o\phi = 0$. Since C_ϕ is invertible, ϕ is nonconstant and hence the range of ϕ is an open set. Thus it follows that $A(fg) = Af \cdot Ag$. According to Theorem 8.3, there exists an analytic map $\psi : G \to G$ such that $A = C_\psi$. Let $f(z) = z$, for every $z \in G$. Then $f \in HV_b(G)$ and we have $(C_\psi C_\phi f)(z) = (f o\phi o\psi)(z) = (\phi o\psi)(z)$, for every $z \in G$. Also, $(C_\phi C_\psi)(z) = (f o\psi o\phi)(z) = (\psi o\phi)(z)$, for every $z \in G$. From this, we conclude that ϕ is invertible with an analytic inverse map as ψ. Hence ϕ is a conformal mapping of G onto itself. □

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