

Research Article

Some External Characterizations of SV-Rings and Hereditary Rings

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We introduce a new class of rings which we call *AI*-rings. By applying this property to endomorphism rings, we give characterizations of semi-Artinian *V*-rings and hereditary rings.

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1. Introduction

In ring theory, the notion of annihilator is an important tool for studying the structures. Many characterizations and structure theorems can be derived by using this notion. On the other hand, certain classes of rings (e.g., Baer rings and Rickart rings) are defined by considering annihilators ideals. In the present work, we introduce a class of rings which is close to the class of Rickart rings. We then investigate endomorphism rings having this property. This will enable us to obtain characterizations of certain classes of rings, namely the SV-rings and the hereditary rings.

We have divided this work into three sections. In the first we give some preliminary results and examples. In the second, we characterize SV-rings. The third section is devoted to hereditary rings.

2. Definitions and preliminary results

All rings are assumed to be associative and unitary. All modules are left unital modules. For definitions and background, we refer to [1] or [2].

Definitions 2.1. (i) Let R be a ring. The right annihilator of $S \subset R$ is the set $\text{Ann}_r(S) = \{a \in R : sa = 0 \text{ for all } s \in S\}$.

(ii) A ring is called a left *AI*-ring if every nonzero right annihilator contains a nonzero idempotent.

Examples 2.2. (i) A ring R is called left Rickart (or left PP-ring), if every left annihilator of an element $s \in R$ is generated by an idempotent. If R is left Rickart, then R is left *AI*. For if $I = \text{Ann}_r(S)$ is a nonzero right annihilator, one can pick a nonzero element a in I and consider $\text{Ann}_l(a)$. Since R is left Rickart, there exists an idempotent $e \neq 1$ in R such that $\text{Ann}_l(a) = Re$. Now $S \subset Re$, implying that $S(1 - e) = 0$. That is, $0 \neq 1 - e \in I$.

(ii) Any prime ring R with nonzero socle is left and right *AI* since $\text{Soc}(R)$ is (left and right) essential, so if $I = \text{Ann}_r(S)$, then $I \cap \text{Soc}(R) \neq 0$. Thus I contains a minimal right ideal which is generated by an idempotent. In particular, if E is a normed vector space, then the algebra of bounded linear operator $\text{BL}(E)$ is left and right *AI*.

PROPOSITION 2.3. *If R is a left *AI*-ring, then R is left nonsingular.*

Proof. Suppose that the left singular ideal of R , $\text{Sing}(R)$, is nonzero, and let $0 \neq x \in \text{Sing}(R)$. Then there exists an essential left ideal I of R such that $Ix = 0$. But R is left *AI*, thus there exists a nonzero idempotent $e \in R$ such that $Ie = 0$. This implies that $I \subset R(1 - e)$, a contradiction. \square

Example 2.4. The converse of the preceding proposition is not true as we see by the following construction. Let K be any field, consider the factor ring $R = K[x, y]/(xy)$, where $K[x, y]$ is the polynomial ring of two commuting indeterminates x, y , and (xy) is the ideal of $K[x, y]$ generated by xy . R is nonsingular but $\text{Ann } x = (y)$ contains no nonzero idempotent.

For rings with some finiteness conditions on idempotents, the condition of being an *AI*-ring is equivalent to that of being Baer ring.

THEOREM 2.5. *Let R be a ring having no infinite set of pairwise orthogonal idempotents. The following assertions are equivalent:*

- (i) R is left *AI*;
- (ii) R is right *AI*;
- (iii) R is Baer;
- (iv) R is left and right Rickart.

For the proof, see [3, Theorem 7.55].

3. Characterization of SV-ring by the *AI* condition

In this section, we will consider modules with *AI* endomorphism ring. This will lead us to an external characterization of the class of semi-Artinian V -rings.

LEMMA 3.1. *Let R be a ring, L, P, Q nonzero R -modules, and Q' the direct sum or the direct product of a denumerable family of copies of Q . Let u, v be the endomorphisms of Q' defined*

by

$$\begin{aligned} u(x_1, x_2, \dots, x_n, \dots) &= (0, x_1, x_2, \dots, x_n, \dots), \\ v(x_1, x_2, \dots, x_n, \dots) &= (x_2, x_3, \dots, x_n, \dots). \end{aligned} \tag{3.1}$$

(i) Let $h : P \rightarrow Q$ and $f : P \rightarrow Q'$ be defined by $f(x) = (h(x), 0, \dots, 0, \dots)$. Consider the endomorphism ϕ of $Q' \oplus P \oplus L$ defined by

$$\phi = \begin{pmatrix} u & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_L \end{pmatrix}. \tag{3.2}$$

Then, if there exists a nonzero projection θ of $Q' \oplus P \oplus L$ such that $\phi\theta = 0$, then there exists a nonzero projection p of P such that $hp = 0$.

(ii) Let $h' : P \rightarrow Q$ and $f' : Q' \rightarrow P$ be defined by $f'(x_1, x_2, \dots, x_n, \dots) = h'(x_1)$. Consider the endomorphism ϕ' of $Q' \oplus P \oplus L$ defined by

$$\phi' = \begin{pmatrix} v & 0 & 0 \\ f' & 0 & 0 \\ 0 & 0 & I_L \end{pmatrix}. \tag{3.3}$$

Then, if there exists a nonzero projection θ' of $Q' \oplus P \oplus L$ such that $\theta'\phi' = 0$, then there exists a nonzero projection p' of P such that $p'h' = 0$.

Proof. We will only prove (i), since (ii) is dual of (i).

Let

$$\theta = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{3.4}$$

be a nonzero projection of $Q' \oplus P \oplus L$ such that $\phi\theta = 0$. Then $c_i = 0$ for $i = 1, 2, 3$. On the other hand, $ua_i + fb_i = 0$ for $i = 1, 2, 3$. Composing on the left by v , using the fact that $vu = I_{Q'}$ and $vf = 0$, we obtain $a_i = 0$ for $i = 1, 2, 3$. The equality $\theta^2 = \theta$ implies $b_2^2 = b_2$. Now $b_2 \neq 0$, since otherwise $\theta^2 = \theta = 0$, a contradiction. So $p = b_2$ is a nonzero idempotent of $\text{End}_R(P)$ such that $fp = 0$ and then $hp = 0$.

Recall that a ring R is said to be a (left) V -ring if every simple R -module is injective. Or equivalently, the Jacobson radical of every R -module is zero. A ring is said to be (left) semi-Artinian if every nonzero module has a nonzero socle.

A ring which is together a V -ring and semi-Artinian is called a semi-artinian V -ring, (SV-ring). Many authors have studied the class of SV-rings, see for example [4, 5], or [6, 7], for SV-modules. □

THEOREM 3.2. *Let R be a ring. Then the following propositions are equivalent.*

- (i) For every R -module M , the endomorphism ring $\text{End}_R(M)$ is left AI.
- (ii) For every R -module M and every nonzero submodule N of M , there exists an idempotent endomorphism p of M such that $p(M) \subset N$.

- (iii) Every nonzero R -module contains a nonzero injective submodule.
- (iv) R is a semi-artinian V -ring.

Proof. The equivalence (iii) \Leftrightarrow (iv) is by [6, Theorem 13].

(ii) \Rightarrow (iii). Take the injective hull of $E(M)$ of M . If N is a submodule of M , then there exists a projection p of $E(M)$ such that $p(E(M)) \subset N$. But $p(E(M))$ is a direct summand of $E(M)$, and it is therefore injective.

(iv) \Rightarrow (i). Let R be semi-Artinian V -ring, M a nonzero R -module, and $S \subset \text{End}_R(M)$ such that $\text{Ann}_r(S) \neq 0$. So, there exists a nonzero $u \in \text{Ann}(S)$ such that $Su = 0$, that is, $u(M) \subset \bigcap_{s \in S} \text{Ker } s$. Since R is semi-Artinian, there exists a simple submodule $N \subset u(M)$. On the other hand, R is a V -ring, thus N is injective and hence a summand of M . Now take an idempotent endomorphism p such that $p(M) = N$, then $Sp = 0$.

(i) \Rightarrow (ii). Let M be an R -module and N a proper submodule of M . In Lemma 3.1, take $L = N, P = M, Q = (M/N)$, and $h = \pi$, where π is the canonical surjection $M \rightarrow Q$. Let

$$\psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \iota \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

where $\iota : N \rightarrow M$, is the canonical injection. Then $\psi \neq 0$ and $\phi\psi = 0$. Since, $\text{End}_R(Q' \oplus P \oplus L)$ is left AI -ring, there exists a nonzero idempotent $\theta \in \text{End}_R(Q' \oplus P \oplus L)$, such that $\phi\theta = 0$. We conclude from Lemma 3.1 that there exists a nonzero projection p of M such that $\pi p = 0$. That is, $p(M) \subset N$. □

Remark 3.3. Dually, one can prove that for a ring R , $\text{End}_R(M)$ is right AI for every R -module M , if and only if R is semisimple.

In [8], Brodski and Grigorjan have given a characterization of left SV -ring which is analogous to Theorem 3.2, by using injective modules. This result can be deduced from our lemma as we will see presently.

THEOREM 3.4 [8, Theorem 2]. *Let R be a ring, then the following propositions are equivalent.*

- (i) For every injective left R -module M and every nonleft invertible $u \in \text{End}_R(M)$, there exists a nonzero idempotent p such that $up = 0$.
- (ii) R is an SV -ring.

Proof. (ii) \Rightarrow (i) is clear. For the converse, take in Lemma 3.1 that $L = P = M$ an injective module, N a submodule of $M, Q = E(M/N)$ the injective hull of $M/N, Q'$ the direct product of a denumerable family of copies of Q . Take $h : M \rightarrow E(M/N)$ defined by $h(x) = \pi(x)$. The endomorphism ϕ in Lemma 3.1 is not injective. Since $Q' \oplus P \oplus L$ is an injective module, then ϕ is not left invertible. By hypothesis, there exists a nonzero idempotent $\theta \in \text{End}_R(Q' \oplus P \oplus L)$ such that $\phi\theta = 0$. We conclude from Lemma 3.1 that there exists a nonzero projection p of M such that $\pi p = 0$. That is, $p(M) \subset N$. □

4. Characterization of hereditary rings

Recall that a ring R is said to be left hereditary if every submodule of a projective module is projective. Or equivalently, if every factor module of an injective module is injective.

The AI property can be used to give a characterization for left hereditary ring.

THEOREM 4.1. *Let R be a ring. Then the following statements are equivalent.*

- (i) R is left hereditary.
- (ii) For every injective R -module M , for every $u \in \text{End}_R(M)$, $\text{Im } u$ is a direct summand of M .
- (iii) For every injective R -module M , $\text{End}_R(M)$ is right Rickart.
- (iv) For every injective R -module M , $\text{End}_R(M)$ is right AI.
- (v) For every injective R -module M , and every nonright regular endomorphism u of M , there exists a nonzero idempotent endomorphism p such that $pu = 0$.

Proof. (i) \Rightarrow (ii) is clear since $\text{Im } u \cong M/\text{Ker } u$. For (ii) \Rightarrow (iii), see [7, Theorem 39.15]. (iii) \Rightarrow (iv) and (iv) \Rightarrow (v) are also clear. So it remains to show that (v) \Rightarrow (i). Let M be an injective R -module and N a submodule of M . We have to show that the factor module M/N is injective. Let $E(M/N)$ be its injective hull. Consider the canonical surjection $\pi : M \rightarrow M/N$ and the canonical injection $\iota : M/N \rightarrow E(M/N)$. Put $h = \iota\pi$ and let us show that h is surjective. Otherwise, there exist an injective R -module S and a homomorphism $g : E(M/N) \rightarrow S$ such that $gh = 0$. Now put in Lemma 3.1 that $L = S$, $P = E(M/N)$, $Q = M$, Q' is the direct product of a denumerable family of copies of M , and consider the endomorphism ϕ' of $Q' \oplus P \oplus L$ of the lemma.

Take

$$\psi' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g & 0 \end{pmatrix}. \tag{4.1}$$

Then $\psi' \neq 0$ and $\psi'\phi' = 0$. Since $Q' \oplus P \oplus L$ is injective, there exists an idempotent endomorphism θ' of $Q' \oplus P \oplus L$ such that $\theta'\phi' = 0$. Lemma 3.1 implies the existence of a nonzero idempotent endomorphism p' of $E(M/N)$ such that $p'h = 0$, that is, $p'(M/N) = 0$. This contradicts the fact that M/N is essential in $E(M/N)$. This contradiction shows that h is surjective. Thus ι is surjective, and hence is bijective. This means that M/N is isomorphic to $E(M/N)$ and is injective. \square

The dualization of the preceding theorem requires the existence of the dual notion of injective hull, namely the projective cover. So, we suppose that the ring R is left perfect.

THEOREM 4.2. *Let R be a left perfect ring. Then the following statements are equivalent.*

- (i) R is left hereditary.
- (ii) For every projective R -module M , and for every $u \in \text{End}_R(M)$, $\text{Ker } u$ is a direct summand of M .
- (iii) For every projective R -module M , $\text{End}_R(M)$ is left Rickart.
- (iv) For every projective R -module M , $\text{End}_R(M)$ is left AI.
- (v) For every projective R -module M , and for every nonleft regular $u \in \text{End}_R(M)$, there exists a nonzero idempotent $p \in \text{End}_R(M)$ such that $up = 0$.

Note that the equivalences of (i), (ii), and (iii) are true without the left perfectness assumption (see [7]).

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