Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 25704, 17 pages
doi:10.1155/2007/25704

Research Article

Matrix Transformations and Quasi-Newton Methods

Boubakeur Benahmed, Bruno de Malafosse, and Adnan Yassine

Received 23 December 2006; Accepted 18 March 2007

Recommended by Narendra K. Govil

We first recall some properties of infinite tridiagonal matrices considered as matrix transformations in sequence spaces of the forms s_ξ, s_ξ°, $s_\xi^{(c)}$, or $l_p(\xi)$. Then, we give some results on the finite section method for approximating a solution of an infinite linear system. Finally, using a quasi-Newton method, we construct a sequence that converges fast to a solution of an infinite linear system.

1. Introduction

In this paper, we are interested in the study of infinite linear systems represented by the matrix equation $AX = B$, where A is an infinite matrix with infinitely many rows and infinitely many columns, B and X are considered as column matrices and X is the unknown. For many applications it is necessary to find an explicit solution of this system whenever it exists. So, for a given matrix transformation A we need to know if the matrix equation $AX = B$ has a solution in a given space. Then, we are interested in the approximation of this solution. Several methods can be used for this purpose; in this paper we will consider the finite section method to construct a natural sequence of finite sequences converging to a solution. We will also consider a new method of approximation which is a direct consequence of the quasi-Newton method, where we construct a sequence that converges fast to the solution.

The plan of this paper is organized as follows. In Section 2, we recall some well-known results on matrix transformations. In Section 3, we deal with the solvability of an infinite system represented by $M(\gamma,a,\eta)X = B$, where $M(\gamma,a,\eta)$ is an infinite tridiagonal matrix, $B \in s_\alpha$, s_α°, $s_\alpha^{(c)}$, $l_p(\alpha)$, and $1 \le p < \infty$. Then in Section 4, we recall some recent results on the finite section method and construct a natural sequence of finite sequences converging to a solution of an infinite linear system. Finally, in Section 5, we recall some results on quasi-Newton methods, specially the symmetric rank-one method and apply it to solve infinite linear systems. Then, a numerical example is given where we construct a sequence that converges fast to the unique solution of the system.

2. Preliminaries and well-known results

Let $A = (a_{nm})_{n,m\geq 1}$ be an infinite matrix and consider the sequence $X = (x_n)_{n\geq 1}$. We define the product $AX = (A_n(X))_{n\geq 1}$ with

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m, \tag{2.1}$$

whenever the series are convergent for all $n \geq 1$. Then, for a given sequence $B = (b_n)_{n\geq 1}$, we will study the equation $AX = B$ which is equivalent to the *infinite linear system of equations*

$$\sum_{m=1}^{\infty} a_{nm}x_m = b_n, \quad n = 1, 2, \dots. \tag{2.2}$$

Throughout Section 3 we use the convention that any term with a subscript less than 1 is equal to naught. Let s denote the set of all complex sequences. We write φ, c_0, c, and l_∞ for the sets of finite, null, convergent, and bounded sequences, respectively. For any given subsets $\mathcal{X}$, $\mathcal{Y}$ of s, we say that the operator represented by the infinite matrix $A = (a_{nm})_{n,m\geq 1}$ *maps* $\mathcal{X}$ *into* $\mathcal{Y}$ *, denoted by* $A \in (\mathcal{X}, \mathcal{Y})$ (cf. [1]), if

(i) the series defined by $A_n(X)$ are convergent for all $n \geq 1$ and for all $X \in \mathcal{X}$;

(ii) $AX \in \mathcal{Y}$ for all $X \in \mathcal{X}$.

For any subset $\mathcal{X}$ of s, we write

$$A\mathcal{X} = \{Y \in s : Y = AX \text{ for some } X \in \mathcal{X}\}. \tag{2.3}$$

Let $\mathcal{X} \subset s$ be a Banach space, with norm $\|\cdot\|_{\mathcal{X}}$. By $\mathcal{B}(\mathcal{X})$ we denote the *set of all bounded linear operators, mapping* $\mathcal{X}$ *into itself*. Thus, we have that $A \in \mathcal{B}(\mathcal{X})$ if and only if $A : \mathcal{X} \mapsto \mathcal{X}$ is a *linear operator* and

$$\|A\|^*_{\mathcal{B}(\mathcal{X})} = \sup_{X\neq 0}\left(\frac{\|AX\|_{\mathcal{X}}}{\|X\|_{\mathcal{X}}}\right) < \infty. \tag{2.4}$$

It is well known that $\mathcal{B}(\mathcal{X})$ is a Banach algebra with the norm $\|A\|^*_{\mathcal{B}(\mathcal{X})}$. A Banach space $\mathcal{X} \subset s$ *is a BK space if the coordinate functionals* $P_n : X \mapsto x_n$ *from* $\mathcal{X}$ *into* $\mathbb{R}$ *are continuous* for all n. A *BK* space $\mathcal{X}$ is *said to have AK if every sequence* $X = (x_k)_{k\geq 1} \in \mathcal{X}$ *has a unique representation* $X = \sum_{k=1}^{\infty} x_k e^{(k)}$, *where* $e^{(k)}$ denotes the sequence with $e_k^{(k)} = 1$ and $e_j^{(k)} = 0$ for $j \neq k$. It is well known that if $\mathcal{X}$ has AK, then $\mathcal{B}(\mathcal{X}) = (\mathcal{X}, \mathcal{X})$ (cf. [2]). In the following we will explicitly give some new properties of particular algebras.

3. Some results from the theory of infinite matrices

In this section, we will give some properties of the equation $AX = B$ for $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{X}$ with $\mathcal{X} \in \{s_\alpha, s_\alpha^\circ, s_\alpha^{(c)}, l_p(\alpha)\}$ and $1 \le p < \infty$.

3.1. The Banach algebra $\mathcal{B}(l_p(\alpha))$ with $1 \le p < \infty$. We write

$$U^+ = \{\alpha = (\alpha_n)_{n\ge 1} \in s : \alpha_n > 0 \ \forall n\}. \tag{3.1}$$

Recall that l_p, for $p > 0$, is the set of sequences $X = (x_n)_{n\ge1}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$. Here for any given $\alpha = (\alpha_n)_{n\ge1} \in U^+$ and $p \ge 1$, we consider the set

$$l_p(\alpha) = \left\{X \in s : \sum_{n=1}^\infty \left(\frac{|x_n|}{\alpha_n}\right)^p < \infty\right\}. \tag{3.2}$$

For $\xi \in s$ let D_ξ be the diagonal matrix defined by $[D_\xi]_{nn} = \xi_n$. We then have $D_\alpha l_p = l_p(\alpha)$. It is easy to see that $l_p(\alpha)$ is a *Banach space* with the norm

$$\|X\|_{l_p(\alpha)} = \left\|D_{1/\alpha}X\right\|_{l_p} = \left[\sum_{n=1}^\infty \left(\frac{|x_n|}{\alpha_n}\right)^p\right]^{1/p}. \tag{3.3}$$

Since $l_p(\alpha)$ *has AK*, we have $\mathcal{B}(l_p(\alpha)) = (l_p(\alpha), l_p(\alpha))$ and $\mathcal{B}(l_p(\alpha))$ *is a Banach algebra with identity* (cf. [3]). So, we get

$$\|AX\|_{l_p(\alpha)} \le \|A\|^*_{\mathcal{B}(l_p(\alpha))}\|X\|_{l_p(\alpha)} \quad \forall X \in l_p(\alpha). \tag{3.4}$$

We have $l_p = l_p(e)$, where $e = (1,\dots,1,\dots)$ and

$$\left\|D_{1/\alpha}AD_\alpha\right\|^*_{\mathcal{B}(l_p)} = \|A\|^*_{\mathcal{B}(l_p(\alpha))} \quad \forall A \in \mathcal{B}(l_p(\alpha)). \tag{3.5}$$

So, we have $A \in \mathcal{B}(l_p(\alpha))$ if and only if $D_{1/\alpha}AD_\alpha \in \mathcal{B}(l_p)$. When $\alpha = (r^n)_{n\ge1}$, for a given real $r > 0$, $l_p(\alpha)$ is denoted by $l_p(r)$. When $p = \infty$, we obtain the next results.

3.2. The Banach algebras S_α and $\mathcal{B}(\mathcal{X})$ for $\mathcal{X} = s_\alpha$, s_α°, or $s_\alpha^{(c)}$. For $\alpha = (\alpha_n)_{n\ge1} \in U^+$, we will write

$$s_\alpha = l_\infty(\alpha) = \left\{X \in s : \frac{x_n}{\alpha_n} = O(1)(n \longrightarrow \infty)\right\} \tag{3.6}$$

(cf. [3–7]). The set s_α is a Banach space with the norm $\|X\|_{s_\alpha} = \sup_{n\ge1}(|x_n|/\alpha_n)$. The set

$$S_\alpha = \left\{A = (a_{nm})_{n,m\ge1} : \|A\|_{S_\alpha} = \sup_{n\ge1}\left(\frac{1}{\alpha_n}\sum_{m=1}^\infty |a_{nm}|\alpha_m\right) < \infty\right\} \tag{3.7}$$

is a Banach algebra with identity normed by $\|A\|_{S_\alpha}$. Recall that if $A \in (s_\alpha, s_\alpha)$, then $\|AX\|_{s_\alpha} \le \|A\|_{S_\alpha}\|X\|_{s_\alpha}$ for all $X \in s_\alpha$. We have $S_\alpha = (s_\alpha, s_\alpha)$ and if we put $B(s_\alpha) = \mathcal{B}(s_\alpha) \bigcap (s_\alpha, s_\alpha)$, then $B(s_\alpha) = S_\alpha$. This means that S_α is a subalgebra of $\mathcal{B}(s_\alpha)$.

As above when $\alpha = (r^n)_{n\geq 1}$, $r > 0$, S_α and s_α are denoted by S_r and s_r. When $r = 1$, $s_1 = l_\infty$ is the set of all bounded sequences.

In the same way, we define the sets

$$s_\alpha^\circ = \left\{X \in s : \frac{x_n}{\alpha_n} \longrightarrow 0 \ (n \longrightarrow \infty)\right\},$$
$$s_\alpha^{(c)} = \left\{X \in s : \frac{x_n}{\alpha_n} \longrightarrow l \ (n \longrightarrow \infty) \text{ for some } l\right\}. \tag{3.8}$$

The sets s_α° *and* $s_\alpha^{(c)}$ *are Banach spaces with the norm* $\|\cdot\|_{s_\alpha}$ (cf. [5]). As a direct consequence of the previous results, the sets $\mathcal{B}(s_\alpha^\circ) = (s_\alpha^\circ, s_\alpha^\circ)$ and $\mathcal{B}(s_\alpha^{(c)})$ are *Banach algebras with the norm* $\|A\|_{\mathcal{B}(s_\alpha)}$.

3.3. An application to infinite tridiagonal matrices. In this subsection, we consider infinite tridiagonal matrices. These matrices are used in many applications, let us cite for instance the case of *continued fractions* (cf. [3]), or the *finite differences method*, (cf. [8]). We deal with some properties of the matrix map $M(\gamma, a, \eta)$ between particular sequence spaces. Then, we will compute the inverse of the matrix $M(\gamma, e, \eta)$ where γ and η are constants. These results will be used in Example 5.8.

Let $\gamma = (\gamma_n)_{n\geq 1}$, $\eta = (\eta_n)_{n\geq 1}$, $a = (a_n)_{n\geq 1}$ be sequences with $a_n \neq 0$ for all n. Consider the infinite tridiagonal matrix

$$M(\gamma, a, \eta) = \begin{pmatrix} a_1 & \eta_1 & & & \\ \gamma_2 & a_2 & \eta_2 & & \mathbf{O} \\ & \cdot & \cdot & \cdot & \\ \mathbf{O} & & \gamma_n & a_n & \eta_n \\ & & & \cdot & \cdot \end{pmatrix} \tag{3.9}$$

and put

$$\Gamma_\alpha = \left\{A = (a_{nm})_{n,m\geq 1} \in S_\alpha : \|I - A\|_{S_\alpha} < 1\right\}, \tag{3.10}$$

for $\alpha = (r^n)_{n\geq 1}$, Γ_α is denoted by Γ_r. Since S_α is a Banach algebra, we immediately see that $A \in \Gamma_\alpha$ implies A is invertible and $A^{-1} \in S_\alpha$. Using the results given in Sections 3.1 and 3.2, we deduce the following proposition.

Proposition 3.1 (see [7, Proposition 17, page 55]). *Assume that* $D_{1/a}M(\gamma, a, \eta) \in \Gamma_\alpha$, *that is,*

$$\sup_n \left[\frac{1}{a_n}\left(|\gamma_n| \frac{\alpha_{n-1}}{\alpha_n} + |\eta_n| \frac{\alpha_{n+1}}{\alpha_n}\right)\right] < 1. \tag{3.11}$$

Then,

(i) (a) $M(\gamma, a, \eta) \in (s_\alpha, s_{|a|\alpha})$,
(b) $M(\gamma, a, \eta)$ *is invertible and* $M(\gamma, a, \eta)^{-1} \in (s_{|a|\alpha}, s_\alpha)$,

(c) *for any* $B \in s_{|a|\alpha}$, *the equation* $M(\gamma,a,\eta)X = B$ *has a unique solution in* s_α *given by*

$$X^* = M(\gamma,a,\eta)^{-1}B, \tag{3.12}$$

(ii) (a) $M(\gamma,a,\eta) \in (s_\alpha^\circ, s_{|a|\alpha}^\circ)$,
(b) $M(\gamma,a,\eta)$ *is invertible and* $M(\gamma,a,\eta)^{-1} \in (s_{|a|\alpha}^\circ, s_\alpha^\circ)$,
(c) *for any* $B \in s_{|a|\alpha}^\circ$, *the equation* $M(\gamma,a,\eta)X = B$ *has a unique solution in* s_α° *given by (3.12),*

(iii) *if*

$$\frac{1}{a_n}\left(\gamma_n\frac{\alpha_{n-1}}{\alpha_n} + \eta_n\frac{\alpha_{n+1}}{\alpha_n}\right) \longrightarrow l \neq 0 \quad (n \longrightarrow \infty), \tag{3.13}$$

then
(a) $M(\gamma,a,\eta) \in (s_\alpha^{(c)}, s_{|a|\alpha}^{(c)})$,
(b) $M(\gamma,a,\eta)$ *is invertible and* $M(\gamma,a,\eta)^{-1} \in (s_{|a|\alpha}^{(c)}, s_\alpha^{(c)})$,
(c) *for any* $B \in s_{|a|\alpha}^{(c)}$, *the equation* $M(\gamma,a,\eta)X = B$ *has a unique solution in* $s_\alpha^{(c)}$ *given by (3.12).*

(iv) *Let* $p \geq 1$ *be a real. If* $\widetilde{K}_{p,\alpha} = K_1 + K_2 < 1$ *with*

$$K_1 = \sup_n\left(\left|\frac{\gamma_n}{a_n}\right|\frac{\alpha_{n-1}}{\alpha_n}\right), \qquad K_2 = \sup_n\left(\left|\frac{\eta_n}{a_n}\right|\frac{\alpha_{n+1}}{\alpha_n}\right), \tag{3.14}$$

then
(a) $M(\gamma,a,\eta) \in (l_p(\alpha), l_p(|a|\alpha))$,
(b) $M(\gamma,a,\eta)$ *is invertible and* $M(\gamma,a,\eta)^{-1} \in (l_p(|a|\alpha), l_p(\alpha))$,
(c) *for any* $B \in l_p(|a|\alpha)$, *the equation* $M(\gamma,a,\eta)X = B$ *has a unique solution in* $l_p(\alpha)$ *given by (3.12).*

We deduce the next corollary.

Corollary 3.2. *If* $\widetilde{K}_{1,\alpha} < 1$, *then* $M(\gamma,a,\eta)$ *is bijective from* $l_1(\alpha)$ *to* $l_1(|a|\alpha)$ *and bijective from* s_α *to* $s_{|a|\alpha}$.

Proof. First, taking $p = 1$ in Proposition 3.1(iv), we deduce that $M(\gamma,a,\eta)$ is bijective from $l_1(\alpha)$ to $l_1(|a|\alpha)$. Then, we get from [7]

$$||I - D_{1/a}M(\gamma,a,\eta)||_{S_\alpha} \leq \widetilde{K}_{1,\alpha} < 1, \tag{3.15}$$

and we conclude that $M(\gamma,a,\eta)$ is bijective from s_α to $s_{|a|\alpha}$. □

Remark 3.3. Note that in the case when $p = 1$, the condition

$$||I - [D_{1/a}M(\gamma,a,\eta)]^t||_{S_\alpha} = \sup_n\left(\left|\frac{\gamma_{n+1}}{a_{n+1}}\right|\frac{\alpha_{n+1}}{\alpha_n} + \left|\frac{\eta_{n-1}}{a_{n-1}}\right|\frac{\alpha_{n-1}}{\alpha_n}\right) < 1 \tag{3.16}$$

also implies that $M(\gamma,a,\eta)$ is bijective from $l_1(\alpha)$ to $l_1(|a|\alpha)$.

When $a = e$, $\gamma_n = \gamma$, and $\eta_n = \eta$, for all n the matrix $M(\gamma, e, \eta)$ is denoted by $M(\gamma, \eta)$. Recall the following result where we explicitly write the inverse of $M(\gamma, \eta)$.

Proposition 3.4 (see [7, Proposition 20, page 57]). *Let γ, η be reals with $0 < \gamma + \eta < 1$. Then,*

(i) *$M(\gamma,\eta) : X \mapsto M(\gamma,\eta)X$ is bijective from $\mathcal{X}$ into itself, for $\mathcal{X} \in \{s_1, c_0, c, l_p\}$, $p \geq 1$.*

(ii) (a) *Let $\mathcal{X}$ be one of the sets s_1, c_0, c, or $l_p(\alpha)$ and put*

$$u = \frac{\left(1 - \sqrt{1 - 4\gamma\eta}\right)}{2\gamma}, \qquad v = \frac{\left(1 - \sqrt{1 - 4\gamma\eta}\right)}{2\eta}. \tag{3.17}$$

Then, for any given $B \in \mathcal{X}$, the equation $M(\gamma,\eta)X = B$ has a unique solution $X^* = (x_n^*)_{n\geq 1}$ in $\mathcal{X}$ given by

$$x_n^* = \left(\frac{uv+1}{uv-1}\right)(-1)^n v^n \sum_{m=1}^{\infty} \left[1 - (uv)^{-l}\right](-1)^m u^m b_m \quad \forall n, \tag{3.18}$$

with $l = \min(n,m)$.

(b) *The inverse $[M(\gamma,\eta)]^{-1} = (a'_{nm})_{n,m\geq 1}$ is given by*

$$a'_{nm} = \left(\frac{uv+1}{uv-1}\right)(-1)^{n+m} v^{n-m}\left[(uv)^l - 1\right] \quad \forall n, m \geq 1,\ l = \min(n,m). \tag{3.19}$$

Until now we have given results on the solvability of a class of systems. Since the solution of the matrix equation can have a complicated expression, we need to approximate this one. So, in the next sections, we will explicitly deal with several methods of approximation.

4. First methods of approximation of a solution of an infinite linear system

In this section, we give useful methods to approximate a solution of an infinite linear system. So, we give some new conditions on A to obtain the convergence of a sequence of finite sequences to the solution of an infinite linear system.

4.1. The finite section method. In the following, we assume that $A \in S_\alpha$ is a matrix with nonzero elements on its main diagonal. For any integer k, let $A'_k = (\eta_{nm})_{n,m\geq 1}$ be the infinite matrix defined by

$$\eta_{nm} = \begin{cases} a_{nm} & \text{if } n, m \leq k, \\ 0 & \text{otherwise;} \end{cases} \tag{4.1}$$

B_k is the matrix obtained from B in the same way. $[A]_k$ denotes the finite matrix $(a_{nm})_{n,m\leq k}$ and $[B]_k = (b_n)_{n\leq k}$. When $[A]_k$ is invertible, we put

$$\widehat{A'}_k = \begin{pmatrix} [A]_k^{-1} & \\ & O \end{pmatrix}. \tag{4.2}$$

Note that $A'_k \widehat{A'}_k = \widehat{A'}_k A'_k = I'_k$.

The replacement of the equation $AX = B$ by $[A]_k[Y]_k = [B]_k$, where $[Y]_k$ is the unknown of the last equation, is called the finite section method. This principle has been used for Toeplitz matrices of the form

$$A = \begin{pmatrix} a_0 & a_{-1} & \cdot & \cdot & \cdot \\ a_1 & a_0 & a_{-1} & \cdot & \cdot \\ a_2 & a_1 & a_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{4.3}$$

$(a_n)_{-\infty<n<+\infty}$ is a given sequence. Note that the matrices used here need not be Toeplitz matrices.

On the other hand, the invertibility of each matrix $[A]_k$ does not imply the invertibility of A (cf. [6, Example 11, page 137]). We see that we must give additional conditions on A, so that the sequence

$$X_k = \widehat{A'}_k B_k \tag{4.4}$$

converges to a limit in a given space as k tends to infinity. Note that this problem was studied in the case when A is a *Toeplitz matrix* mapping l_2 to l_2 and was connected to the notion of stability (cf. [9]). Recall that the sequence of matrices $([A]_k)_{k\geq 1}$ is *stable* if each matrix $[A]_k$ is invertible for all sufficiently large k, for $k \geq k_0$ say, and for a suitably chosen norm $\|\cdot\|$, we have

$$\sup_{k\geq k_0} \left(\left\|[A]_k^{-1}\right\|\right) < \infty. \tag{4.5}$$

Here, we are interested in the case when $A \in (s_\alpha, s_\alpha)$ and we will see that the condition of stability is given by Definition 4.1(i).

4.1.1. First method of approximation of a solution of an infinite linear system. We first need a definition.

Definition 4.1. Let $\alpha = (\alpha_n)_{n\geq 1} \in U^+$ be a decreasing sequence with

$$\alpha_n \leq 1 \quad \forall n. \tag{4.6}$$

An infinite matrix $A \in S_\alpha$ is called α-invertible if the following conditions are satisfied.

(i) The matrix $[A]_k$ is invertible for every k and putting $[A]_k^{-1} = (a'_{nm}(k))_{n,m\leq k}$, we have

$$\tau_k = \sup_{n,m\leq k} \left|a'_{nm}(k)\right| = O(1) \quad (k \longrightarrow \infty). \tag{4.7}$$

(ii)

$$q = \left(\sup_{n \geq k} \left(\sum_{m=1}^{k-1} \frac{|a_{nm}|}{\alpha_n} \right) \right)_{k \geq 2} \in l_1. \tag{4.8}$$

When $\alpha = (r^n)_{n \geq 1}$, with $r \in]0,1]$, we say that A is r-invertible. We can state the following result.

Theorem 4.2 (see [6, Theorem 13, page 138], [4, Theorem 4, page 95]). *Let A be α-invertible. For every $B \in \varphi$, there is a solution $X^* \in s_\alpha$ of the equation $AX = B$ such that*

$$\lim_{k \to \infty} ||X_k - X^*||_{s_\alpha} = 0, \tag{4.9}$$

that is $X_k \to X^*$ $(k \to \infty)$ in s_α.

As a direct consequence of Theorem 4.2, we obtain the next example.

Example 4.3 (cf. [4, Example 7, page 98]). Let $A = (\sigma^{|m-n|m})_{n,m \geq 1}$ with $\sigma \in]0, 1/3[$. First, we see that

$$\sigma^{|m-n|m} = \left(\sigma^{|m-n|}\right)^m \leq \sigma^{|m-n|} \quad \forall m, n. \tag{4.10}$$

Then,

$$\sum_{m \geq 1, m \neq n} \sigma^{|m-n|m} \leq \sum_{i=1}^{n-1} \sigma^i + \sum_{i=1}^{\infty} \sigma^i \leq \frac{2\sigma}{1-\sigma}, \qquad ||I - A||_{S_1} = \sup_n \left(\sum_{m \geq 1, m \neq n} \sigma^{|m-n|m} \right) \leq \frac{2\sigma}{1-\sigma}. \tag{4.11}$$

So, $A \in \Gamma_1$ for $0 < \sigma < 1/3$. We deduce (cf. [4, Proposition 5, page 97, Corollary 6, page 98], that the matrix $[A]_k$ is invertible for each k, that

$$\tau_k \leq \frac{1}{1 - ||I - A||_{S_u}} \quad \forall k \text{ and for every given } u \geq 1 \tag{4.12}$$

and that Definition 4.1(i) holds. Then, we get for $\alpha = e$ in Theorem 4.2

$$q_k = \sup_{n \geq k} \left(\sum_{m=1}^{k-1} \sigma^{|m-n|m} \right) \leq \sum_{m=1}^{k-1} \sigma^{|m-k|m} \quad \forall k, \tag{4.13}$$

and since $\sigma^{|k-m|m} = \sigma^{(k-m)m} \leq \sigma^{k-1}$ for $m = 1, 2, \ldots, k-1$, we get

$$q_k \leq \sum_{m=1}^{k-1} \sigma^{(k-m)m} \leq (k-1)\sigma^{k-1} \quad \forall k. \tag{4.14}$$

Then $q \in l_1$, condition (ii) in Definition 4.1 holds, and A is 1-invertible. So, $X_k = \widehat{A'}_k B_k \to X^*(k \to \infty)$ in s_α. Note that this matrix is of Poòlya type (cf. [4, 10]), which proves that this system has infinitely many solutions. Here, we have shown that the unique bounded solution X^* of $AX = B$, $(B \in \varphi)$, can be approximated by the sequence X_k.

Remark 4.4. The results in the previous example also hold for $1/2 < \sigma < 2/3$. Indeed, we have

$$\sum_{m\geq 1, m\neq n} \sigma^{|m-n|m} \leq -1 + \sum_{m=1}^{\infty} \sigma^m = \frac{-1+2\sigma}{1-\sigma}, \tag{4.15}$$

and $\|I-A\|_{S_1} \leq (-1+2\sigma)/(1-\sigma) < 1$ for $1/2 < \sigma < 2/3$.

4.1.2. Second method of approximation of a solution of an infinite system. When we suppose that all the diagonal elements are equal to 1, we can give a similar result, where the solution

$$X^* = \sum_{n=0}^{\infty} (I-A)^n B \tag{4.16}$$

of the equation $AX = B$ can be approximated by the sequence

$$X'_k = (A_k^*)^{-1} B = \sum_{n=0}^{\infty} (I - A_k^*)^n B, \tag{4.17}$$

with

$$A_k^* = \begin{pmatrix} [A]_k & & & \\ & 1 & & \mathbf{0} \\ \mathbf{0} & & 1 & \\ & & & \cdot \end{pmatrix}. \tag{4.18}$$

The advantage of this method is that it yields an upper bound for $\|X'_k - X^*\|_{s_r}$. Note that for any given $B \in \varphi$, we have $X'_k = X_k$ for all k. Now let $r > 0$ and put

$$\gamma_k = \sup_{n\leq k}\left(\frac{1}{r^n}\sum_{m=k+1}^{\infty} |a_{nm}| r^m\right), \qquad \gamma'_k = \sup_{n\geq k+1}\left(\frac{1}{r^n}\sum_{m=1, m\neq n}^{\infty} |a_{nm}| r^m\right). \tag{4.19}$$

Then, we can state the following result based on the fact that $\|A - A_k^*\|_{S_r} = \sup(\gamma_k, \gamma'_k)$.

Proposition 4.5 (cf. [6, Proposition 14, page 140], [4, Proposition 9, page 99]). *Assume that $A \in \Gamma_r$ and $(\gamma_k)_{k\geq 1}$, $(\gamma'_k)_{k\geq 1} \in c_0$. Then, $X'_k \to X^* (k \to \infty)$ in s_r for all $B \in s_r$ and*

$$\|X'_k - X^*\|_{s_r} \leq \sup(\gamma_k, \gamma'_k) \frac{\|B\|_{s_r}}{(1-\rho)^2} \quad \forall k, \tag{4.20}$$

where $\rho = \|I - A\|_{S_r}$.

Example 4.6. Proposition 4.5 can be applied to the matrix $A = (\sigma^{|m-n|m})_{n,m\geq 1}$, defined in Example 4.3 with $0 < \sigma < 1/3$. As we have just seen, since $A \in \Gamma_1$, we will take $r = 1$. We get for every k, putting $\varkappa_k = \sigma^k/(1-\sigma^k)$,

$$\gamma_k \le \sup_{n\le k}\left(\sum_{i=1}^{\infty}\sigma^{(k+1)(k+i-n)}\right) \le \varkappa_{k+1},$$
$$\gamma'_k \le \sup_{n\ge k+1}\left((n-1)\sigma^{n-1} + \frac{\sigma^{n+1}}{1-\sigma^{n+1}}\right) \le k\sigma^k + \varkappa_{k+2}. \tag{4.21}$$

Since we have $(\varkappa_k)_{k\ge1}$, $(k\sigma^k)_{k\ge1}$, we also have $(\gamma_k)_{k\ge1}$, $(\gamma'_k)_{k\ge1} \in c_0$.

Then, there is an integer N, such that

$$k\sigma^k + \varkappa_{k+2} - \varkappa_{k+1} = k\sigma^k\left[1 + \frac{\sigma^2}{k(1-\sigma^{k+2})} - \frac{\sigma}{k(1-\sigma^{k+1})}\right] > 0 \tag{4.22}$$

for all $k \ge N$. Then using Proposition 4.5 and the inequality $\|I-A\|_{S_1} \le 2\sigma/(1-\sigma)$, we conclude that

$$\left\|X'_k - X^*\right\|_{s_1} \le \left(k\sigma^k + \varkappa_{k+2}\right)\left(\frac{1-\sigma}{1-3\sigma}\right)^2 \|B\|_{s_1} \quad \forall k \ge N. \tag{4.23}$$

Remark 4.7. We note that an r-invertible matrix does not necessarily satisfy the conditions of Proposition 4.5. Indeed, take a real ρ, $0<\rho<1$ and put

$$A = \begin{pmatrix} 1 & \rho & & \\ & 1 & \rho & \mathbf{0} \\ \mathbf{0} & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}. \tag{4.24}$$

It can easily be seen that A is 1-invertible, but $\gamma_k = \rho$ does not tend to 0. This shows that the first condition of the previous proposition is not satisfied.

In the following, we will deal with another method of approximation where we construct a sequence that converges fast to a solution of an infinite linear system.

5. Quasi-Newton methods

5.1. Well-known results. Quasi-Newton methods play an important role in numerically solving unconstrained optimization problems, systems of linear and nonlinear equations in Euclidean spaces.

Let $\mathscr{X}$ be a Hilbert space *with inner product* $\langle\cdot,\cdot\rangle$. We use the norm

$$\|X\| = \sqrt{\langle X, X\rangle} \tag{5.1}$$

induced by the inner product.

In this section, we are interested in the solvability by *quasi-Newton methods, of systems of linear equations in the infinite Hilbert space* $\mathscr{X}$. These systems can be written in the form

$$F(X) = AX - B = 0, \tag{5.2}$$

where $A \in \mathscr{B}(\mathscr{X})$ and $B, X \in \mathscr{X}$.

Since systems of linear equations can be regarded as special cases of *systems of nonlinear equations*, we will describe *quasi-Newton methods* in the general case of systems of nonlinear equations represented by

$$F(X) = 0, \tag{5.3}$$

where $F : \mathscr{X} \mapsto \mathscr{X}$ is a *differentiable function*. We make the following classical assumptions.

Assumption A. (i) (5.3) has a solution X^*.

(ii) The derivative F' exists and is *Lipschitz* continuous in a neighborhood V of X^*.

(iii) The operator $F'(X^*)$ has a bounded inverse and $[F'(X^*)]^{-1} \in \mathscr{B}(\mathscr{X})$.

We are led to give an explicit algorithm.

5.1.1. The iterative schemes for quasi-Newton methods. Every *quasi-Newton method is an iterative scheme which generates a sequence* $(X_k)_{k\geq 1}$ *in* $\aleph$ *approximating* X^* *and a sequence* $(H_k)_{k\geq 1}$ *in* $\mathscr{B}(\mathscr{X})$ *approximating* $[F'(X^*)]^{-1}$ by means of the formulas

$$X_{k+1} = X_k - H_k F(X_k), \quad k = 1,2,\ldots, \tag{5.4}$$

$$H_{k+1} = H_k + E_k, \quad k = 1,2,\ldots, \tag{5.5}$$

where $E_k \in \mathscr{B}(\mathscr{X})$ is a *correction term* depending on X_k, X_{k+1}, and H_k. The algorithm terminates when $\|F(X_k)\| < \epsilon$ for given small $\epsilon > 0$.

A quasi-Newton method is defined by formula (5.5) to compute H_{k+1}.

5.1.2. The (SR1) method. Here, we need to define the *outer product* $X \otimes Y$ of two vectors X and Y (cf. [11]), which is a rank-one operator in $\mathscr{B}(\mathscr{X})$, such that

$$(X \otimes Y)Z = \langle Y, Z\rangle X \quad \forall Z \in \aleph. \tag{5.6}$$

The notation $X \otimes Y$ generalizes the classical outer product XY^T used in finite dimensional spaces.

In the remainder of the paper, we will use the *symmetric rank-one* (SR1) formula given by

$$H_{k+1} = H_k + \frac{1}{\langle S_k - H_k Y_k, Y_k\rangle}(S_k - H_k Y_k) \otimes (S_k - H_k Y_k) \tag{5.7}$$

with

$$S_k = X_{k+1} - X_k, \quad Y_k = F(X_{k+1}) - F(X_k), \quad k = 1,2,\ldots. \tag{5.8}$$

Note that the *(SR1) method* was first published by Broyden (1967) in the finite-dimensional case (cf. [12]). Many authors have given a generalization of quasi-Newton methods to infinite Hilbert spaces (cf., e.g., Horwitz-Sarachik [13] and Sachs [14]).

In the next section, we recall some convergence results for quasi-Newton methods in infinite-dimensional spaces.

5.2. Convergence results. First, we need some definitions on convergence rates. Let $(X_k)_{k\geq 1}$ be a sequence in $\mathscr{X}$ generated by the quasi-Newton method defined by (5.4) and (5.5).

Definition 5.1. (i) A sequence $(X_k)_{k\geq 1}$ is said to be locally convergent to X^*, if there are $\epsilon, \delta > 0$ such that whenever X_1 and H_1 satisfy

$$||X_1 - X^*|| < \epsilon, \qquad ||H_1 - F'(X^*)|| < \delta, \tag{5.9}$$

then $(X_k)_{k\geq 1}$ converges to X^*.

(ii) If for each X_1, H_1, the sequence $(X_k)_{k\geq 1}$ is convergent to X^*, we say that the sequence $(X_k)_{k\geq 1}$ is globally convergent to X^*.

Note that global convergence implies local convergence, but the converse is false.

Definition 5.2 (see [11]). Let $(X_k)_{k\geq 1}$ be a sequence converging to X^*.

(i) The convergence rate is called linear if there are $\gamma \in (0,1)$ and $k_0 \in \mathbb{N}$ such that

$$||X_{k+1} - X^*|| \leq \gamma ||X_k - X^*|| \quad \forall k \geq k_0, \tag{5.10}$$

that is

$$\lim_{k\to\infty} \frac{||X_{k+1} - X^*||}{||X_k - X^*||} = \gamma. \tag{5.11}$$

(ii) The convergence rate is called superlinear if

$$\lim_{k\to\infty} \frac{||X_{k+1} - X^*||}{||X_k - X^*||} = 0. \tag{5.12}$$

Here, superlinear convergence implies linear convergence.

Now, we consider a result on local convergence that can be used for the (SR1) method and can be applied to other quasi-Newton methods. We put

$$\sigma_k = \max\{||X_{k+1} - X^*||, ||X_k - X^*||\}. \tag{5.13}$$

Theorem 5.3 (see [15, Lemma 2.2, pages 4–7]). *Suppose that F satisfies Assumption A. If there are $\alpha_1, \alpha_2 > 0$ such that*

$$||H_{k+1} - [F'(X^*)]^{-1}|| \leq (1 + \alpha_1 \sigma_k) ||H_k - [F'(X^*)]^{-1}|| + \alpha_2 \sigma_k \quad \forall k, \tag{5.14}$$

then the sequence $(X_k)_{k\geq 1}$ defined by a quasi-Newton method is well defined and converges locally and linearly to X^*.

Furthermore, H_k^{-1} exists for all k and $(\|H_k\|)_{k\geq 1}$, $(\|H_k^{-1}\|)_{k\geq 1} \in l_\infty$.

To obtain superlinear convergence, we need the additional condition of compactness of the operator

$$E_1 = H_1 - [F'(X^*)]^{-1}. \tag{5.15}$$

More precisely, we state the next result which is a direct consequence of [16].

Theorem 5.4. *Assume that Assumption A and identity (5.14) hold. If E_1 is compact, then the sequence $(X_k)_{k\geq 1}$ generated by any quasi-Newton method is superlinearly convergent to X^*.*

Remark 5.5. The compactness assumption is trivially satisfied in the finite-dimensional case but is necessary to have superlinear convergence in the infinite-dimensional case. In several quasi-Newton methods, examples were given where we only have linear convergence for a noncompact operator E_1 (cf. [16]).

In the remainder of this paper, we will consider the special case of systems of linear equations. As we have seen, they are represented by (5.2) with $A \in \mathcal{B}(\mathcal{X})$ and $B, X \in \mathcal{X}$ and we will assume that A is nonsingular. Since it can easily be proved that the operators $(H_k)_{k\geq 1}$ generated by the (SR1) method satisfy (5.14), by Theorem 5.3 we have local and linear convergence for this method. More precisely, we can state the following result.

Corollary 5.6. *Let $A \in \mathcal{B}(\mathcal{X})$ be a nonsingular operator. Then the sequence $(X_k)_{k\geq 1}$ obtained by the (SR1) method defined by (5.4), (5.7), and (5.8) converges locally and linearly to X^*. The convergence is superlinear under the additional condition of compactness of $E_1 = H_1 - A^{-1}$.*

Remark 5.7. Let $\mathbf{B}_1, A \in \mathcal{B}(\mathcal{X})$ be nonsingular. Since

$$H_1 - A^{-1} = -H_1(\mathbf{B}_1 - A)A^{-1} \tag{5.16}$$

is the product of a compact operator with bounded operators, we deduce that if $D_1 = \mathbf{B}_1 - A$ is compact, so is the $E_1 = H_1 - A^{-1} = \mathbf{B}_1^{-1} - A^{-1}$.

5.3. An application to infinite linear systems. Quasi-Newton methods have been applied to several problems in infinite-dimensional Hilbert spaces such as to approximate the solutions of nonlinear integral equations, elliptic boundary value problems, *unconstrained optimal control problems* (cf. [17–19]), *identification of a parabolic system, parabolic inverse problem* (cf. [15, 20]), and so forth but it seems that the quasi-Newton methods have never been applied directly to solving infinite linear systems.

In the special case of a Hilbert space $\mathcal{X}$, an infinite linear system can be represented by (5.2) where we assume $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{X}$ are given and A is invertible in $\mathcal{B}(\mathcal{X})$. So for any $B \in \mathcal{X}$, the equation $AX = B$ has a unique solution given by $X^* = A^{-1}B$. Note that the computation of A^{-1} is very difficult in many cases, so it is natural to use an iterative method to obtain an approximation of the solution of this equation. In the next example, we construct a sequence which is obtained by a quasi-Newton method and converges to the solution X^* given in Proposition 3.4 in the particular case when $A = 3M(\gamma, \eta)$ and $\gamma = \eta = 1/3$.

Example 5.8. Take $\mathscr{X} = l_2$. Then, we have

$$\langle X, Y\rangle = \sum_{n=1}^{\infty} x_n y_n, \quad \|X\| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2} \tag{5.17}$$

for all $X = (x_n)_{n\geq 1}$, $Y = (y_n)_{n\geq 1} \in l_2$.

We consider $B = e^{(1)}$ and

$$A = 3M\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{pmatrix} 3 & 1 & 0 & & & & \\ 1 & 3 & 1 & 0 & & \mathbf{0} & \\ 0 & 1 & 3 & 1 & 0 & & \\ 0 & 0 & 1 & 3 & 1 & 0 & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & & \end{pmatrix}. \tag{5.18}$$

As we have seen in Proposition 3.1, the matrix $(1/3)A = M(1/3, 1/3)$, considered as operator from l_2 into itself is bijective and the unique solution of the equation $M(1/3, 1/3)X = (1/3)B$ is determined by Proposition 3.4(ii)(a). The approximation methods considered in Section 4 cannot be applied. Indeed, since $q_k = 1/3$, the matrix $M(1/3, 1/3)$ is not α-invertible and since $\gamma_k = r/3$, the hypothesis of Proposition 4.5 are not satisfied. Here using the quasi-Newton method, we will construct a sequence $(X_k)_{k\geq 1}$ tending to this solution. We will use the (SR1) method defined by (5.4), (5.7), and (5.8). Note that the outer product is defined in (5.7) by

$$\left[(X \otimes Y)Z\right]_n = \left[\langle Y, Z\rangle X\right]_n = x_n \sum_{m=1}^{\infty} y_m z_m \quad \forall n. \tag{5.19}$$

By Corollary 5.6, the sequence $(X_k)_{k\geq 1}$ *converges locally and linearly* to the unique solution X^*.

To start the algorithm, we take $X_1 = 0$ (the zero vector in l_2) and $H_1 = I$ (the identity operator in $\mathscr{B}(l_2)$). Since $B = e^{(1)}$, we see that for each k, the infinite matrix $H_k = (h_{nm}^k)_{n,m\geq 1}$ is defined by $h_{nn}^k = 1$ for $n > k+1$, and $h_{nm}^k = 0$ for $n, m > k+1$ and $n \neq m$. So, we can do the calculations considering H_k as a finite matrix of order $k+1$. We have a similar result for X_k. Here, the calculations are made on Matlab with finite matrices the size of which increases by one in each iteration. For the convenience of the reader we explicitly give the algorithm as follows:

$$\begin{gathered} X_1 = [0, 0, 0, \dots]^T \longrightarrow X_2 = [1, 0, 0, \dots]^T \longrightarrow X_3 = [0.4285, -0.2857, 0, \dots]^T \longrightarrow \cdots \longrightarrow \\ X_6 = [0.3820, -0.1461, 0.0563, -0.0230, 0.0129, 0, \dots]^T \longrightarrow \cdots, \end{gathered} \tag{5.20}$$

and for H_k we have

$$H_1=\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \cdot & & \mathbf{O} & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & \mathbf{O} & & 1 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix} \longrightarrow H_2=\begin{bmatrix} 0.4286 & -0.2857 & & & & & \\ -0.2857 & 0.8571 & & & & & \\ & & 1 & & \mathbf{O} & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & \mathbf{O} & & 1 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}$$

$$\longrightarrow H_3 = \begin{bmatrix} 0.3872 & -0.1614 & 0.0967 & & & & \\ -0.1614 & 0.4840 & -0.2902 & & & & \\ 0.0967 & -0.2902 & 0.7743 & & \mathbf{O} & & \\ & & & 1 & & & \\ & & & & \cdot & & \\ & & \mathbf{O} & & & \cdot & \\ & & & & & & 1 \\ & & & & & & & \cdot \end{bmatrix}, \tag{5.21}$$

and so forth. More precisely, we obtain the following. For $k = 1$, we have $X_1, B_1 \in \mathbb{R}^2$ and $H_1, A_1 \in M_2$ such that

$$X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.22}$$

Using Matlab, identity (5.4) gives X_2 and identities (5.7) and (5.8) yield H_2 with

$$X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.4286 & -0.2857 \\ -0.2857 & 0.8571 \end{bmatrix}. \tag{5.23}$$

For $k = 2$, it is natural to put

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.4286 & -0.2857 & 0 \\ -0.2857 & 0.8571 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.24}$$

and to consider $A_2 \in M_3$ and $B_2 \in \mathbb{R}^3$ as follows:

$$A_2 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{5.25}$$

Again, by identities (5.4), (5.7), and (5.8), we obtain

$$X_3 = \begin{bmatrix} 0.4285 \\ -0.2857 \\ 0 \end{bmatrix}, \qquad H_3 = \begin{bmatrix} 0.3872 & -0.1614 & 0.0967 \\ -0.1614 & 0.4840 & -0.2902 \\ 0.0967 & -0.2902 & 0.7743 \end{bmatrix}. \tag{5.26}$$

As above, we write for $k = 2$

$$X_3 = \begin{bmatrix} 0.4285 \\ -0.2857 \\ 0 \\ 0 \end{bmatrix}, \qquad H_3 = \begin{bmatrix} 0.3872 & -0.1614 & 0.0967 & 0 \\ -0.1614 & 0.4840 & -0.2902 & 0 \\ 0.0967 & -0.2902 & 0.7743 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.27}$$

and consider

$$A_3 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5.28}$$

and so forth. For each step, we must verify the condition $\|F(X_k)\| < \varepsilon$. If we take $\varepsilon = 10^{-3}$, we then obtain for $k = 7$ the following:

$$X_7 = \begin{bmatrix} 0.3820 \\ -0.1459 \\ 0.0558 \\ -0.0215 \\ 0.0088 \\ -0.0049 \\ 0 \end{bmatrix},$$

$$H_7 = \begin{bmatrix} 0.3828 & -0.1465 & 0.0565 & -0.0223 & 0.0091 & -0.0034 & 0.0001 \\ -0.1465 & 0.4387 & -0.1688 & 0.0662 & -0.0274 & 0.0133 & 0.0017 \\ 0.0565 & -0.1688 & 0.4487 & -0.1750 & 0.0736 & -0.0425 & -0.0091 \\ -0.0223 & 0.0662 & -0.1750 & 0.4572 & -0.1944 & 0.1234 & 0.0316 \\ 0.0091 & -0.0274 & 0.0736 & -0.1944 & 0.5108 & -0.3393 & -0.0932 \\ -0.0035 & 0.0133 & -0.0425 & 0.1234 & -0.3393 & -0.9092 & 0.2575 \\ 0.0001 & 0.0017 & -0.0091 & 0.0316 & -0.0932 & 0.2575 & 1.3096 \end{bmatrix} \tag{5.29}$$

with $\|F(X_7)\|^2 = 0.00005885$. We conclude that the vector

$$\widetilde{X_7} = (0.3820, -0.1459, 0.0558, -0.0215, 0.0088, -0.0049, 0, \dots) \tag{5.30}$$

is a good approximation of the unique solution of $AX = B$ and of the solution explicitly given in Proposition 3.4.

References

[1] I. J. Maddox, *Infinite Matrices of Operators*, vol. 786 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1980.

[2] A. M. Jarrah and E. Malkowsky, "Ordinary, absolute and strong summability and matrix transformations," *Filomat*, vol. 2003, no. 17, pp. 59–78, 2003.

[3] B. de Malafosse, "On the Banach algebra $\mathcal{B}(l_p(\alpha))$," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 60, pp. 3187–3203, 2004.
[4] B. de Malafosse, "Some new properties of sequence spaces and application to the continued fractions," *Matematichki Vesnik*, vol. 53, no. 3-4, pp. 91–102, 2001.
[5] B. de Malafosse, "On some BK spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 28, pp. 1783–1801, 2003.
[6] B. de Malafosse, "The Branch algebra s_α and applications," *Acta Scientiarum Mathematicarum*, vol. 70, no. 1-2, pp. 125–145, 2004.
[7] B. de Malafosse, "The Banach algebra $\mathcal{B}(X)$, where X is a BK space and applications," *Matematichki Vesnik*, vol. 57, no. 1-2, pp. 41–60, 2005.
[8] R. Labbas and B. de Malafosse, "On some Banach algebra of infinite matrices and applications," *Demonstratio Mathematica*, vol. 31, no. 1, pp. 153–168, 1998.
[9] S. R. Treil', "Invertibility of a Toeplitz operator does not imply its invertibility by the projection method," *Doklady Akademii Nauk SSSR*, vol. 292, no. 3, pp. 563–567, 1987 (Russian).
[10] R. G. Cooke, *Infinite Matrices and Sequence Spaces*, Macmillan, London, UK, 1950.
[11] W. A. Gruver and E. Sachs, *Algorithmic Methods in Optimal Control*, vol. 47 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1981.
[12] C. G. Broyden, "Quasi-Newton methods and their application to function minimisation," *Mathematics of Computation*, vol. 21, no. 99, pp. 368–381, 1967.
[13] L. B. Horwitz and P. E. Sarachik, "Davidon's method in Hilbert space," *SIAM Journal on Applied Mathematics*, vol. 16, no. 4, pp. 676–695, 1968.
[14] E. W. Sachs, "Broyden's method in Hilbert space," *Mathematical Programming*, vol. 35, no. 1, pp. 71–82, 1986.
[15] W.-H. Yu, "A quasi-Newton approach to identification of a parabolic system," *Journal of Australian Mathematical Society. Series B*, vol. 40, no. 1, pp. 1–22, 1998.
[16] A. Griewank, "The local convergence of Broyden-like methods on Lipschitzian problems in Hilbert spaces," *SIAM Journal on Numerical Analysis*, vol. 24, no. 3, pp. 684–705, 1987.
[17] C. T. Kelley and E. W. Sachs, "Broyden's method for approximate solution of nonlinear integral equations," *Journal of Integral Equations*, vol. 9, no. 1, pp. 25–43, 1985.
[18] C. T. Kelley and E. W. Sachs, "A quasi-Newton method for elliptic boundary value problems," *SIAM Journal on Numerical Analysis*, vol. 24, no. 3, pp. 516–531, 1987.
[19] C. T. Kelley and E. W. Sachs, "Quasi-Newton methods and unconstrained optimal control problems," *SIAM Journal on Control and Optimization*, vol. 25, no. 6, pp. 1503–1516, 1987.
[20] W.-H. Yu, "A quasi-Newton method in infinite-dimensional spaces and its application for solving a parabolic inverse problem," *Journal of Computational Mathematics*, vol. 16, no. 4, pp. 305–318, 1998.

Boubakeur Benahmed: Laboratoire Mathématiques Appliquées du Havre (LMAH) Université du Havre, IUT Le Havre, BP 4006, 76610 Le Havre, France
Current address: Département de Mathématiques et d'Informatique, ENSET d'Oran, BP 1523, 31000 Oran, Algeria
Email address: boubakeur.benahmed@enset-oran.dz

Bruno de Malafosse: Laboratoire Mathématiques Appliquées du Havre (LMAH) Université du Havre, IUT Le Havre, BP 4006, 76610 Le Havre, France
Email address: bdemalaf@wanadoo.fr

Adnan Yassine: Institut Supérieur d'Études Logistique (ISEL), Université du Havre, Quai Frissard, BP 1137, 76063 Le Havre, France
Email address: adnan.yassine@univ-lehavre.fr