Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 24873, 6 pages
doi:10.1155/2007/24873

Research Article

The Wiener Polynomial of the kth Power Graph

Omar A. AbuGhneim, Hasan Al-Ezeh, and Mahmoud Al-Ezeh

Received 11 March 2007; Accepted 11 June 2007

Recommended by Pentti Haukkanen

We presented a formula for the Wiener polynomial of the kth power graph. We use this formula to find the Wiener polynomials of the kth power graphs of paths, cycles, ladder graphs, and hypercubes. Also, we compute the Wiener indices of these graphs.

1. Introduction

The graphs in this paper are connected and simple. Let $d(u,v)$ denote the distance between two vertices $u,v \in V(G)$. The Wiener polynomial and the Wiener index of G are defined as follows.

Definition 1.1. The Wiener polynomial of G is

$$W(G;q) = \sum_{\{u,v\} \subseteq V(G)} q^{d(u,v)}. \tag{1.1}$$

The Wiener index of G is

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v), \tag{1.2}$$

where the sum is taken over all unordered distinct pairs of vertices $\{u,v\}$ in G.

Observe that the degree of the Wiener polynomial is equal to the diameter of G and the coefficient of q is equal to the number of the edges in G. Also, notice that

$$W(G) = W'(G;1). \tag{1.3}$$

The Wiener index is first presented by Wiener in [1] to determine the boiling point of paraffin. Also, the index has some applications related to properties of molecules and some applications related to mathematics. For more information, the reader is advised to see [2–4].

Wiener polynomials were first introduced by Hosoya in [5]. Some authors call these polynomials Hosoya's polynomials as an honor of Haruo Hosoya. Many papers have been devoted to compute the Wiener polynomial for different types of graphs. More information can be found in [6–9].

In this paper, we will present the Wiener polynomials and the Wiener indices of the kth power graphs.

2. Main result

First, we give the definition of the kth power graph.

Definition 2.1. The kth power graph, denoted by G^k, has the same vertex set as G. Two vertices are adjacent in G^k if their distance in G is at most k.

For instance, if the diameter of G is k, then G^k is complete. In the definition of the Wiener polynomial, one can collect the coefficients that have the same power and get

$$W(G;q)=\sum_{\{u,v\}}q^{d(u,v)}=\sum_{i=1}^{n}a_iq^i, \tag{2.1}$$

where a_i denotes the number of distinct pairs of vertices in G at distance i, and n denotes the diameter of G.

Now, we give the main result in this paper.

Theorem 2.2. *If G is a graph with Wiener polynomial $\sum_{i=1}^{n}a_iq^i$, then the Wiener polynomial of G^k is given by*

$$\sum_{i=0}^{[n/k]-1}\sum_{j=1}^{k}a_{j+ik}q^{i+1}+\left(a_{1+[n/k]k}+\cdots+a_n\right)q^{[n/k]+1}, \tag{2.2}$$

where $[n/k]$ is the greatest integer less than or equal to n/k, $a_{n+1}=a_{n+2}=\cdots=0$, and $n\geq k$. If $[n/k]=n/k$, then the Wiener polynomial of G^k becomes

$$\sum_{i=1}^{[n/k]}\left(\sum_{(i-1)k+1\leq j\leq ik}a_j\right)q^i. \tag{2.3}$$

Proof. The distinct pairs of vertices in G that are at distance $1,2,\ldots,k$ become adjacent in G^k. There are $a_1,a_2,\ldots,a_k$ vertices that are at distance $1,2,\ldots,k$, respectively, in G. These vertices become at distance one in G^k. Hence the coefficient of q is $\sum_{j=1}^{k}a_j$ in G^k.

One can generalize this idea by taking the distinct pairs of vertices in G whose distance lies in the set $A_i=\{ik+j,\ j=1,2,\ldots,k\}$, where $0\leq i\leq[n/k]-1$. There are $a_{ik+1}+\cdots+a_{ik+k}$ distinct pairs of vertices in G whose distance lies in A_i. These distinct pairs of vertices become at distance $i+1$ in G^k. Hence we have $a_{ik+1}+\cdots+a_{ik+k}$ distinct pairs of vertices in G^k that are at distance $i+1$. This gives the Wiener polynomial of G^k. $\square$

Next, we state Wiener polynomials for paths, cycles, ladder graphs, and hypercubes. One can find the proof of the next theorem in [8, 9].

Theorem 2.3. *(1) $W(P_n,q)=(n-1)q+(n-2)q^2+\cdots+q^{n-1}$, where P_n is a path with n vertices.*

(2) $W(C_{2n+1},q)=(2n+1)(q+q^2+\cdots+q^n)$, where C_{2n+1} is a cycle with $2n+1$ vertices.

(3) $W(C_{2n},q)=2n(q+q^2+\cdots+q^{n-1})+nq^n$, where C_{2n} is a cycle with $2n$ vertices.

(4) $W(L_n,q)=(3n-2)q+\sum_{i=2}^{n}(2i-3)q^{n-i-2}$, where L_n is the ladder graph.

(5) $W(Q_n,q)=2^{n-1}[(1+q)^n]-1$, where Q_n is the hypercube.

Now, we use Theorem 2.2 to find Wiener polynomials and Wiener indices of the kth power graphs of paths, cycles, ladder graphs, and hypercubes.

Corollary 2.4. *The Wiener polynomial of the graph P_n^k is given by*

$$\begin{aligned} W(P_n^k;q) &= \sum_{i=1}^{[(n-1)/m]}\frac{k}{2}(2n-(2i-1)k-1)q^i \\ &\quad+\left(\frac{1}{2}\left((n-1)-\left[\frac{n-1}{k}\right]k\right)\left(n-\left[\frac{n-1}{k}\right]k\right)\right)q^{[(n-1)/k]+1}, \quad n-1\geq k. \end{aligned} \tag{2.4}$$

The Wiener index of P_n^k is given by

$$\begin{aligned} W(P_n^k) &= \frac{n(n-1)}{2}+\left(\frac{n(n-1)}{2}-kn+\frac{k}{4}+\frac{k^2}{12}\right)\left[\frac{n-1}{k}\right] \\ &\quad+\left(\frac{k^2}{4}-\frac{kn}{2}+\frac{k}{4}\right)\left[\frac{n-1}{k}\right]^2+\left(\frac{k^2}{6}\right)\left[\frac{n-1}{k}\right]^3, \quad n-1\geq k, \end{aligned} \tag{2.5}$$

where P_n is a path with n vertices and P_n^k is the kth power graph of P_n.

Proof. Using Theorems 2.2 and 2.3, we will get the coefficient of q in P_n^k which is

$$(n-1)+(n-2)+\cdots+(n-k)=\frac{k}{2}(2n-k-1). \tag{2.6}$$

Again, using Theorems 2.2 and 2.3, we get the coefficient of q^i in P_n^k which is

$$(n-(i-1)k-1)+(n-(i-1)k-2)+\cdots+(n-ik)=\frac{k}{2}(2n-(2i-1)k-1). \tag{2.7}$$

The coefficient of $q^{[(n-1)/k]+1}$, which is the highest power in P_n^k, will be

$$\begin{aligned} &\left(n-\left[\frac{n-1}{k}\right]k\right)+\left(n-\left[\frac{n-1}{k}\right]k-1\right)+\cdots+1 \\ &\quad=\frac{1}{2}\left(n-\left[\frac{n-1}{k}\right]k\right)\left(\left(n-\left[\frac{n-1}{k}\right]k\right)+1\right). \end{aligned} \tag{2.8}$$

Hence we get the formula for the Wiener polynomial of P_n^k. For the Wiener index of P_n^k, we have

$$\begin{aligned} W(P_n^k) = W'(P_n^k;1) = & \sum_{i=1}^{[(n-1)/k]} \frac{k}{2}(2n-(2i-1)k-1)i \\ & + \left(\frac{1}{2}\left((n-1)-\left[\frac{n-1}{k}\right]k\right)\left(n-\left[\frac{n-1}{k}\right]k\right)\right)\left[\frac{n-1}{k}\right]+1. \end{aligned} \tag{2.9}$$

Simplifying and using the identities $\sum_{i=1}^n i = (n(n+1))/2$ and $\sum_{i=1}^n i^2 = (n(n+1)(2n+1))/6$ will give the formula for the Wiener index of P_n^k. □

We give the following example as an application of Corollary 2.4.

Example 2.5. Using Theorem 2.3, we have $P_{10} = 9q+8q^2+7q^3+6q^4+5q^5+4q^6+3q^7+2q^8+q^9$. Using Corollary 2.4, we get

$$\begin{aligned} &W(P_{10}^2;q) = 17q+13q^2+9q^3+5q^4+q^5, \quad W(P_{10}^2) = 95, \\ &W(P_{10}^3;q) = 24q+15q^2+6q^3, \quad W(P_{10}^3) = 72, \\ &W(P_{10}^4;q) = 30q+14q^2+q^3, \quad W(P_{10}^4) = 61, \\ &W(P_{10}^5;q) = 35q+10q^2, \quad W(P_{10}^5) = 55, \\ &W(P_{10}^6;q) = 39q+6q^2, \quad W(P_{10}^6) = 51, \\ &W(P_{10}^7;q) = 42q+3q^2, \quad W(P_{10}^7) = 48, \\ &W(P_{10}^8;q) = 44q+q^2, \quad W(P_{10}^8) = 46, \\ &W(P_{10}^9;q) = 45q, \quad W(P_{10}^9) = 45. \end{aligned} \tag{2.10}$$

We leave the check of the next three corollaries to the reader.

Corollary 2.6. *The Wiener polynomial and the Wiener index of the graph C_{2n+1}^k are given by*

$$\begin{aligned} W(C_{2n+1}^k;q) &= \sum_{i=1}^{[n/k]}(2n+1)kq^i + \left(n-\left[\frac{n}{k}\right]k\right)(2n+1)q^{[n/k]+1}, \quad n \ge k, \\ W(C_{2n+1}^k) &= n(2n+1)\left(\left[\frac{n}{k}\right]+1\right). \end{aligned} \tag{2.11}$$

The Wiener polynomial and the Wiener index of C_{2n}^k are given by

$$\begin{aligned} W(C_{2n}^k;q) &= \sum_{i=1}^{[(n-1)/k]}(2n)kq^i + \left(\left(n-1-\left[\frac{n-1}{k}\right]k\right)(2n)+n\right)q^{[(n-1)/k]+1}, \quad n-1 \ge k, \\ W(C_{2n}^k) &= n(2n-1)\left(\left[\frac{n-1}{k}\right]+1\right). \end{aligned} \tag{2.12}$$

We give the following example as an application of Corollary 2.6.

Example 2.7. Using Theorem 2.3, we have $C_{15} = 15(q+q^2+q^3+q^4+q^5+q^6+q^7)$. Using Corollary 2.6, we get

$$\begin{aligned}
&W(C_{15}^2;q) = 30(q+q^2+q^3)+15q^4, \quad W(C_{15}^2) = 240,\\
&W(C_{15}^3;q) = 45(q+q^2)+15q^3, \quad W(C_{15}^3) = 180,\\
&W(C_{15}^4;q) = 60q+45q^2, \quad W(C_{15}^4) = 150,\\
&W(C_{15}^5;q) = 75q+30q^2, \quad W(C_{15}^5) = 135,\\
&W(C_{15}^6;q) = 90q+15q^2, \quad W(C_{15}^6) = 120,\\
&W(C_{15}^7;q) = 105q, \quad W(C_{15}^7) = 105.
\end{aligned} \tag{2.13}$$

Corollary 2.8. *The Wiener polynomial of graph L_n^k is given by*

$$\begin{aligned}
W(L_n^k;q) = {}&(2k(2n-k)-n)q + \sum_{i=2}^{[n/k]} 2k(2n+(1-2i)k)q^i\\
&+2\left(n-\left[\frac{n}{k}\right]k\right)^2 q^{[n/k]+1}, \quad n \ge k.
\end{aligned} \tag{2.14}$$

The Wiener index of L_n^k is given by

$$W(L_n^k) = (2n^2-n)+\left(2n^2+\frac{k^2}{3}-2nk\right)\left[\frac{n}{k}\right]+(k^2-2nk)\left[\frac{n}{k}\right]^2+\frac{2k^2}{3}\left[\frac{n}{k}\right]^3, \tag{2.15}$$

where L_n is the ladder graph.

Corollary 2.9. *The Wiener polynomial and the Wiener index of the graph Q_n^k are given by*

$$\begin{aligned}
W(Q_n^k;q) = {}&\sum_{i=0}^{[n/k]-1}\sum_{j=1}^{k}\binom{n}{j+ik}2^{n-1}q^i\\
&+\left(\binom{n}{1+[n/k]k}+\binom{n}{2+[n/k]k}+\cdots+\binom{n}{n}\right)q^{[n/k]+1}, \quad n \ge k,\\
W(Q_n^k) = {}&\sum_{i=0}^{[n/k]-1}\sum_{j=1}^{k}\binom{n}{j+ik}2^{n-1}i\\
&+\left(\binom{n}{1+[n/k]k}+\binom{n}{2+[n/k]k}+\cdots+\binom{n}{n}\right)\left(\left[\frac{n}{k}\right]+1\right).
\end{aligned} \tag{2.16}$$

References

[1] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, no. 1, pp. 17–20, 1947.

[2] F. Buckly and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood, Calif, USA, 1990.

[3] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, Germany, 1986.

[4] I. Gutman, “A new method for the calculation of the Wiener number of acyclic molecules,” *Journal of Molecular Structure: THEOCHEM*, vol. 285, no. 1, pp. 137–142, 1993.
[5] H. Hosoya, “On some counting polynomials in chemistry,” *Discrete Applied Mathematics*, vol. 19, no. 1–3, pp. 239–257, 1988.
[6] I. Gutman, “Hosoya polynomial and the distance of the total graph of a tree,” *Publikacije Elektrotehničkog Fakulteta, Univerzitet u Beogradu*, vol. 10, pp. 53–58, 1999.
[7] M. Lepović and I. Gutman, “A collective property of trees and chemical trees,” *Journal of Chemical Information and Modeling*, vol. 38, no. 5, pp. 823–826, 1998.
[8] B. E. Sagan, Y.-N. Yen, and P. Zhang, “The Wiener polynomial of a graph,” *International Journal of Quantum Chemistry*, vol. 60, no. 5, pp. 959–969, 1996.
[9] D. Stevanović, “Hosoya polynomial of composite graphs,” *Discrete Mathematics*, vol. 235, no. 1–3, pp. 237–244, 2001.

Omar A. AbuGhneim: Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
Email address: o.abughneim@ju.edu.jo

Hasan Al-Ezeh: Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
Email address: alezeh@ju.edu.jo

Mahmoud Al-Ezeh: Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
Email address: mah_azzeh@yahoo.com