

Research Article

Logarithm of the Discrete Fourier Transform

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The discrete Fourier transform defines a unitary matrix operator. The logarithm of this operator is computed, along with the projection maps onto its eigenspaces. A geometric interpretation of the discrete Fourier transform is also given.

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Recall that the *discrete Fourier transform* (DFT) in dimension n is the complex-linear transformation $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$, with $F(\mathbf{u}) = \hat{\mathbf{u}}$, where $\hat{\mathbf{u}} = (\hat{u}_0, \dots, \hat{u}_{n-1})$ has components $\hat{u}_j = (1/\sqrt{n}) \sum_{k=0}^{n-1} u_k e^{-2\pi jk/n}$, where $i \doteq \sqrt{-1}$. In the standard basis for \mathbb{C}^n , the DFT can be represented as an $n \times n$ matrix $F = (F_{jk})$, where

$$F_{jk} = \frac{1}{\sqrt{n}} e^{-2\pi jk/n}. \tag{1}$$

Our choice of normalization factor ensures that F is unitary: $F^\dagger \circ F = I$, where I is the identity transformation, and F^\dagger is the Hermitian conjugate of F , that is, $(F^\dagger)_{jk} = \overline{F_{kj}}$.

Also recall that the exponential of a matrix M is given by the infinite series $(M) \doteq \sum_{p=0}^{\infty} (1/p!)M^p$ (provided it converges). Thus a complex-linear map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a *logarithm* for F if $F = \exp(f)$, and we write $f = \log(F)$. For more information on the exponential and logarithm maps see, [1, Chapter 4].

THEOREM 1. *In any dimension n , we may take $\log(F) = (1/2)i\pi G_1 + i\pi G_2 - (1/2)i\pi G_3$, where $G_1 \doteq (1/4)(I - iF - F^2 + iF^3)$, $G_2 \doteq (1/4)(I - F + F^2 - F^3)$, $G_3 \doteq (1/4)(I + iF - F^2 - iF^3)$. The G_j are projections which satisfy $G_j \circ G_k = 0$ for $j \neq k$. Moreover, the image of G_k has dimension q_k , where if $n \equiv 0 \pmod{4}$, then $q_1 = n/4 - 1$ and $q_2 = q_3 = n/4$; if*

$n \equiv 1 \pmod{4}$, then $q_1 = q_2 = q_3 = (n - 1)/4$; if $n \equiv 2 \pmod{4}$, then $q_1 = q_3 = (n - 2)/4$ and $q_2 = (n + 2)/4$; and if $n \equiv 3 \pmod{4}$, then $q_1 = (n - 3)/4$ and $q_2 = q_3 = (n + 1)/4$.

COROLLARY 2. $\log(F) = (1/4)i\pi(I - (1 + i)F + F^2 - (1 - i)F^3)$.

COROLLARY 3. $F = (I - (1 - i)G_1)(I - 2G_2)(I - (1 + i)G_3)$, where G_1, G_2, G_3 are as in Theorem 1, and all factors commute.

Corollary 3 has a nice geometric interpretation. Since each G_j is a projection, we have $G_j^p = G_j$ for $p > 0$; and thus for any $\theta \in \mathbb{R}$, $\exp(i\theta G_j) = I + (e^{i\theta} - 1)G_j$, using the definition of the matrix exponential and the Taylor series expansion of $e^{i\theta}$. In particular, for vectors $\mathbf{u} \in \text{image}(G_j)$, we have $\exp(i\theta G_j)\mathbf{u} = e^{i\theta}\mathbf{u}$; and for vectors $\mathbf{v} \in \text{image}(G_j)^\perp$, the orthogonal complement to the image of G_j , we have $\exp(i\theta G_j)\mathbf{v} = \mathbf{v}$. Consequently, the factors in Corollary 3 are rotations by $\pi/2$, π , and $-\pi/2$ in the planes $\text{image}(G_1)$, $\text{image}(G_2)$, and $\text{image}(G_3)$, respectively; these planes are orthogonal to each other. Note that F is the identity (and $\log(F)$ is trivial) in the orthogonal complement to the span of $\text{image}(G_1)$, $\text{image}(G_2)$, and $\text{image}(G_3)$.

Before we prove Theorem 1, we first state some known facts concerning the $n \times n$ DFT matrix F :

$$F^4 = I, \tag{2}$$

$$\text{trace}(F) = 1 - i, 1, 0, -i, \quad \text{according to } n \equiv 0, 1, 2, 3 \pmod{4}, \tag{3}$$

$$\text{trace}(F^2) = 2, 1 \quad \text{according to } n \equiv 0, 1 \pmod{2}. \tag{4}$$

Note that as consequences, we have $F^3 = F^\dagger$ and $\text{trace}(F^3) = \overline{\text{trace}(F)}$. We remark that (2) and (4) are straight-forward to verify from (1). See also [2, page 244], for instance (although our choice in normalization for F is different from the one used there). Equation (3) is a rearrangement of the *Gauss sum*:

$$\sum_{k=0}^{n-1} e^{ik^2/n} = \epsilon_n \sqrt{n}, \quad \text{where } \epsilon_n = 1 + i, 1, 0, i, \quad \text{according to } n \equiv 0, 1, 2, 3 \pmod{4}, \tag{5}$$

which Gauss proved in the 1800's [2, page 76]. Proofs can be found in [3, page 177–180], and in ([4], p. 214–216).

Proof of Theorem 1. As F is unitary, it is diagonalizable; and by (2), its eigenvalues are $\pm 1, \pm i$ (the fourth roots of unity). The spectral theorem for a diagonalizable matrix (see [1, page 517] thus implies that $F = G_0 + iG_1 - G_2 - iG_3$, where G_0, \dots, G_3 are eigenspace projections with $G_j \circ G_k = 0$ if $j \neq k$, and $G_0 + \dots + G_3 = I$. For given $\theta_0, \dots, \theta_3 \in \mathbb{R}$, define $f \doteq i\theta_0 G_0 + \dots + i\theta_3 G_3$. Since $\exp(f) = e^{i\theta_0} G_0 + \dots + e^{i\theta_3} G_3$, it follows that if we take $\theta_0 = 0, \theta_1 = (1/2)\pi, \theta_2 = \pi$, and $\theta_3 = -(1/2)\pi$, then $\exp(f) = F$. Although there exist explicit formulae for the projections [1, page 529], it is perhaps more expedient to observe

that $(F - I)^p = [(i\theta_1 - 1)G_1 + (i\theta_2 - 1)G_2 + (i\theta_3 - 1)G_3]^p = (i\theta_1 - 1)^p G_1 + (i\theta_2 - 1)^p G_2 + (i\theta_3 - 1)^p G_3$ and in particular, for $p = 1, 2, 3$. Thus, we need only to solve the (Vandermonde) linear system

$$\begin{pmatrix} (i\theta_1 - 1) & (i\theta_2 - 1) & (i\theta_3 - 1) \\ (i\theta_1 - 1)^2 & (i\theta_2 - 1)^2 & (i\theta_3 - 1)^2 \\ (i\theta_1 - 1)^3 & (i\theta_2 - 1)^3 & (i\theta_3 - 1)^3 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} F - I \\ (F - I)^2 \\ (F - I)^3 \end{pmatrix}. \quad (6)$$

Inverting this equation yields the formulae for G_1, G_2, G_3 given in the statement of the theorem. For the stated projection image dimensions, recall that the image of the projection P has dimension given by $\text{trace}(P)$; thus $q_j = \text{trace}(G_j)$, which can be computed using (3) and (4). \square

As with the logarithm for complex numbers, the logarithm of a matrix is not unique. For example, if we had taken $\theta_3 = 3\pi/2$ in the above, then we would have obtained the formula $\log(F) = (1/4)i\pi(3I - (1 - i)F - F^2 - (1 + i)F^3)$ instead of that in Corollary 2.

References

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