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*Research Article*

# Expression of a Tensor Commutation Matrix in Terms of the Generalized Gell-Mann Matrices

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We have expressed the tensor commutation matrix $n \otimes n$ as linear combination of the tensor products of the generalized Gell-Mann matrices. The tensor commutation matrices $3 \otimes 2$ and $2 \otimes 3$ have been expressed in terms of the classical Gell-Mann matrices and the Pauli matrices.

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## 1. Introduction

When we had worked on Raoelina Andriambololona idea on the use of tensor product in Dirac equation [1, 2], we had met the unitary matrix

$$U_{2\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

This matrix is frequently found in quantum information theory [3–5] where one writes, by using the Pauli matrices [3–5],

$$U_{2\otimes 2} = \frac{1}{2} I_2 \otimes I_2 + \frac{1}{2}\sum_{i=1}^{3} \sigma_i \otimes \sigma_i \tag{1.2}$$

with $I_2$ the $2 \times 2$ unit matrix. We call this matrix a tensor commutation matrix $2 \otimes 2$. The tensor commutation matrix $3 \otimes 3$ is expressed by using the Gell-Mann matrices under the following form [6]:

$$U_{3\otimes 3}=\frac{1}{3}I_3\otimes I_3+\frac{1}{2}\sum_{i=1}^{8}\lambda_i\otimes\lambda_i. \tag{1.3}$$

We have to talk a bit about different types of matrices because in the generalization of the above formulas, we will consider a commutation matrix as a matrix of fourth-order tensor and in expressing the commutation matrices $U_{3\otimes 2}$, $U_{2\otimes 3}$, at the last section, a commutation matrix will be considered as matrix of second-order tensor.

$\mathcal{M}_{m\times n}(\mathbb{C})$ denotes the set of $m\times n$ matrices whose elements are complex numbers.

## 2. Tensor product of matrices

**2.1. Matrices.** If the elements of a matrix are considered as the components of a second-order tensor, we adopt the habitual notation for a matrix, without parentheses inside, whereas if the elements of the matrix are, for instance, considered as the components of sixth-order tensor, three times covariant and three times contravariant, then we represent the matrix of the following way, for example:

$$M=\begin{pmatrix}
\begin{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}&\begin{pmatrix}1&1\\3&2\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}1&1\\1&1\end{pmatrix}\end{pmatrix} & \begin{pmatrix}\begin{pmatrix}1&0\\1&2\end{pmatrix}&\begin{pmatrix}7&8\\9&0\end{pmatrix}\\\begin{pmatrix}3&4\\5&6\end{pmatrix}&\begin{pmatrix}9&8\\7&6\end{pmatrix}\end{pmatrix}\\
\begin{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}&\begin{pmatrix}0&0\\3&2\end{pmatrix}\\\begin{pmatrix}4&5\\1&6\end{pmatrix}&\begin{pmatrix}1&7\\8&9\end{pmatrix}\end{pmatrix} & \begin{pmatrix}\begin{pmatrix}5&4\\3&2\end{pmatrix}&\begin{pmatrix}1&0\\1&2\end{pmatrix}\\\begin{pmatrix}3&4\\5&6\end{pmatrix}&\begin{pmatrix}7&8\\9&0\end{pmatrix}\end{pmatrix}\\
\begin{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}&\begin{pmatrix}9&8\\7&6\end{pmatrix}\\\begin{pmatrix}5&6\\7&8\end{pmatrix}&\begin{pmatrix}5&4\\3&2\end{pmatrix}\end{pmatrix} & \begin{pmatrix}\begin{pmatrix}9&8\\7&6\end{pmatrix}&\begin{pmatrix}5&4\\3&2\end{pmatrix}\\\begin{pmatrix}1&0\\1&2\end{pmatrix}&\begin{pmatrix}3&4\\5&6\end{pmatrix}\end{pmatrix}
\end{pmatrix},$$

$$\begin{aligned}M=\left(M^{i_1i_2i_3}_{j_1j_2j_3}\right)\quad & i_1i_2i_3=111,112,121,122,211,212,221,222,311,312,321,322 \text{ row indices},\\ & j_1j_2j_3=111,112,121,122,211,212,221,222 \text{ column indices}.\end{aligned} \tag{2.1}$$

The first indices $i_1$ and $j_1$ are the indices of the outside parenthesis which we call the first-order parenthesis; the second indices $i_2$ and $j_2$ are the indices of the next parentheses which we call the second-order parentheses; the third indices $i_3$ and $j_3$ are the indices of the most interior parentheses, of this example, which we call third-order parentheses. So, for instance, $M^{321}_{121}=5$.

If we delete the third-order parenthesis, then the elements of the matrix $M$ are considered as the components of a fourth-order tensor, twice contravariant and twice covariant.

A matrix is a diagonal matrix if deleting the interior parentheses, we have a habitual diagonal matrix.

A matrix is a symmetric (resp., antisymmetric) matrix if deleting the interior parentheses, we have a habitual symmetric (resp., antisymmetric) matrix.

We identify one matrix to another matrix if after deleting the interior parentheses, they are the same matrices.

### 2.2. Tensor product of matrices

*Definition 2.1.* Consider $A = (A^i_j) \in \mathcal{M}_{m\times n}(\mathbb{C})$, $B = (B^i_j) \in \mathcal{M}_{p\times r}(\mathbb{C})$. The matrix defined by

$$A \otimes B = \begin{pmatrix} A^1_1 B & \dots & A^1_j B & \dots & A^1_n B \\ \vdots & & \vdots & & \vdots \\ A^i_1 B & \dots & A^i_j B & \dots & A^i_n B \\ \vdots & & \vdots & & \vdots \\ A^m_1 B & \dots & A^m_j B & \dots & A^m_n B \end{pmatrix} \tag{2.2}$$

is called the tensor product of the matrix $A$ by the matrix $B$,

$$\begin{gathered} A \otimes B \in \mathcal{M}_{mp\times nr}(\mathbb{C}), \\ A \otimes B = \left(C^{i_1 i_2}_{j_1 j_2}\right) = \left(A^{i_1}_{j_1} B^{i_2}_{j_2}\right), \end{gathered} \tag{2.3}$$

(cf., e.g., [3]) where, $i_1 i_2$ are row indices and $j_1 j_2$ are column indices.

## 3. Generalized Gell-Mann matrices

Let us fix $n \in \mathbb{N}$, $n \geq 2$ for all continuations. The generalized Gell-Mann matrices or $n\times n$-Gell-Mann matrices are the traceless Hermitian $n \times n$ matrices $\Lambda_1, \Lambda_2, \dots, \Lambda_{n^2-1}$ which satisfy the relation $Tr(\Lambda_i \Lambda_j) = 2\delta_{ij}$, for all $i, j \in \{1,2,\dots,n^2-1\}$, where $\delta_{ij} = \delta^{ij} = \delta^i_j$ is the Kronecker symbol [7].

However, for the demonstration of Theorem 4.3, denote, for $1 \leq i < j \leq n$, the $C^2_n = (n!/2!(n-2)!)$ $n\times n$-Gell-Mann matrices which are symmetric with all elements 0 except the $i$th row $j$th column and the $j$th row $i$th column which are equal to 1, by $\Lambda^{(ij)}$; the $C^2_n = (n!/2!(n-2)!)$ $n \times n$-Gell-Mann matrices which are antisymmetric with all elements are 0 except the $i$th row $j$th column which is equal to $-i$ and the $j$th row $i$th column which is equal to $i$, by $\Lambda^{[ij]}$ and by $\Lambda^{(d)}$, $1 \leq d \leq n-1$, the following $(n-1)$ $n \times n$-Gell-Mann matrices are diagonal:

$$\Lambda^{(1)} = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & -1 & & & & \\ & & 0 & & & \vdots \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 0 & & \cdots & & & 0 \end{pmatrix},$$

$$\Lambda^{(2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 1 & & & & \\ & & -2 & & & \vdots \\ \vdots & & & 0 & & \\ & & & & \ddots & \\ 0 & & \cdots & & & 0 \end{pmatrix}, \ldots, \Lambda^{(n-1)} = \frac{1}{\sqrt{C_n^2}} \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 1 & & & & \\ & & 1 & & & \vdots \\ \vdots & & & \ddots & & \\ & & & & 1 & \\ 0 & & \cdots & & & -(n-1) \end{pmatrix}. \tag{3.1}$$

For $n = 2$, we have the Pauli matrices.

## 4. Tensor commutation matrices

*Definition 4.1.* For $p, q \in \mathbb{N}$, $p \geq 2$, $q \geq 2$, call the tensor commutation matrix $p \otimes q$ the permutation matrix $U_{p\otimes q} \in \mathcal{M}_{pq\times pq}(\mathbb{C})$ formed by 0 and 1, verifying the property

$$U_{p\otimes q} \cdot (a \otimes b) = b \otimes a \tag{4.1}$$

for all $a \in \mathcal{M}_{p\times 1}(\mathbb{C})$, $b \in \mathcal{M}_{q\times 1}(\mathbb{C})$.

Considering $U_{p\otimes q}$ as a matrix of a second-order tensor, one can construct it by using the following rule [6].

*Rule 4.2.* Let us start in putting 1 at first row and first column, after that let us pass into second column in going down at the rate of $p$ rows and put 1 at this place, then pass into third column in going down at the rate of $p$ rows and put 1, and so on until there are only for us $p-1$ rows for going down (then we have obtained number of $1 : q$). Then pass into the next column which is the $(q+1)$th column, put 1 at the second row of this column and repeat the process until we have only $p-2$ rows for going down (then we have obtained number of $1 : 2q$). After that pass into the next column which is the $(2q+1)$th column, put 1 at the third row of this column and repeat the process until we have only $p-3$ rows for going down (then we have obtained number of $1 : 3q$). Continuing in this way, we will have that the element at $p \times q$th row and $p \times q$th column is 1. The other elements are 0.

Theorem 4.3. *One has*

$$U_{n\otimes n} = \frac{1}{n} I_n \otimes I_n + \frac{1}{2} \sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i. \tag{4.2}$$

*Proof.* One has

$$\begin{gathered} I_n \otimes I_n = (\delta^{i_1 i_2}_{j_1 j_2}) = (\delta^{i_1}_{j_1} \delta^{i_2}_{j_2}), \\ U_{n\otimes n} = (\delta^{i_1}_{j_2} \delta^{i_2}_{j_1}), \end{gathered} \tag{4.3}$$

where, $i_1 i_2$ are row indices and $j_1 j_2$ are column indices [3].

Consider at first the $C_n^2$ symmetric $n \times n$ Gell-Mann matrices which can be written as

$$\begin{aligned} \Lambda^{(ij)} &= (\Lambda^{(ij)}{}^l_k)_{1\le l\le n, 1\le k\le n} \\ &= (\delta^{il}\delta^j_k)_{1\le l\le n, 1\le k\le n} + (\delta^{jl}\delta^i_k)_{1\le l\le n, 1\le k\le n} \\ &= (\delta^{il}\delta^j_k + \delta^{jl}\delta^i_k)_{1\le l\le n, 1\le k\le n}. \end{aligned} \tag{4.4}$$

Then

$$\Lambda^{(ij)} \otimes \Lambda^{(ij)} = \left( (\Lambda^{(ij)} \otimes \Lambda^{(ij)})^{l_1 l_2}_{k_1 k_2} \right) = (\delta^{il_1}\delta^j_{k_1} + \delta^{jl_1}\delta^i_{k_1})(\delta^{il_2}\delta^j_{k_2} + \delta^{jl_2}\delta^i_{k_2}), \tag{4.5}$$

where $l_1 l_2$ are row indices and $k_1 k_2$ are column indices.

That is,

$$(\Lambda^{(ij)} \otimes \Lambda^{(ij)})^{l_1 l_2}_{k_1 k_2} = \delta^{il_1}\delta^j_{k_1}\delta^{il_2}\delta^j_{k_2} + \delta^{il_1}\delta^j_{k_1}\delta^{jl_2}\delta^i_{k_2} + \delta^{jl_1}\delta^i_{k_1}\delta^{il_2}\delta^j_{k_2} + \delta^{jl_1}\delta^i_{k_1}\delta^{jl_2}\delta^i_{k_2}. \tag{4.6}$$

The $C_n^2$ antisymmetric $n \times n$ Gell-Mann matrices can be written as

$$\Lambda^{[ij]} = (\Lambda^{[ij]}{}^l_k)_{1\le l\le n, 1\le k\le n} = (-i\delta^{il}\delta^j_k + i\delta^{jl}\delta^i_k)_{1\le l\le n, 1\le k\le n}. \tag{4.7}$$

Then

$$\begin{aligned} \Lambda^{[ij]} \otimes \Lambda^{[ij]} &= \left( (\Lambda^{[ij]} \otimes \Lambda^{[ij]})^{l_1 l_2}_{k_1 k_2} \right), \\ (\Lambda^{[ij]} \otimes \Lambda^{[ij]})^{l_1 l_2}_{k_1 k_2} &= -\delta^{il_1}\delta^j_{k_1}\delta^{il_2}\delta^j_{k_2} + \delta^{il_1}\delta^j_{k_1}\delta^{jl_2}\delta^i_{k_2} + \delta^{jl_1}\delta^i_{k_1}\delta^{il_2}\delta^j_{k_2} - \delta^{jl_1}\delta^i_{k_1}\delta^{jl_2}\delta^i_{k_2}, \\ \sum_{1\le i<j\le n} (\Lambda^{(ij)} \otimes \Lambda^{(ij)})^{l_1 l_2}_{k_1 k_2} &+ \sum_{1\le i<j\le n} (\Lambda^{[ij]} \otimes \Lambda^{[ij]})^{l_1 l_2}_{k_1 k_2} \\ &= 2 \sum_{1\le i<j\le n} (\delta^{il_1}\delta^j_{k_1}\delta^{jl_2}\delta^i_{k_2} + \delta^{jl_1}\delta^i_{k_1}\delta^{il_2}\delta^j_{k_2}) = 2 \sum_{i\neq j} \delta^{il_1}\delta^j_{k_1}\delta^{jl_2}\delta^i_{k_2} \end{aligned} \tag{4.8}$$

is the $l_1 l_2$th row, $k_1 k_2$th column of the matrix

$$\sum_{1\le i<j\le n} \Lambda^{(ij)} \otimes \Lambda^{(ij)} + \sum_{1\le i<j\le n} \Lambda^{[ij]} \otimes \Lambda^{[ij]}. \tag{4.9}$$

Now, consider the diagonal $n \times n$ Gell-Mann matrices. Let $d \in \mathbb{N}$, $1 \le d \le n-1$,

$$\Lambda^{(d)} = \frac{1}{\sqrt{C_{d+1}^2}} \left( \delta_k^l \sum_{p=1}^{d} \delta_k^p - d\delta_k^l \delta_k^{d+1} \right) \tag{4.10}$$

and the $l_1 l_2$th row, $k_1 k_2$th of the matrix $\Lambda^{(d)} \otimes \Lambda^{(d)}$ is

$$\begin{aligned}(\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1k_2}^{l_1l_2} = {} & \frac{1}{C_{d+1}^2}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2}\left(\sum_{q=1}^{d}\sum_{p=1}^{d}\delta_{k_1}^{q}\delta_{k_2}^{p}\right) - \frac{1}{C_{d+1}^2}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2}\left(d\delta_{k_2}^{d+1}\sum_{p=1}^{d}\delta_{k_1}^{p}\right) \\ & - \frac{1}{C_{d+1}^2}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2}\left(d\delta_{k_1}^{d+1}\sum_{p=1}^{d}\delta_{k_2}^{p}\right) + \frac{1}{C_{d+1}^2}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2}\left(d^2\delta_{k_1}^{d+1}\delta_{k_2}^{d+1}\right),\end{aligned} \tag{4.11}$$

$\Lambda^{(d)} \otimes \Lambda^{(d)}$ is a diagonal matrix, so all that we have to do is to calculate the elements on the diagonal where $l_1 = k_1$ and $l_2 = k_2$. Then,

$$\begin{aligned}\sum_{d=1}^{n-1}(\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1k_2}^{l_1l_2} = {} & \sum_{d=1}^{n-1}\frac{1}{C_{d+1}^2}\left(\sum_{q=1}^{d}\delta_{k_1}^{q}\right)\left(\sum_{p=1}^{d}\delta_{k_2}^{p}\right) - \sum_{d=1}^{n-1}\frac{1}{C_{d+1}^2}d\delta_{k_2}^{d+1}\sum_{p=1}^{d}\delta_{k_1}^{p} \\ & - \sum_{d=1}^{n-1}\frac{1}{C_{d+1}^2}d\delta_{k_1}^{d+1}\sum_{p=1}^{d}\delta_{k_2}^{p} + \sum_{d=1}^{n-1}\frac{1}{C_{d+1}^2}d^2\delta_{k_1}^{d+1}\delta_{k_2}^{d+1}\end{aligned} \tag{4.12}$$

is the $l_1 l_2$th row, $k_1 k_2$th column of the diagonal matrix $\sum_{d=1}^{n-1} \Lambda^{(d)} \otimes \Lambda^{(d)}$ with $l_1 = k_1$ and $l_2 = k_2$.

Let us distinguish two cases.

*Case 1.* $k_1 \neq 1$ or $k_2 \neq 1$.

*Case 1.1.* $k_1 \neq k_2$.

If $k_1 < k_2$,

$$\sum_{d=1}^{n-1}(\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1k_2}^{l_1l_2} = \sum_{d=k_2}^{n-1}\frac{1}{C_{d+1}^2} - \frac{k_2-1}{C_{k_2}^2} = 2\left[\sum_{d=k_2}^{n-1}\left(\frac{1}{d} - \frac{1}{d+1}\right) - \frac{1}{k_2}\right] = -\frac{2}{n}. \tag{4.13}$$

Similarly, if $k_1 > k_2$,

$$\sum_{d=1}^{n-1}(\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1k_2}^{l_1l_2} = -\frac{2}{n}. \tag{4.14}$$

*Case 1.2.* $k_1 = k_2 \neq 1$:

$$\sum_{d=1}^{n-1}(\Lambda^{(d)} \otimes \Lambda^{(d)})_{k_1k_2}^{l_1l_2} = \sum_{d=k_2}^{n-1}\frac{1}{C_{d+1}^2} + \frac{(k_2-1)^2}{C_{k_2}^2} = \frac{2}{k_2} - \frac{2}{n} + \frac{(k_2-1)^2}{C_{k_2}^2} = 2 - \frac{2}{n}. \tag{4.15}$$

*Case 2.* $k_1 = k_2 = 1$:

$$\sum_{d=1}^{n-1} \left(\Lambda^{(d)} \otimes \Lambda^{(d)}\right)_{k_1 k_2}^{l_1 l_2} = \sum_{d=1}^{n-1} \frac{1}{C_{d+1}^2} = 2 - \frac{2}{n}. \tag{4.16}$$

We can condense these cases in one formula as

$$\sum_{d=1}^{n-1} \left(\Lambda^{(d)} \otimes \Lambda^{(d)}\right)_{k_1 k_2}^{l_1 l_2} = -\frac{2}{n}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2} + 2\sum_{i=1}^{n} \delta^{il_1}\delta_{k_1}^{i}\delta^{il_2}\delta_{k_2}^{i}, \tag{4.17}$$

which yields the diagonal of the diagonal matrix $\sum_{d=1}^{n-1} \Lambda^{(d)} \otimes \Lambda^{(d)}$.

For all the $n \times n$ Gell-Mann matrices, we have

$$\begin{aligned}
&\sum_{1\le i<j\le n} \left(\Lambda^{(ij)} \otimes \Lambda^{(ij)}\right)_{k_1 k_2}^{l_1 l_2} + \sum_{1\le i<j\le n} \left(\Lambda^{[ij]} \otimes \Lambda^{[ij]}\right)_{k_1 k_2}^{l_1 l_2} + \sum_{d=1}^{n-1} \left(\Lambda^{(d)} \otimes \Lambda^{(d)}\right)_{k_1 k_2}^{l_1 l_2} \\
&\qquad = -\frac{2}{n}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2} + 2\sum_{i=1}^{n} \delta^{il_1}\delta_{k_1}^{i}\delta^{il_2}\delta_{k_2}^{i} + 2\sum_{i\neq j} \delta^{il_1}\delta_{k_1}^{j}\delta^{jl_2}\delta_{k_2}^{i} \\
&\qquad = -\frac{2}{n}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2} + 2\sum_{j=1}^{n}\sum_{i=1}^{n} \delta^{il_1}\delta_{k_1}^{j}\delta^{jl_2}\delta_{k_2}^{i} \\
&\qquad = -\frac{2}{n}\delta_{k_1}^{l_1}\delta_{k_2}^{l_2} + 2\delta_{k_2}^{l_1}\delta_{k_1}^{l_2}
\end{aligned} \tag{4.18}$$

for all $l_1, l_2, k_1, k_2 \in \{1, 2, \dots, n\}$.

Hence, by using (4.3),

$$\sum_{i=1}^{n^2-1} \Lambda_i \otimes \Lambda_i = -\frac{2}{n} I_n \otimes I_n + 2U_{n\otimes n} \tag{4.19}$$

and the theorem is proved. □

## 5. Expression of $U_{3\otimes 2}$ and $U_{2\otimes 3}$

In this section, we derive formulas for $U_{3\otimes 2}$ and $U_{2\otimes 3}$, naturally in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.1}$$

and the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.2}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

For $r \in \mathbb{N}^*$, define $E_{ij}^{(r)}$ as the elementary $r \times r$ matrix whose elements are zeros except the $i$th row and $j$th column which is equal to 1. We construct $U_{3\otimes 2}$ by using Rule 4.2, and then we have

$$U_{3\otimes 2} = E_{11}^{(6)} + E_{23}^{(6)} + E_{35}^{(6)} + E_{42}^{(6)} + E_{54}^{(6)} + E_{66}^{(6)}. \tag{5.3}$$

Take

$$E_{11}^{(6)} = E_{11}^{(3)} \otimes E_{11}^{(2)}. \tag{5.4}$$

Let

$$E_{11}^{(3)} = \alpha_0 I_3 + \alpha_3 \lambda_3 + \alpha_8 \lambda_8 \tag{5.5}$$

with $\alpha_0, \alpha_3, \alpha_8 \in \mathbb{C}$, then

$$\alpha_0 = \frac{1}{3}, \qquad \alpha_3 = \frac{1}{2}, \qquad \alpha_8 = \frac{\sqrt{3}}{6},$$
$$E_{11}^{(3)} = \frac{1}{3} I_3 + \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{6}\lambda_8. \tag{5.6}$$

Let

$$E_{11}^{(2)} = \beta_0 I_2 + \beta_3 \sigma_3 \tag{5.7}$$

with $\beta_0, \beta_3 \in \mathbb{C}$, then

$$\beta_0 = \frac{1}{2}, \qquad \beta_3 = \frac{1}{2},$$
$$E_{11}^{(2)} = \frac{1}{2} I_2 + \frac{1}{2}\sigma_3. \tag{5.8}$$

So we have

$$E_{11}^{(6)} = \left(\frac{1}{3} I_3 + \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{6}\lambda_8\right) \otimes \left(\frac{1}{2} I_2 + \frac{1}{2}\sigma_3\right). \tag{5.9}$$

In a similar way, we have

$$
\begin{aligned}
E_{23}^{(6)} &= \left(\frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2\right) \otimes \left(\frac{1}{2}\sigma_1 - \frac{i}{2}\sigma_2\right),\\
E_{35}^{(6)} &= \left(\frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7\right) \otimes \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3\right),\\
E_{42}^{(6)} &= \left(\frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2\right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3\right),\\
E_{54}^{(6)} &= \left(\frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7\right) \otimes \left(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2\right),\\
E_{66}^{(6)} &= \left(\frac{1}{3}I_3 - \frac{\sqrt{3}}{3}\lambda_8\right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3\right).
\end{aligned}
\tag{5.10}
$$

Hence

$$
\begin{aligned}
U_{3\otimes 2} = {} & \left(\frac{1}{3}I_3 + \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{6}\lambda_8\right) \otimes \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3\right) + \left(\frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2\right) \otimes \left(\frac{1}{2}\sigma_1 - \frac{i}{2}\sigma_2\right)\\
& + \left(\frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7\right) \otimes \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3\right) + \left(\frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2\right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3\right)\\
& + \left(\frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7\right) \otimes \left(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2\right) + \left(\frac{1}{3}I_3 - \frac{\sqrt{3}}{3}\lambda_8\right) \otimes \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3\right).
\end{aligned}
\tag{5.11}
$$

In an analogous way,

$$
\begin{aligned}
U_{2\otimes 3} = {} & \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3\right) \otimes \left(\frac{1}{3}I_3 + \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{6}\lambda_8\right) + \left(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2\right) \otimes \left(\frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2\right)\\
& + \left(\frac{1}{2}I_2 + \frac{1}{2}\sigma_3\right) \otimes \left(\frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7\right) + \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3\right) \otimes \left(\frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2\right)\\
& + \left(\frac{1}{2}\sigma_1 - \frac{i}{2}\sigma_2\right) \otimes \left(\frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7\right) + \left(\frac{1}{2}I_2 - \frac{1}{2}\sigma_3\right) \otimes \left(\frac{1}{3}I_3 - \frac{\sqrt{3}}{3}\lambda_8\right).
\end{aligned}
\tag{5.12}
$$

One can develop these formulas in employing the distributivity of the tensor product.


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