Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 20138, 6 pages
doi:10.1155/2007/20138

Research Article

Polynomial Rings over Pseudovaluation Rings

V. K. Bhat

Received 5 March 2007; Accepted 1 August 2007

Recommended by Howard E. Bell

Let R be a ring. Let σ be an automorphism of R. We define a σ-divided ring and prove the following. (1) Let R be a commutative pseudovaluation ring such that $x \notin P$ for any $P \in \operatorname{Spec}(R[x,\sigma])$. Then $R[x,\sigma]$ is also a pseudovaluation ring. (2) Let R be a σ-divided ring such that $x \notin P$ for any $P \in \operatorname{Spec}(R[x,\sigma])$. Then $R[x,\sigma]$ is also a σ-divided ring. Let now R be a commutative Noetherian Q-algebra (Q is the field of rational numbers). Let δ be a derivation of R. Then we prove the following. (1) Let R be a commutative pseudovaluation ring. Then $R[x,\delta]$ is also a pseudovaluation ring. (2) Let R be a divided ring. Then $R[x,\delta]$ is also a divided ring.

1. Introduction

All rings are associative with identity 1. Now let R be a ring. $N(R)$ denotes the set of all nilpotent elements of R. $Z(R)$ denotes the centre of R. Q denotes the field of rational numbers unless otherwise stated. We recall that as in Hedstrom and Houston [1], an integral domain R with quotient field F, is called a pseudovaluation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). In Badawi et al. [2], the study of pseudovaluation domains was generalized to arbitrary rings in the following way.

A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion) for all $a,b \in R$. A commutative ring R is said to be a pseudovaluation ring (PVR) if each prime ideal P of R is strongly prime. We note that a commutative PVR is quasilocal by Badawi et al. [2, Lemma 1(b)].

An integral domain is a PVR if and only if it is a PVD by Anderson [3, Proposition 3.1], Anderson [4, Proposition 4.2], and Badawi [5, Proposition 3]. We recall that a prime ideal

P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided. We denote the set of prime ideals of R by $\mathrm{Spec}(R)$ and the set of strongly prime ideals of R by $S \cdot \mathrm{Spec}(R)$.

In Badawi [6], another generalization of PVDs is given in the following way:

For a ring R with total quotient ring Q such that $N(R)$ is a divided prime ideal of R, let $\phi : Q \to R_{N(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R\backslash Z(R)$. Then ϕ is a ring homomorphism from Q into $R_{N(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{N(R)}$ given by $\phi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by T. A prime ideal P of $\phi(R)$ is called a T-strongly prime ideal if $xy \in P$, $x \in T$, $y \in T$ implies that either $x \in P$ or $y \in P$. $\phi(R)$ is said to be a T-pseudovaluation ring (T-PVR) if each prime ideal of $\phi(R)$ is T-strongly prime. A prime ideal S of R is called ϕ-strongly prime ideal if $\phi(S)$ is a T-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a ϕ-pseudovaluation ring (ϕ-PVR).

Also recall from Badawi [7], a ring R is called a ϕ-chained ring (ϕ-CR) if $N(R)$ is a divided prime ideal of R and for every $a \in T\backslash\phi(R)$, we have $a^{-1} \in \phi(R)$. In Badawi [8, Proposition 2.6], it is shown that if $N(R)$ is a divided prime ideal of R, and P is a regular ϕ-strongly prime ideal of R. Then the total quotient ring Q of R is ϕ-CR.

This article concerns the study of skew polynomial rings over PVDs. Let R be a ring and σ be an automorphism of R. We denote the skew polynomial ring $R[x,\sigma]$ by $S(R)$. If I is an ideal of R such that I is σ-stable; that is, $\sigma(I) = I$, then we denote $I[x,\sigma]$ by $S(I)$. We would like to mention that $R[x,\sigma]$ is the usual set of polynomials with coefficients in R, that is, $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$.

Let R be a ring and σ be an automorphism of R. We denote the skew Laurent polynomial ring $R[x,x^{-1},\sigma]$ by $L(R)$. We would also like to mention that $L(R) = \{\sum_{i=-m}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$. If I is an ideal of R such that $\sigma(I) = I$, then we denote $I[x,x^{-1},\sigma]$ by $L(I)$.

Let R be a ring and δ be a derivation of R. We denote the differential operator ring $R[x,\delta]$ by $D(R)$. If I is an ideal of R such that $\delta(I) \subseteq I$, then we denote $I[x,\delta]$ by $D(I)$. We would like to mention that $D(R)$ is the usual set of polynomials with coefficients in R, that is, $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = xa + \delta(a)$ for all $a \in R$.

Ore-extensions including skew polynomial rings and differential operator rings have been of interest to many authors. See [9–12].

We define a σ-divided ring (σ is an automorphism of R) in the following way.

Let R be a ring. We say that a prime ideal P of R is σ-divided if it is comparable (under inclusion) to every σ-stable ideal I of R. A ring R is called a σ-divided ring if every prime ideal of R is σ-divided.

Let now R be a ring. Let σ be an automorphism of R. Then we prove the following.

(1) Let R be a commutative pseudovaluation ring such that $x \notin P$ for any $P \in \mathrm{Spec}(S(R))$. Then $R[x,\sigma]$ is also a pseudovaluation ring.

(2) Let R be a σ-divided ring such that $x \notin P$ for any $P \in \mathrm{Spec}(S(R))$. Then $R[x,\sigma]$ is also a σ-divided ring.

These results are proved in Theorems 2.6 and 2.8, respectively.

Let now R be a commutative Noetherian Q-algebra. Let δ be a derivation of R. Then we prove the following.

(1) Let R be a commutative pseudovaluation ring. Then $R[x,\delta]$ is also a pseudovaluation ring.

(2) Let R be a divided ring. Then $R[x,\delta]$ is also a divided ring.

These results are proved in Theorems 2.10 and 2.11, respectively.

2. Polynomial rings

We begin with the following known results.

Lemma 2.1. *Let R be a ring. Let σ be an automorphism of R.*

(1) *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*

(2) *If Q is a prime ideal of R such that $\sigma(Q) = Q$, then $S(Q)$ is a prime ideal of $S(R)$ and $S(Q) \cap R = Q$.*

Proof. The proof follows on the same lines as in McConnell and Robson [13, 14, Lemma 10.6.4]. □

Lemma 2.2. *Let R be a commutative Noetherian Q-algebra. Let δ be a derivation of R. Then:*

(1) *If P is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of R and $\delta(P \cap R) \subseteq P \cap R$.*

(2) *If U is a prime ideal of R such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.*

Proof. See Goodearl and Warfield [15, Theorem 2.22]. □

Lemma 2.3. *Let R be a Noetherian ring. Let σ be an automorphism of R. If I is a prime ideal of R such that $\sigma(I) \subseteq I$, then $L(I)$ is an ideal of $L(R)$ and if J is an ideal of $L(R)$, then $J \cap R$ is an ideal of R and $\sigma(J \cap R) \subseteq J \cap R$.*

Proof. See Goodearl and Warfield [15, Example 2ZA]. □

Let R be a ring. Let α be an automorphism of R and ρ be an α-derivation of R, that is, $\rho(ab) = \rho(a)\alpha(b) + a\rho(b)$, for a, $b \in R$. Then Ore-extension $R[x,\alpha,\rho]$ is the usual set of polynomials with coefficients in R, that is, $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\alpha(a) + \rho(a)$ for all $a \in R$.

Theorem 2.4 (Hilbert Basis theorem). *Let R be a right/left Noetherian ring. Let α and ρ be as above. Then the ore-extension $O(R) = R[x,\alpha,\rho]$ is right/left Noetherian. Also $R[x,x^{-1},\alpha]$ is right/left Noetherian.*

Proof. See Goodearl and Warfield [15, Theorems 1.12 and 1.17]. □

Proposition 2.5. *Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. Then the following hold.*

(1) *For any strongly prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P) = P[x,\sigma,\delta]$ is a strongly prime ideal of $O(R)$.*

(2) *For any strongly prime ideal U of $O(R)$, $U \cap R$ is a strongly prime ideal of R.*

Proof. (1) Let P be a strongly prime ideal of R. Now let $f(x) = \sum_{i=0}^{n} x^i a_i \in O(R)$ and $g(x) = \sum_{j=0}^{m} x^j b_j \in O(R)$ be such that $f(x)g(x) \in O(P)$. Suppose $f(x) \notin O(P)$. We will show that $g(x) \in O(P)$. We use induction on n and m. For $n = m = 1$, the verification is easy. We check for $n = 2$ and $m = 1$. Let $f(x) = x^2 a + xb + c$ and $g(x) = xu + v$. Now $f(x)g(x) \in O(P)$ with $f(x) \notin O(P)$. The possibilities are $a \notin P$ or $b \notin P$ or $c \notin P$ or any two out of these three do not belong to P or all of them do not belong to P. We verify case by case.

Let $a \notin P$. Since $x^3\sigma(a)u + x^2(\delta(a)u + \sigma(b)u + av) + x(\delta(b)u + \sigma(c)u + bv) + \delta(c)u + cv \in O(P)$, we have $\sigma(a)u \in P$, and so $u \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies $av \in P$, and so $v \in P$. Therefore, $g(x) \in O(P)$.

Let $b \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then $\sigma(a) \in P$ and therefore $a, \delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$ which in turn implies that $b \in P$, which is not the case. Therefore, we have $u \in P$. Now $\delta(b)u + \sigma(c)u + bv \in P$ implies that $bv \in P$ and therefore $v \in P$. Thus, we have $g(x) \in O(P)$.

Let $c \notin P$. Now $\sigma(a)u \in P$. Suppose $u \notin P$, then as above $a, \delta(a) \in P$. Now $\delta(a)u + \sigma(b)u + av \in P$ implies that $\sigma(b)u \in P$. Now $u \notin P$ implies that $\sigma(b) \in P$; that is, $b, \delta(b) \in P$. Also $\delta(b)u + \sigma(c)u + bv \in P$ implies $\sigma(c)u \in P$ and therefore $\sigma(c) \in P$ which is not the case. Thus, we have $u \in P$. Now $\delta(c)u + cv \in P$ implies $cv \in P$, and so $v \in P$. Therefore, $g(x) \in O(P)$.

Now suppose that the result is true for k, $n = k > 2$ and $m = 1$. We will prove for $n = k + 1$. Let $f(x) = x^{k+1}a_{k+1} + x^k a_k + \cdots x a_1 + a_0$, and $g(x) = xb_1 + b_0$ be such that $f(x)g(x) \in O(P)$, but $f(x) \notin O(P)$. We will show that $g(x) \in O(P)$. If $a_{k+1} \notin P$, then equating coefficients of x^{k+2}, we get $\sigma(a_{k+1})b_1 \in P$, which implies that $b_1 \in P$. Now equating coefficients of x^{k+1}, we get $\sigma(a_k)b_1 + a_{k+1}b_0 \in P$, which implies that $a_{k+1}b_0 \in P$, and therefore $b_0 \in P$. Hence $g(x) \in O(P)$.

If $a_j \notin P$, $0 \le j \le k$, then using induction hypothesis, we get that $g(x) \in O(P)$. Therefore, the statement is true for all n. Now using the same process, it can be easily seen that the statement is true for all m also. We leave the details to the reader.

(2) Let U be a strongly prime ideal of $O(R)$. Suppose $a, b \in R$ are such that $ab \in (U \cap R)$ with $a \notin (U \cap R)$. This means that $a \notin U$ as $a \in R$. Thus we have $ab \in (U \cap R) \subseteq U$, with $a \notin U$. Therefore, we have $b \in U$, and thus $b \in (U \cap R)$. □

Theorem 2.6. *Let R be a commutative PVR such that $x \notin P$ for any $P \in \mathrm{Spec}(S(R))$. Then $S(R)$ is also a PVR.*

Proof. Let $J \in \mathrm{Spec}(S(R))$. Then by Lemma 2.1, $J \cap R \in \mathrm{Spec}(R)$ and $\sigma(J \cap R) = J \cap R$. Now R is a commutative PVR, therefore $J \cap R \in S \cdot \mathrm{Spec}(R)$. Now Proposition 2.5 implies that $S(J \cap R) \in S \cdot \mathrm{Spec}(D(R))$. Now it is easy to see that $S(J \cap R) = J$. Therefore, $J \in S \cdot \mathrm{Spec}(D(R))$. Hence, $S(R)$ is a PVR. □

Corollary 2.7. *Let R be a commutative Noetherian ring which is also a PVR and $\sigma(P) = P$ for all $P \in Spec(R)$. Then $L(R)$ is also a PVR.*

Proof. Use Proposition 2.5 and Goodearl and Warfield [15, Example 2ZA]. □

Theorem 2.8. *Let R be a σ-divided Noetherian ring such that $x \notin P$ for any $P \in \mathrm{Spec}(S(R))$. Then $S(R)$ is also σ-divided Noetherian.*

Proof. We note that σ can be extended to an automorphism of $S(R)$ such that $\sigma(x) = x$. Also $S(R)$ is Noetherian by Theorem 2.4. Let $J \in \text{Spec}(S(R))$ and $0 \neq K$ be a proper ideal of $S(R)$ such that $\sigma(K) = K$. Now by McConnell and Robson [13, 14, Lemma 10.6.4], $J \cap R \in \text{Spec}(R)$ and $\sigma(J \cap R) = (J \cap R)$. Also by McConnell and Robson [13, 14, Lemma 10.6.3], $K \cap R$ is an ideal of R and $\sigma(K \cap R) = (K \cap R)$. Now R is σ-divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $(J \cap R) \subseteq (K \cap R)$. Therefore, $S(J \cap R) \subseteq S(K \cap R)$. Thus $J \subseteq K$. Hence, $S(R)$ is σ-divided Noetherian. □

Corollary 2.9. *Let R be a divided Noetherian ring and $\sigma(P) = P$ for all $P \in \text{Spec}(R)$. Then $L(R)$ is also divided.*

Proof. Use Goodearl and Warfield [15, Example 2ZA]. □

Theorem 2.10. *Let R be a commutative Noetherian Q-algebra which is also a PVR. Then $D(R)$ is also a PVR.*

Proof. Let $J \in \text{Spec}(D(R))$. Then by Lemma 2.2, $J \cap R \in \text{Spec}(R)$ and $\delta(J \cap R) \subseteq J \cap R$. Now R is a PVR, therefore $J \cap R \in S \cdot \text{Spec}(R)$. Now Proposition 2.5 implies that $D(J \cap R) \in S \cdot \text{Spec}(D(R))$; but $D(J \cap R) = J$ by Lemma 2.2. Therefore, $J \in S \cdot \text{Spec}(D(R))$. Hence $D(R)$ is a PVR. □

Theorem 2.11. *Let R be a divided commutative Noetherian Q-algebra. Then $D(R)$ is also divided Noetherian.*

Proof. $D(R)$ is Noetherian by Theorem 2.4. Let $J \in \text{Spec}(D(R))$ and $0 \neq K$ be a proper ideal of $D(R)$. Now by Goodearl and Warfield [15, Theorem 2.22], $J \cap R \in \text{Spec}(R)$ and $\delta(J \cap R) \subseteq (J \cap R)$. Also $K \cap R$ is an ideal of R and $\delta(K \cap R) \subseteq (K \cap R)$ by Goodearl and Warfield [15, Lemma 2.18]. Now R is divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $(J \cap R) \subseteq (K \cap R)$. Therefore, $D(J \cap R) \subseteq D(K \cap R)$. Thus, $J \subseteq K$. Hence, $D(R)$ is divided Noetherian. □

Question 1. Let R be a commutative PVR. Let σ be an automorphism of R and δ be a σ-derivation of R. Is $O(R) = R[x, \sigma, \delta]$ a PVR (even if R is Noetharian)?

References

[1] J. R. Hedstrom and E. G. Houston, "Pseudo-valuation domains," *Pacific Journal of Mathematics*, vol. 75, no. 1, pp. 137–147, 1978.

[2] A. Badawi, D. F. Anderson, and D. E. Dobbs, "Pseudo-valuation rings," in *Commutative Ring Theory (Fès, 1995)*, vol. 185 of *Lecture Notes in Pure and Appl. Math.*, pp. 57–67, Marcel Dekker, New York, NY, USA, 1997.

[3] D. F. Anderson, "Comparability of ideals and valuation overrings," *Houston Journal of Mathematics*, vol. 5, no. 4, pp. 451–463, 1979.

[4] D. F. Anderson, "When the dual of an ideal is a ring," *Houston Journal of Mathematics*, vol. 9, no. 3, pp. 325–332, 1983.

[5] A. Badawi, "On domains which have prime ideals that are linearly ordered," *Communications in Algebra*, vol. 23, no. 12, pp. 4365–4373, 1995.

[6] A. Badawi, "On ϕ-pseudo-valuation rings," in *Advances in Commutative Ring Theory (Fez, 1997)*, vol. 205 of *Lecture Notes in Pure and Appl. Math.*, pp. 101–110, Marcel Dekker, New York, NY, USA, 1999.

[7] A. Badawi, "On ϕ-chained rings and ϕ-pseudo-valuation rings," *Houston Journal of Mathematics*, vol. 27, no. 4, pp. 725–736, 2001.
[8] A. Badawi, "On the complete integral closure of rings that admit a ϕ-strongly prime ideal," in *Commutative Ring Theory and Applications (Fez, 2001)*, vol. 231 of *Lecture Notes in Pure and Appl. Math.*, pp. 15–22, Marcel Dekker, New York, NY, USA, 2003.
[9] S. Annin, "Associated primes over skew polynomial rings," *Communications in Algebra*, vol. 30, no. 5, pp. 2511–2528, 2002.
[10] W. D. Blair and L. W. Small, "Embedding differential and skew polynomial rings into Artinian rings," *Proceedings of the American Mathematical Society*, vol. 109, no. 4, pp. 881–886, 1990.
[11] C. Y. Hong, N. K. Kim, and T. K. Kwak, "Ore extensions of Baer and p.p.-rings," *Journal of Pure and Applied Algebra*, vol. 151, no. 3, pp. 215–226, 2000.
[12] T. K. Kwak, "Prime radicals of skew polynomial rings," *International Journal of Mathematical Sciences*, vol. 2, no. 2, pp. 219–227, 2003.
[13] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Pure and Applied Mathematics, John Wiley & Sons, Chichester, UK, 1987.
[14] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, vol. 30 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, revised edition, 2001.
[15] K. R. Goodearl and R. B. Warfield Jr., *An Introduction to Noncommutative Noetherian Rings*, vol. 16 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, UK, 1989.

V. K. Bhat: School of Applied Physics and Mathematics, Shri Mata Vaishno Devi University, P/o Kakryal, Katra 182301, India
Email address: vijay.bhat@smvdu.ac.in