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Research Article

The Interplay between Linear Representations of the Braid Group

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We consider Wada's representation as a twisted version of the standard action of the braid group, B_n, on the free group with n generators. Constructing a free group, G_{nm}, of rank nm, we compose Cohen's map $B_n \to B_{nm}$ and the embedding $B_{nm} \to \mathrm{Aut}(G_{nm})$ via Wada's map. We prove that the composition factors of the obtained representation are one copy of Burau representation and $m-1$ copies of the standard representation after changing the parameter t to t^k in the definitions of the Burau and standard representations. This is a generalization of our previous result concerning the standard Artin representation of the braid group.

1. Introduction

There are many kinds of representations of B_n, the braid group on n strings. The earliest was the Artin representation, which is an embedding $B_n \to \mathrm{Aut}(F_n)$, the automorphism group of a free group on n generators [1, page 25]. A certain type of representation, introduced by F. R. Cohen and studied by him and others, is the map $B_n \to B_{nm}$ which is defined on geometric braids by replacing each string with m strings [2, page 208].

In Section 2 of this paper, we present an infinite series of representations generalizing the standard Artin representation, which were discovered by M. Wada [3]. More precisely, for an arbitrary nonzero integer k, the automorphism corresponding to the braid generator σ_i takes x_i to ${x_i}^k x_{i+1} {x_i}^{-k}$; x_{i+1} to x_i, and fixes all other free generators. Utilizing Fox derivatives, we have a twisted version of the Burau representation. Shpilrain has shown that these representations are indeed faithful [3, page 773]. In [4], it was shown that Wada's representations are unitary.

In Section 3, we compose Cohen's map with Wada's representation and we get a linear representation of degree nm which has a subrepresentation isomorphic to the Burau representation, and the quotient is isomorphic to the direct sum of $m-1$ copies of the standard representation, which was studied by Sysoeva [5]. This is done after we change the indeterminate t to t^k in the definitions of the Burau and standard representations. As a corollary, by letting $k=1$, we get our previous result concerning the standard Artin representation of the braid group. For more details, see [6].

2. Notation and preliminaries

The braid group on n strings, B_n, is an abstract group which has a presentation with generators

$$\sigma_1,\dots,\sigma_{n-1} \tag{2.1}$$

and defining relations

$$\begin{gathered}\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}\quad \text{for } i=1,2,\dots,n-2,\\ \sigma_i\sigma_j=\sigma_j\sigma_i\quad \text{if } |i-j|\geq 2.\end{gathered} \tag{2.2}$$

The generators $\sigma_1,\dots,\sigma_{n-1}$ are called the standard generators of B_n. Let t be an indeterminate and let $\mathbb{C}[t^{\pm 1}]$ represent the Laurent polynomial ring over complex numbers.

Definition 2.1. The *Burau representation* $\beta_n(t): B_n \to GL_n(\mathbb{C}[t^{\pm 1}])$ is defined by

$$\beta_n(t)(\sigma_i)=\left(\begin{array}{c|cc|c} I_{i-1} & \multicolumn{2}{c|}{0} & 0\\ \hline 0 & 1-t & t & 0\\ & 1 & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_{n-i-1}\end{array}\right)\quad \text{for } i=1,\dots,n-1. \tag{2.3}$$

The *standard representation* $\gamma_n(t): B_n \to GL_n(\mathbb{C}[t^{\pm 1}])$ is defined by

$$\gamma_n(t)(\sigma_i)=\left(\begin{array}{c|cc|c} I_{i-1} & \multicolumn{2}{c|}{0} & 0\\ \hline 0 & 0 & t & 0\\ & 1 & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_{n-i-1}\end{array}\right)\quad \text{for } i=1,\dots,n-1. \tag{2.4}$$

For more details about the standard representation, see [5].

There is a well-known standard representation (due to Artin) of group B_n in group $\mathrm{Aut}(F_n)$ of automorphisms of the free group F_n generated by $x_1,\dots,x_n$. The automorphism $\overline{\sigma_i}$ corresponding to the braid generator σ_i takes $x_i \to x_ix_{i+1}x_i^{-1}$; $x_{i+1}\to x_i$, and fixes all other free generators.

A twisted version of the standard action of the braid group on the free group is Wada's representation; thus we have the following definition.

Definition 2.2. Wada's representations are *generalizations of the standard Artin representation*, discovered by M. Wada, and assert that the automorphism corresponding to σ_i

takes

$$\begin{aligned} x_i &\longrightarrow x_i{}^k x_{i+1} x_i{}^{-k}, \\ x_{i+1} &\longrightarrow x_i, \\ x_j &\longrightarrow x_j \quad \text{for } j \neq i,\ i+1. \end{aligned} \tag{2.5}$$

Definition 2.3 [7, page 104]. Let G be an arbitrary group and let $\mathbb{Z}G$ be the group ring of G with respect to the ring of integers $\mathbb{Z}$. A mapping $D : \mathbb{Z}G \to \mathbb{Z}G$ is said to be a *derivative* if and only if

(1) $D(f+h) = Df + Dh$ and

(2) $D(fh) = (Df)(\epsilon h) + f(Dh)$ (product rule) for all f and h in $\mathbb{Z}G$.

Here, ϵ is the augmentation homomorphism: $\mathbb{Z}G \to \mathbb{Z}$ defined by $\epsilon(\sum_{g\in G} n_g g) = \sum_{g\in G} n_g$.

Let F_n be a free group of rank n, with free basis $x_1,\dots,x_n$. We define for $j = 1,2,\dots,n$ the *free derivatives* on the group $\mathbb{Z}F_n$ by

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(x_{\mu_1}^{\epsilon_1}\cdots x_{\mu_r}^{\epsilon_r}\right) &= \sum_{i=1}^{r} \epsilon_i \delta_{\mu_i,j} x_{\mu_1}^{\epsilon_1}\cdots x_{\mu_i}^{(1/2)(\epsilon_i-1)}, \\ \frac{\partial}{\partial x_j}\left(\sum a_g g\right) &= \sum a_g \frac{\partial g}{\partial x_j}, \quad g \in F_n,\ a_g \in \mathbb{Z}, \end{aligned} \tag{2.6}$$

where $\epsilon_i = \pm 1$ and $\delta_{i,j}$ is the Kronecker symbol.

The following properties hold true.

(i) $\partial x_i/\partial x_j = \delta_{i,j}$.

(ii) $\partial x_i^{-1}/\partial x_j = -\delta_{i,j} x_i^{-1}$.

(iii) $(\partial/\partial x_j)(uv) = (\partial u/\partial x_j)\epsilon(v) + u(\partial v/\partial x_j)$ $u, v \in \mathbb{Z}F_n$.

Note that if $v \in F_n$, then $\epsilon(v) = 1$. For simplicity, we denote $\partial/\partial x_j$ by d_j.

Using the Magnus representation, the automorphism σ_i under Wada's representation is mapped onto the $n \times n$ matrix $[\phi((\partial/\partial x_r)\sigma_i(x_j))]$ which differs from the identity only by a 2×2 block with the top left corner in the (i,i)th place. More precisely,

$$\sigma_i(t) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1-t^k & t^k & 0 \\ & 1 & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array}\right) \quad \text{for } i = 1,2,\dots,n-1. \tag{2.7}$$

Given a positive integer k, we introduce indeterminates $y_1,\dots,y_n$ defined as $y_1 = x_1{}^k$, $y_2 = x_2{}^k,\dots,y_n = x_n{}^k$ and let G_n be the free group of rank n with free basis $y_1,\dots,y_n$.

If ϕ is an arbitrary homomorphism acting on F_n defined as $\phi(x_i) = t$, then $\phi(y_i) = t^k$ for $i = 1,\dots,n$. Let $G_n{}^{\phi}$ denote the image of G_n under ϕ.

Under Wada's representation, the action of the generators of B_n on the free group F_n induces an action on the free subgroup G_n. That is, we have a faithful representation of B_n as a subgroup of $\mathrm{Aut}(G_n)$.

Lemma 2.4. *Under Wada's representation, the action of σ_i on the basis of G_n, namely, $\{y_1,\dots,y_n\}$, is given by*

$$\begin{aligned} y_i &\longrightarrow y_i y_{i+1} y_i{}^{-1}, \\ y_{i+1} &\longrightarrow y_i, \\ y_r &\longrightarrow y_r, \quad r \neq i,\ i+1. \end{aligned} \tag{2.8}$$

Proof. $\sigma_i(y_i) = \sigma_i(x_i{}^k) = (\sigma_i(x_i))^k = x_i{}^k x_{i+1} x_i{}^{-k} x_i{}^k x_{i+1} x_i{}^{-k} \cdots x_i{}^k x_{i+1} x_i{}^{-k} = x_i{}^k x_{i+1}{}^k x_i{}^{-k} = y_i y_{i+1} y_i{}^{-1}$.

The action of σ_i on the other generators follows easily. □

Using Lemma 2.4 and the Magnus representation of B_n as a subgroup of $\mathrm{Aut}(G_n)$, the automorphism σ_i is mapped onto the $n \times n$ matrix $[\phi((\partial/\partial y_r)\sigma_i(y_s))]$. Direct computations show that it is the same matrix as in (2.7). Therefore, we get the following corollary.

Corollary 2.5. *Under Wada's representation, the $n \times n$ matrices obtained by letting B_n act on F_n or on G_n are exactly the same.*

Proof. This follows easily from Lemma 2.4 and the fact that we have defined $\phi(y_i) = t^k$. □

3. Automorphisms of G_{nm}

Definition 3.1 [2, page 208]. The *Cohen representation* is the map $B_n \to B_{nm}$ defined as follows:

$$\sigma_i \longrightarrow 1 \times \sigma_i = (\sigma_{mi}\sigma_{mi+1}\cdots\sigma_{mi+m-1})(\sigma_{mi-1}\sigma_{mi}\cdots\sigma_{mi+m-2})\cdots(\sigma_{mi-m+1}\sigma_{mi-m+2}\cdots\sigma_{mi}). \tag{3.1}$$

Here, $1 \times \sigma_i$ is the braid obtained by replacing each string of the geometric braid, σ_i, with m parallel strings. Cohen called $1 \times \sigma_i$ a tensor product.

Putting $k = 1$ in the definition of Wada's map, we get the result in [6], which asserts that by composing Cohen's map with Artin's representation of the braid group, we get a linear representation: $B_n \to B_{nm} \to GL_{nm}(\mathbb{Z}[t^{\pm 1}])$ which has a subrepresentation isomorphic to the Burau representation, and the quotient is isomorphic to the direct sum of $m-1$ copies of the standard representation, which was studied by Sysoeva [5].

In this paper, we generalize the result by taking any positive integer k and consider Wada's representation, which is a twisted version of the standard action of the braid group on the free group.

Given the free generators $x_1,\dots,x_{nm}$, we let $y_i = x_i{}^k$ for $i = 1,\dots,nm$. We take G_{nm} to be the free group generated by $y_1,\dots,y_{nm}$.

Let τ_i be the image of the braid generator σ_i of B_n under the Cohen map. Using Lemma 2.4, there is an induced action of τ_i on the free subgroup G_{nm}. As in Section 2, we show that the $(nm)\times(nm)$ matrix obtained by letting τ_i as act on F_{nm} with generators $x_1,\dots,x_{nm}$ is exactly the same as that obtained by having τ_i act on G_{nm} with generators $x_1{}^k,\dots,x_{nm}{}^k$ instead. Therefore, we get the following theorem.

Theorem 3.2. *The action of the image of the generator of B_n under Cohen's map, namely, τ_i, on F_{nm} gives an $(nm)\times(nm)$ matrix which is the same as the one obtained under the action of τ_i on the free subgroup G_{nm}.*

Proof. Let

$$\tau_i = (\sigma_{mi}\sigma_{mi+1}\cdots\sigma_{mi+m-1})(\sigma_{mi-1}\sigma_{mi}\cdots\sigma_{mi+m-2})\cdots(\sigma_{mi-m+1}\sigma_{mi-m+2}\cdots\sigma_{mi}). \tag{3.2}$$

Let us see the action of τ_i on F_{nm} with generators $x_1,\dots,x_{nm}$.

It is clear that we need to see the action of τ_i especially on the $2m$ elements, namely,

$$x_{mi-m+1}, x_{mi-m+2},\dots,x_{mi}, x_{mi+1}, x_{mi+2},\dots,x_{mi+m}. \tag{3.3}$$

As for the other elements, the action of τ_i is trivial. Direct computations show that

$$\tau_i(x_{mi-m+s}) = (x_{mi-m+1}{}^k\cdots x_{mi}{}^k)x_{mi+s}(x_{mi-m+1}{}^k\cdots x_{mi}{}^k)^{-1} \quad \text{for } s = 1,\dots,m. \tag{3.4}$$

Also, we have that

$$\tau_i(x_{mi+s}) = x_{mi+s-m} \quad \text{for } s = 1,\dots,m. \tag{3.5}$$

The action of τ_i on the free subgroup G_{nm} with generators $y_1,\dots,y_{nm}$, where $y_j = x_j{}^k$ for $j = 1,\dots,nm$, is given by

$$\tau_i(y_{mi-m+s}) = (y_{mi-m+1}\cdots y_{mi})y_{mi+s}(y_{mi-m+1}\cdots y_{mi})^{-1} \quad \text{for } s = 1,\dots,m. \tag{3.6}$$

Also, we have that

$$\tau_i(y_{mi+s}) = y_{mi+s-m} \quad \text{for } s = 1,\dots,m. \tag{3.7}$$

Next, we apply Magnus representation to get the matrices corresponding to τ_i, namely, $[\phi((\partial/\partial x_r)\tau_i(x_s))]$ and $[\phi((\partial/\partial y_r)\tau_i(y_s))]$. Using Fox derivatives and having defined $\phi(x_j) = t$ and $\phi(y_j) = t^k$ for $j = 1,\dots,nm$, we get that the matrices are the same. To see this, we make some computations.

For fixed values of i and m, we denote $\phi((\partial/\partial y_r)\tau_i(y_{mi-m+s}))$ or $\phi((\partial/\partial x_r)\tau_i(x_{mi-m+s}))$ by $d_r(\tau_i(y_{mi-m+s}))$ or $d_r(\tau_i(x_{mi-m+s}))$. Direct computations show that these derivatives are equal. More precisely, we have that

$$d_{mi-m+1}(\tau_i(y_{mi-m+s})) = 1 - t^k, \qquad d_{mi-m+2}(\tau_i(y_{mi-m+s})) = t^k - t^{2k},$$
$$d_{mi-m+3}(\tau_i(y_{mi-m+s})) = t^{2k} - t^{3k}, \dots, d_{mi}(\tau_i(y_{mi-m+s})) = t^{(m-1)k} - t^{mk}. \tag{3.8}$$

For $2 \leq s \leq m$, we have

$$d_{mi+1}(\tau_i(y_{mi-m+s})) = \cdots = d_{mi+s-1}(\tau_i(y_{mi-m+s})) = 0. \tag{3.9}$$

Also, we have that for $1 \leq s \leq m$

$$d_{mi+s}(\tau_i(y_{mi-m+s})) = t^{mk}. \tag{3.10}$$

If $s \leq m-1$, then

$$d_{mi+s+1}(\tau_i(y_{mi-m+s})) = \cdots = d_{mi+m}(\tau_i(y_{mi-m+s})) = 0. \tag{3.11}$$

As for the elements y_{mi+s}, we have that

$$d_p(\tau_i(y_{mi+s})) = \delta_{p,mi+s-m} \tag{3.12}$$

($\delta_{i,j}$ is the Kronecker symbol). □

Notice that for $m = 1$, we get Corollary 2.5.

Throughout our work, we will then treat the generators of B_n as automorphisms of the free group G_{nm} with generators $y_1, \dots, y_{nm}$, where $y_i = x_i{}^k$ rather than automorphisms of F_{nm}.

Next, we proceed as in [6] by choosing elements $z_{i,j}$ of G_{nm}, each of which is a word in these y_i's. More precisely, for $1 \leq i \leq m$ and $1 \leq j \leq n$ we define $z_{i,j}$ as follows:

$$z_{i,j} = y_{1+mj-m}\, y_{2+mj-m} \cdots y_{mj-i+1}. \tag{3.13}$$

It is then clear that for fixed choices of a positive integer, m, and an integer $i : 1 \leq i \leq m$, the length of $z_{i,j}$ is $m - i + 1$. In other words, the generators $\{z_{i,j}\}$ are defined as follows:

$$\begin{array}{llll}
z_{1,1} = y_1 \cdots y_m, & z_{2,1} = y_1 \cdots y_{m-1}, & \dots, & z_{m,1} = y_1, \\
z_{1,2} = y_{1+m} \cdots y_{2m}, & z_{2,2} = y_{1+m} \cdots y_{2m-1}, & \dots, & z_{m,2} = y_{1+m}, \\
\vdots & \vdots & & \vdots \\
z_{1,n} = y_{1+(n-1)m} \cdots y_{nm}, & z_{2,n} = y_{1+(n-1)m} \cdots y_{nm-1}, & \dots, & z_{m,n} = y_{1+(n-1)m}.
\end{array} \tag{3.14}$$

Lemma 3.3. *$\{z_{i,j}\}$ is a basis of G_{nm}.*

Let $\overline{\tau_r}$ be the automorphism on G_{nm} that corresponds to τ_r which is the image of the braid generator σ_r of B_n under the Cohen map. When there is no danger of confusion, we will still denote the automorphism $\overline{\tau_r}$ by τ_r.

Using Lemma 2.4 in Section 2 of our work and [6, Theorem 3.1, page 172], we easily get the following theorem.

Theorem 3.4. *For $1 \leq r \leq n-1$ and $1 \leq i \leq m$, the action of τ_r on the basis $\{z_{i,j}\}$ of G_{nm} is given by*

(1) $z_{i,r} \to z_{1,r} z_{i,r+1} z_{1,r}{}^{-1}$,
(2) $z_{i,r+1} \to z_{i,r}$,
(3) $z_{i,j} \to z_{i,j}$, $1 \leq j \leq n$ $(j \neq r, r+1)$.

Let $\phi(z_{i,j}) = t^k$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $D_{i,j} = \phi(\partial/\partial z_{i,j})$. Now to find the linear representation

$$B_n \longrightarrow B_{nm} \longrightarrow GL(nm, \mathbb{Z})[t^{\pm 1}], \tag{3.15}$$

we determine the Jacobian matrix of the image of the braid generator σ_r under Cohen map, namely the automorphism τ_r on the group G_{nm}. But first, we give an order to the generators of G_{nm} as follows:

$$z_{1,1}, z_{1,2}, \ldots, z_{1,n}, z_{2,1}, z_{2,2}, \ldots, z_{2,n}, \ldots, z_{m,1}, z_{m,2}, \ldots, z_{m,n}. \tag{3.16}$$

Then we define the Jacobian matrix as follows:

$$J(\tau_r) = \begin{pmatrix} D_{1,1}(\tau_r(z_{1,1})) & \cdots & D_{m,n}(\tau_r(z_{1,1})) \\ \vdots & & \vdots \\ D_{1,1}(\tau_r(z_{m,n})) & \cdots & D_{m,n}(\tau_r(z_{m,n})) \end{pmatrix}. \tag{3.17}$$

We now prove our main theorem.

Theorem 3.5. *The linear representation obtained by composing the Cohen representation with Wada's representation has a subrepresentation isomorphic to the Burau representation of B_n, and the quotient is isomorphic to the direct sum of $m-1$ copies of the standard representation of B_n after changing the parameter t to t^k in the definitions of the Burau and standard representations. More precisely,*

$$\sigma_r \longrightarrow \begin{pmatrix} \beta_n(t^k)(\sigma_r) & 0 & \cdots & 0 \\ & \gamma_n(t^k)(\sigma_r) & & \vdots \\ & & \ddots & 0 \\ & & & \gamma_n(t^k)(\sigma_r) \end{pmatrix}. \tag{3.18}$$

Proof. Using Definition 2.3 for free derivatives and Theorem 3.4, we get for $1 \leq i \leq m$

$$D_{1,r}(\tau_r(z_{i,r})) = 1 - t^k, \qquad D_{i,r+1}(\tau_r(z_{i,r})) = t^k. \tag{3.19}$$

Also notice that

$$D_{i,r}\left(\tau_r\left(z_{i,r+1}\right)\right) = 1 \tag{3.20}$$

(here $\phi(z_{i,j}) = t^k$).

We take this subrepresentation as the one specified by the basis $\{z_{1,1},\dots,z_{1,n}\}$. The direct summands of the quotient are generated by the images of $\{z_{i,1},\dots,z_{i,n}\}$ for $i = 2$, $\dots$, m. In other words, the Jacobian matrix of τ_r is given by

$$\begin{pmatrix}
1 & 0 & & & & & \cdots & & & & & & & & & & & 0 \\
0 & \ddots & & & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & & & \\
 & & & 1-t^k & t^k & & & & & & & & & & & & & \\
 & & & 1 & 0 & & & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & & & \\
\vdots & & & & & & \ddots & & & & & & & & & & & \vdots \\
 & & & & & & & 1 & & & & & & & & & & \\
 & & & 1-t^k & 0 & & & & 0 & t^k & & & & & & & & \\
 & & & 0 & 0 & & & & 1 & 0 & & & & & & & & \\
 & & & & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & & & \ddots & & & & & & \\
\vdots & & & \vdots & \vdots & & & & & & & & 1 & & & & & \vdots \\
 & & & 1-t^k & 0 & & & & & & & & & 0 & t^k & & & \\
 & & & 0 & 0 & & & & & & & & & 1 & 0 & & & \\
 & & & & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & & & & \ddots & 0 \\
0 & & & & & & \cdots & & & & & & & & & & 0 & 1
\end{pmatrix}. \tag{3.21}$$

Recalling Definition 2.1, we have then proved our theorem. □

Notice that, for $k = 1$, we get the result that was proved in [6].

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