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Research Article

Statistical Convergence of Double Sequences on Probabilistic Normed Spaces

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The concept of statistical convergence was presented by Steinhaus in 1951. This concept was extended to the double sequences by Mursaleen and Edely in 2003. Karakus has recently introduced the concept of statistical convergence of ordinary (single) sequence on probabilistic normed spaces. In this paper, we define statistical analogues of convergence and Cauchy for double sequences on probabilistic normed spaces. Then we display an example such that our method of convergence is stronger than usual convergence on probabilistic normed spaces. Also we give a useful characterization for statistically convergent double sequences.

1. Introduction

An interesting and important generalization of the notion of metric space was introduced by Menger [1] under the name of statistical metric, which is now called probabilistic metric space. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. The theory of probabilistic metric space was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [3, 4]. An important family of probabilistic metric spaces are probabilistic normed spaces. The theory of probabilistic normed spaces is important as a generalization of deterministic results of linear normed spaces. The concept of statistical convergence of ordinary (single) sequence on probabilistic normed spaces was introduced by Karakus in [5]. In this paper, we extended in [5] the concept of statistical convergence from single to multiple sequences and proved some basic results.

Now we recall some notations and definitions which we use in the paper.

Definition 1.1. A function $f:\mathbb{R}\to\mathbb{R}_0^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t)=0$ and $\sup_{t\in\mathbb{R}} f(t)=1$.

We will denote the set of all distribution functions by D.

Definition 1.2. A *triangular norm*, briefly t-norm, is a *binary operation on* $[0,1]$ which is continuous, commutative, associative, nondecreasing and has 1 as neutral element, that is, it is the continuous mapping $*:[0,1]\times[0,1]\to[0,1]$ such that for all $a,b,c\in[0,1]$:

(1) $a*1=a$,
(2) $a*b=b*a$,
(3) $c*d\geq a*b$ if $c\geq a$ and $d\geq b$,
(4) $(a*b)*c=a*(b*c)$.

Example 1.3. The $*$ operations $a*b=\max\{a+b-1,0\}$, $a*b=ab$, and $a*b=\min\{a,b\}$ on $[0,1]$ are t-norms.

Definition 1.4. A triple $(X,N,*)$ is called a probabilistic normed space (briefly, a PN-space) if X is a real vector space, N is a mapping from X into D (for $x\in X$, the distribution function $N(x)$ is denoted by N_x, and $N_x(t)$ is the value of N_x at $t\in\mathbb{R}$) and $*$ is a t-norm satisfying the following conditions:

(PN-1) $N_x(0)=0$,
(PN-2) $N_x(t)=1$ for all $t>0$ if and only if $x=0$,
(PN-3) $N_{\alpha x}(t)=N_x(t/|\alpha|)$ for all $\alpha\in\mathbb{R}\backslash\{0\}$,
(PN-4) $N_{x+y}(s+t)\geq N_x(s)*N_y(t)$ for all $x,y\in X$, and $s,t\in\mathbb{R}_0^+$.

Example 1.5. Suppose that $(X,\|\cdot\|)$ is a normed space $\mu\in D$ with $\mu(0)=0$ and $\mu\neq h$, where

$$h(t)=\begin{cases}0, & t\leq 0,\\ 1, & t>0.\end{cases}\tag{1.1}$$

Define

$$N_x(t)=\begin{cases}h(t), & x=0,\\ \mu\left(\dfrac{t}{\|x\|}\right), & x\neq 0,\end{cases}\tag{1.2}$$

where $x\in X$, $t\in\mathbb{R}$. Then $(X,N,*)$ is a PN-space. For example if we define the functions μ and μ' on $\mathbb{R}$ by

$$\mu(x)=\begin{cases}0, & x\leq 0,\\ \dfrac{x}{1+x}, & x>0,\end{cases}\qquad \mu'(x)=\begin{cases}0, & x\leq 0,\\ \exp\left(\dfrac{-1}{x}\right), & x>0,\end{cases}\tag{1.3}$$

then we obtain the following well-known $*$-norms:

$$N_x(t)=\begin{cases}h(t), & x=0,\\ \dfrac{t}{t+\|x\|}, & x\neq 0,\end{cases}\qquad N'_x(t)=\begin{cases}h(t), & x=0,\\ \exp\left(\dfrac{-\|x\|}{t}\right), & x\neq 0.\end{cases}\tag{1.4}$$

We recall the concepts of convergence and Cauchy for single sequences in a probabilistic normed space.

Definition 1.6. Let $(X,N,*)$ be a PN-space. Then, a sequence $x=(x_n)$ is said to be convergent to $L\in X$ with respect to the probabilistic norm N if, for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exists a positive integer k_0 such that $N_{x_n-L}(\varepsilon)>1-\lambda$ whenever $n\geq k_0$. It is denoted by $N-\lim x=L$ or $x_n\xrightarrow{N}L$ as $n\to\infty$.

Definition 1.7. Let $(X,N,*)$ be a PN-space. Then, a sequence $x=(x_n)$ is called a Cauchy sequence with respect to the probabilistic norm N if, for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exists a positive integer k_0 such that $N_{x_n-x_m}(\varepsilon)>1-\lambda$ for all $n,m\geq k_0$.

Remark 1.8 [6]. Let $(X,\|\cdot\|)$ be a real normed space, and $N_x(t)=t/(t+\|x\|)$, where $x\in X$ and $t\geq 0$ (standard $*$-norm induced by $\|\cdot\|$). Then it is not hard to see that $x_n\xrightarrow{\|\cdot\|}x$ if and only if $x_n\xrightarrow{N}x$.

Definitions 1.6 and 1.7 for double sequences on probabilistic normed space are as follows.

Definition 1.9 [5]. Let $(X,N,*)$ be a PN-space. Then, a double sequence $x=(x_{jk})$ is said to be convergent to $L\in X$ with respect to the probabilistic norm N if, for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exists a positive integer k_0 such that $N_{x_{jk}-L}(\varepsilon)>1-\lambda$ whenever $j,k\geq k_0$. It is denoted by $N_2-\lim x=L$ or $x_{jk}\overset{N}{\to}L$ as $j,k\to\infty$.

Definition 1.10 [5]. Let $(X,N,*)$ be a PN-space. Then, a double sequence $x=(x_{jk})$ is said to be a Cauchy sequence with respect to the probabilistic norm N if, for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exist $M'=M'(\varepsilon)$ and $M=M(\varepsilon)$ such that $N_{x_{jk}-x_{pq}}(\varepsilon)>1-\lambda$ for all $j,p\geq M'$, $k,q\geq M$.

2. Statistical convergence of double sequence on PN-spaces

Steinhaus [7] introduced the idea of statistical convergence (see also Fast [8]). If K is a subset of $\mathbb{N}$, the set of natural numbers, then the asymptotic density of K denoted by $\delta(K)$ is given by

$$\delta(K):=\lim_n\frac{1}{n}\left|\{k\leq n:k\in K\}\right| \tag{2.1}$$

whenever the limit exists, where $|A|$ denotes the cardinality of the set A. A sequence $x=(x_k)$ of numbers is statistically convergent to L if

$$\delta(\{k\in\mathbb{N}:|x_k-L|\geq\varepsilon\})=0 \tag{2.2}$$

for every $\varepsilon>0$. In this case we write $\mathrm{st}-\lim x=L$.

Statistical convergence has been investigated in a number of papers [9–11].

Now we recall the concept of statistical convergence of double sequences.

Let $K\subseteq\mathbb{N}\times\mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers of (i,j) in K such that $i\leq n$ and $j\leq m$. Then the two-dimensional analog of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{nm}. \tag{2.3}$$

In case the sequence $(K(n,m)/nm)$ has a limit in Pringsheim's sense [12], then we say that K has a double natural density and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K). \tag{2.4}$$

If we consider the set of $K = \{(i,j) : i,j \in \mathbb{N}\}$, then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0. \tag{2.5}$$

Also, if we consider the set of $\{(i,2j) : i,j \in \mathbb{N}\}$ has double natural density 1/2.

If we set $n = m$, we have a two-dimensional natural density considered by Christopher [13].

Now we recall the concepts of statistical convergence and statistical Cauchy for double sequences as follows.

Definition 2.1 [14]. A real double sequence $x = (x_{jk})$ is said to be statistically convergent to a number ℓ provided that, for each $\varepsilon > 0$, the set

$$\{(j,k),\ j \leq n,\ k \leq m : |x_{jk} - \ell| \geq \varepsilon\} \tag{2.6}$$

has double natural density zero. In this case, one writes $\mathrm{st}_2 - \lim_{j,k} x_{jk} = \ell$.

Definition 2.2 [14]. A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy provided that, for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j,p \geq N$, $k,q \geq M$, the set

$$\{(j,k),\ j \leq n,\ k \leq m : |x_{jk} - x_{pq}| \geq \varepsilon\} \tag{2.7}$$

has double natural density zero.

The statistical convergence for double sequences is also studied by Móricz [15].

Now we give the analogues of these definitions with respect to the probabilistic norm N.

Definition 2.3. Let $(X,N,*)$ be a PN-space. A double sequence $x = (x_{jk})$ is statistically convergent to $L \in X$ with respect to the probabilistic norm N provided that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$,

$$K = \{(j,k),\ j \leq n,\ k \leq m : N_{x_{jk}-L}(\varepsilon) \leq 1 - \lambda\} \tag{2.8}$$

has double natural density zero, that is, if $K(n,m)$ become the numbers of (j,k) in K:

$$\lim_{n,m} \frac{K(n,m)}{nm} = 0. \tag{2.9}$$

In this case, one writes $\mathrm{st}_{N_2} - \lim_{j,k} x_{jk} = L$, where L is said to be $\mathrm{st}_{N_2}-$ limit. Also, one denotes the set of all statistically convergent double sequences with respect to the probabilistic norm N by st_{N_2}.

Now we give a useful lemma as follows.

Lemma 2.4. *Let $(X,N,*)$ be a PN-space. Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ the following statements are equivalent:*

(i) $\mathrm{st}_{N_2} - \lim_{j,k} x_{jk} = L$,

(ii) $\delta_2\{(j,k),\ j \leq n \text{ and } k \leq m : N_{x_{jk}-L}(\varepsilon) \leq 1-\lambda\} = 0$,

(iii) $\delta_2\{(j,k),\ j \leq n \text{ and } k \leq m : N_{x_{jk}-L}(\varepsilon) > 1-\lambda\} = 1$,

(iv) $\mathrm{st}_2 - \lim N_{x_{jk}-L}(\varepsilon) = 1$.

Proof. The first three parts are equivalent is trivial from Definition 2.3. It follows from Definition 2.1 that

$$\begin{aligned} &\{(j,k),\ j \leq n,\ k \leq m : |N_{x_{jk}-L}(\varepsilon) - 1| \geq \lambda\} \\ &\quad = \{(j,k),\ j \leq n,\ k \leq m : N_{x_{jk}-L}(\varepsilon) \geq 1+\lambda\} \cup \{(j,k),\ j \leq n,\ k \leq m : N_{x_{jk}-L}(\varepsilon) \leq 1-\lambda\}. \end{aligned} \tag{2.10}$$

Also, Definition 1.4 implies that (ii) and (iv) are equivalent. □

Theorem 2.5. *Let $(X,N,*)$ be a PN-space. If a double sequence $x = (x_{jk})$ is statistically convergent with respect to the probabilistic norm N, then the $\mathrm{st}_{N_2}-$limit is unique.*

Proof. Let $x = (x_{jk})$ be a double sequence. Suppose that $\mathrm{st}_{N_2} - \lim x = L_1$ and $\mathrm{st}_{N_2} - \lim x = L_2$. Let $\varepsilon > 0$ and $\lambda > 0$. Choose $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) \geq 1-\lambda$. Then, we define the following sets:

$$\begin{aligned} K_{N,1}(\gamma,\varepsilon) &:= \{(j,k) \in \mathbb{N}\times\mathbb{N} : N_{x_{jk}-L_1}(\varepsilon) \leq 1-\gamma\}, \\ K_{N,2}(\gamma,\varepsilon) &:= \{(j,k) \in \mathbb{N}\times\mathbb{N} : N_{x_{jk}-L_2}(\varepsilon) \leq 1-\gamma\}. \end{aligned} \tag{2.11}$$

Since $\mathrm{st}_{N_2} - \lim x = L_1$, we have $\delta_2\{K_{N,1}(\gamma,\varepsilon)\} = 0$ for all $\varepsilon > 0$. Furthermore, using $\mathrm{st}_{N_2} - \lim x = L_2$, we get $\delta_2\{K_{N,2}(\gamma,\varepsilon)\} = 0$ for all $\varepsilon > 0$. Now let $K_N(\gamma,\varepsilon) := \{K_{N,1}(\gamma,\varepsilon)\} \cap \{K_{N,1}(\gamma,\varepsilon)\}$. Then observe that $\delta_2\{K_N(\gamma,\varepsilon)\} = 0$ which implies

$$\delta_2\{\mathbb{N}\times\mathbb{N} \mid K_N(\gamma,\varepsilon)\} = 1. \tag{2.12}$$

If $(j,k) \in \mathbb{N}\times\mathbb{N}/K_N(\gamma,\varepsilon)$, then we have

$$N_{L_1-L_2}(\varepsilon) \geq N_{x_{jk}-L_1}\left(\frac{\varepsilon}{2}\right) * N_{x_{jk}-L_2}\left(\frac{\varepsilon}{2}\right) > (1-\gamma) * (1-\gamma) \geq 1-\lambda. \tag{2.13}$$

Since $\lambda > 0$ was arbitrary, we get $N_{L_1-L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which yields $L_1 = L_2$. Therefore, we conclude that the $\mathrm{st}_{N_2}-$ limit is unique. □

Theorem 2.6. *Let $(X,N,*)$ be a PN-space. If $N_2 - \lim x = L$ for a double sequence $x = (x_{jk})$, then $\mathrm{st}_{N_2} - \lim x = L$.*

Proof. By hypothesis, for every $\lambda \in (0,1)$ and $\varepsilon > 0$, there is a number $k_0 \in \mathbb{N}$ such that $N_{x_{jk}-L}(\varepsilon) > 1-\lambda$ for all $j \geq k_0$ and $k \geq k_0$. This guarantees that the set $\{(j,k) \in \mathbb{N}\times\mathbb{N} : N_{x_{jk}-L}(\varepsilon) \leq 1-\lambda\}$ has at most finitely many terms. Since every finite subset of the natural numbers has double density zero, we immediately see that

$$\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : N_{x_{jk}-L}(\varepsilon) \leq 1-\lambda\} = 0, \tag{2.14}$$

whence the result. □

The following example shows that the converse of Theorem 2.6 does not hold in general.

Example 2.7. Let $(\mathbb{R}, |\cdot|)$ be a real normed space, and $N_x(t) = t/(t+|x|)$, where $x \in X$ and $t \geq 0$ (standard $*$-norm induced by $|\cdot|$). In this case, observe that $(X,N,*)$ is a PN-space. Now we define a sequence $x = (x_{jk})$ whose terms are given by

$$x_{jk} := \begin{cases} \sqrt{jk}, & \text{if } j \text{ and } k \text{ are squares,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

Then, for every $\lambda \in (0,1)$ and for any $\varepsilon > 0$, let

$$K_{(\lambda,\varepsilon)}(n,m) := \{(j,k),\ j \leq n,\ k \leq m : N_{x_{jk}}(\varepsilon) \leq 1-\lambda\}. \tag{2.16}$$

Since

$$\begin{aligned} K_{(\lambda,\varepsilon)}(n,m) &= \left\{(j,k),\ j \leq n,\ k \leq m : \frac{t}{t+|x_{jk}|} \leq 1-\lambda\right\} \\ &= \left\{(j,k),\ j \leq n,\ k \leq m : |x_{jk}| \geq \frac{\lambda t}{1-\lambda} > 0\right\} \\ &= \left\{(j,k),\ j \leq n,\ k \leq m : x_{jk} = \sqrt{jk}\right\} \\ &= \{(j,k),\ j \leq n,\ k \leq m : j,\ k \text{ are squares}\}, \end{aligned} \tag{2.17}$$

we get

$$\begin{aligned} \frac{1}{nm}|K_{(\lambda,\varepsilon)}(n,m)| &\leq \frac{1}{nm}|\{(j,k),\ j \leq n,\ k \leq m : j,\ k \text{ are squares}\}| \\ &\leq \frac{\sqrt{n}\sqrt{m}}{nm} = 0, \end{aligned} \tag{2.18}$$

which implies that $\delta_2\{K_{(\lambda,\varepsilon)}(n,m)\} = 0$. Hence, by Definition 2.3, we get $\mathrm{st}_{N_2} - \lim x = 0$. However, since the sequence $x = (x_{jk})$ given by (2.15) is not convergent in the space $(\mathbb{R}, |\cdot|)$, by Remark 1.8, we also see that x is not convergent with respect to the probabilistic norm N.

Theorem 2.8. *Let $(X,N,*)$ be a PN-space and let $x = (x_{jk})$ be a double sequence. Then $\mathrm{st}_{N_2} - \lim x = L$ if and only if there exists a subset $K = \{(j,k) : j,k = 1,2,\ldots\} \subseteq \mathbb{N}\times\mathbb{N}$, such that $\delta_2(K) = 1$ and $N_2 - \lim_{(j,k)\in K}^{j,k\to\infty} x_{jk} = L$.*

Proof. We first assume that $\text{st}_{N_2} - \lim x = L$. Now, for any $\varepsilon > 0$ and $r \in \mathbb{N}$, let

$$\begin{aligned} K(r,\varepsilon) &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{x_{jk}-L}(\varepsilon) \leq 1 - \frac{1}{r}\right\}, \\ M(r,\varepsilon) &= \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{x_{jk}-L}(\varepsilon) > 1 - \frac{1}{r}\right\}. \end{aligned} \tag{2.19}$$

Then $\delta_2\{K(r,\varepsilon)\} = 0$ and

(1) $M(1,\varepsilon) \supset M(2,\varepsilon) \supset \cdots \supset M(i,\varepsilon) \supset M(i+1,\varepsilon) \supset \ldots$,

(2) $\delta_2\{M(r,\varepsilon)\} = 1$, $r = 1,2,\ldots$.

Now we have to show that for $(j,k) \in M(r,\varepsilon)$, (x_{jk}) is N_2-convergent to L. Suppose that (x_{jk}) is not N_2-convergent to L. Therefore there is $\lambda > 0$ such that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{x_{jk}-L}(\varepsilon) \leq 1 - \lambda\} \tag{2.20}$$

for infinitely many terms.

Let

$$M(\lambda,\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{x_{jk}-L}(\varepsilon) > 1 - \lambda\}, \quad \lambda > \frac{1}{r}\ (r = 1,2,\ldots). \tag{2.21}$$

Then

(3) $\delta_2\{M(\lambda,\varepsilon)\} = 0$,

and by (1), $M(r,\varepsilon) \subset M(\lambda,\varepsilon)$. Hence $\delta_2\{M(r,\varepsilon)\} = 0$ which contradicts (2). Therefore (x_{jk}) is N_2-convergent to L.

Conversely, suppose that there exists a subset $K = \{(j,k) : j,k = 1,2,\ldots\} \subset \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $N_2 - \lim_{j,k \in K} x_{jk} = L$, that is, there exists $k_0 \in \mathbb{N}$ such that for every $\lambda \in (0,1)$ and $\varepsilon > 0$

$$N_{x_{jk}-L}(\varepsilon) > 1 - \lambda, \quad \forall\, j,k \geq k_0. \tag{2.22}$$

Now

$$\begin{aligned} M(\lambda,\varepsilon) &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{x_{jk}-L}(\varepsilon) \leq 1 - \lambda\} \\ &\subseteq \mathbb{N} \times \mathbb{N} - \{(j_{k_0+1}, k_{k_0+1}), (j_{k_0+2}, k_{k_0+2}), \ldots\}. \end{aligned} \tag{2.23}$$

Therefore, $\delta_2\{M(\lambda,\varepsilon)\} \leq 1 - 1 = 0$. Hence, we conclude that $\text{st}_{N_2} - \lim x = L$. □

Definition 2.9. Let $(X,N,*)$ be a PN-space. It is assumed that a double sequence $x = (x_{jk})$ is statistically Cauchy with respect to the probabilistic norm N provided that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exist $M' = M'(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j,p \geq M'$, $k,q \geq M$, the set

$$\{(j,k),\ j \leq n,\ k \leq m : N_{x_{jk}-x_{pq}}(\varepsilon) \leq 1 - \lambda\} \tag{2.24}$$

has double natural density zero.

Now using a similar technique in the proof of Theorem 2.8, one can get the following result at once.

Theorem 2.10. *Let $(X,N,*)$ be a PN-space, and let $x = (x_{jk})$ be a double sequence whose terms are in the vector space X. Then, the following conditions are equivalent:*

(i) *x is a statistically Cauchy sequence with respect to the probabilistic norm N;*

(ii) *there exists an increasing index sequence $K = \{(j,k) : j,k = 1,2,\ldots\} \subseteq \mathbb{N}\times\mathbb{N}$ such that $\delta_2(K) = 1$ and the subsequence $\{x_{jk}\}_{(j,k)\in K}$ is a Cauchy sequence with respect to the probabilistic norm N.*

Now we show that statistical convergence of double sequences on probabilistic normed spaces has some arithmetical properties similar to properties of the usual convergence on $\mathbb{R}$.

Lemma 2.11. *Let $(X,N,*)$ be a PN-space.*

(i) *If $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$ and $\mathrm{st}_{N_2} - \lim y_{jk} = \eta$, then $\mathrm{st}_{N_2} - \lim(x_{jk} + y_{jk}) = \xi + \eta$.*

(ii) *If $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$ and $\alpha \in \mathbb{R}$, then $\mathrm{st}_{N_2} - \lim \alpha x_{jk} = \alpha\xi$.*

(iii) *If $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$ and $\mathrm{st}_{N_2} - \lim y_{jk} = \eta$, then $\mathrm{st}_{N_2} - \lim(x_{jk} - y_{jk}) = \xi - \eta$.*

Proof. (i) Let $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$, $\mathrm{st}_{N_2} - \lim y_{jk} = \eta$, $\varepsilon > 0$ and $\lambda \in (0,1)$. Choose $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) \geq 1-\lambda$. Then we define the following sets:

$$\begin{aligned} K_{N,1}(\gamma,\varepsilon) &:= \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : N_{x_{jk}-\xi}(\varepsilon) \leq 1-\gamma\right\}, \\ K_{N,2}(\gamma,\varepsilon) &:= \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : N_{x_{jk}-\eta}(\varepsilon) \leq 1-\gamma\right\}. \end{aligned} \tag{2.25}$$

Since $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$, we have

$$\delta_2\{K_{N,1}(\gamma,\varepsilon)\} = 0 \quad \forall \varepsilon > 0. \tag{2.26}$$

Similarly, since $\mathrm{st}_{N_2} - \lim y_{jk} = \eta$, we get

$$\delta_2\{K_{N,2}(\gamma,\varepsilon)\} = 0 \quad \forall \varepsilon > 0. \tag{2.27}$$

Now let $K_N(\gamma,\varepsilon) := K_{N,1}(\gamma,\varepsilon) \cap K_{N,2}(\gamma,\varepsilon)$. Then observe that $\delta_2\{K_N(\gamma,\varepsilon)\} = 0$ which implies $\delta_2\{\mathbb{N}\times\mathbb{N}/K_N(\gamma,\varepsilon)\} = 1$. If $(j,k) \in \mathbb{N}\times\mathbb{N}/K_N(\gamma,\varepsilon)$, then we have

$$\begin{aligned} N_{(x_{jk}-\xi)+(y_{jk}-\eta)}(\varepsilon) &\geq N_{x_{jk}-\xi}\left(\frac{\varepsilon}{2}\right) * N_{y_{jk}-\eta}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\gamma) * (1-\gamma) \geq 1-\lambda. \end{aligned} \tag{2.28}$$

This shows that

$$\delta_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{(x_{jk}-\xi)+(y_{jk}-\eta)}(\varepsilon) \le 1-\lambda\}) = 0 \tag{2.29}$$

so $\mathrm{st}_{N_2} - \lim(x_{jk} + y_{jk}) = \xi + \eta$.

(ii) Let $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$, $\lambda \in (0,1)$ and $\varepsilon > 0$. First of all, we consider the case of $\alpha = 0$. In this case

$$N_{0x_{jk}-0\xi}(\varepsilon) = N_0(\varepsilon) = 1 > 1-\lambda. \tag{2.30}$$

So we obtain $N_2 - \lim 0x_{jk} = 0$. Then from Theorem 2.6 we have $\mathrm{st}_{N_2} - \lim 0x_{jk} = 0$.

Now we consider the case of $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$. Since $\mathrm{st}_{N_2} - \lim x_{jk} = \xi$, if we define the set

$$K_N(\gamma,\varepsilon) := \{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{x_{jk}-\xi}(\varepsilon) \le 1-\lambda\}, \tag{2.31}$$

then we can say $\delta_2(K_N(\gamma,\varepsilon)) = 0$ for all $\varepsilon > 0$. In this case $\delta_2(\mathbb{N} \times \mathbb{N}/K_N(\gamma,\varepsilon)) = 1$. If $(j,k) \in \mathbb{N} \times \mathbb{N}/K_N(\gamma,\varepsilon)$ then

$$\begin{aligned} N_{\alpha x_{jk}-\alpha\xi}(\varepsilon) &= N_{x_{jk}-\xi}\left(\frac{\varepsilon}{|\alpha|}\right) \ge N_{x_{jk}-\xi}(\varepsilon) * N_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right) \\ &= N_{x_{jk}-\xi}(\varepsilon) * 1 = N_{x_{jk}-\xi}(\varepsilon) > 1-\lambda \end{aligned} \tag{2.32}$$

for $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$. This shows that

$$\delta_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{\alpha x_{jk}-\alpha\xi}(\varepsilon) \le 1-\lambda\}) = 0 \tag{2.33}$$

so $\mathrm{st}_{N_2} - \lim \alpha x_{jk} = \alpha\xi$.

(iii) The proof is clear from (i) and (ii). □

Definition 2.12. Let $(X,N,*)$ be a PN-space. For $x = (x_{jk}) \in X$, $t > 0$ and $0 < r < 1$, the ball centered at x with radius r is defined by

$$B(x,r,t) = \{y \in X : N_{x-y}(t) > 1-r\}. \tag{2.34}$$

Definition 2.13. A subset Y of PN-space $(X,N,*)$ is called bounded on PN-spaces if for every $r \in (0,1)$, there exists $t_0 > 0$ such that $N_{x_{jk}}(t_0) > 1-r$ for all $x = (x_{jk}) \in Y$.

It follows from Lemma 2.11 that the set of all bounded statistically convergent double sequences on PN-space is a linear subspace of the linear normed space $\ell_\infty^{N_2}(X)$ of all bounded sequences on PN-space.

Theorem 2.14. *Let $(X,N,*)$ be a PN-space and the set $\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)$ is closed linear subspace of the set $\ell_\infty^{N_2}(X)$.*

Proof. It is clear that $\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X) \subset \overline{\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)}$. Now we show $\overline{\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)} \subset \mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)$. Let $y \in \overline{\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)}$. Since $B(y,r,t) \cap (\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)) \neq \varnothing$, there is an $x \in B(y,r,t) \cap (\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X))$.

Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r) * (1-r) \geq 1-\varepsilon$. Since $x \in B(y,r,t) \cap (\mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X))$, there is a set $K \subseteq \mathbb{N} \times \mathbb{N}$ with $\delta_2(K) = 1$ such that

$$N_{y_{jk}-x_{jk}}\left(\frac{t}{2}\right) > 1-r, \qquad N_{x_{jk}}\left(\frac{t}{2}\right) > 1-r \tag{2.35}$$

for all $(j,k) \in K$. Then we have

$$\begin{aligned} N_{y_{jk}}(t) &= N_{y_{jk}-x_{jk}+x_{jk}}(t) \\ &\geq N_{y_{jk}-x_{jk}}\left(\frac{t}{2}\right) * N_{x_{jk}}\left(\frac{t}{2}\right) \\ &> (1-r) * (1-r) \geq 1-\varepsilon \end{aligned} \tag{2.36}$$

for all $(j,k) \in K$. Hence

$$\delta_2\left(\{(j,k) \in \mathbb{N} \times \mathbb{N} : N_{y_{jk}}(t) > 1-\varepsilon\}\right) = 1 \tag{2.37}$$

and thus $y \in \mathrm{st}_{N_2}(X) \cap \ell_\infty^{N_2}(X)$. □

3. Conclusion

In this paper, we obtained results on statistical convergence for double sequences on probabilistic normed spaces. As every ordinary norm induces a probabilistic norm, the obtained results here are more general than the corresponding results of normed spaces.

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