INTUITIONISTIC H-FUZZY RELATIONS

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Received 14 February 2005 and in revised form 19 June 2005

We introduce the category **IRel**(H) consisting of intuitionistic fuzzy relational spaces on sets and we study structures of the category **IRel**(H) in the viewpoint of the topological universe introduced by Nel. Thus we show that **IRel**(H) satisfies all the conditions of a topological universe over **Set** except the terminal separator property and **IRel**(H) is cartesian closed over **Set**.

1. Introduction

In 1965, Zadeh [30] introduced a concept of a fuzzy set as the generalization of a crisp set. Also, in 1971, he introduced a fuzzy relation naturally, as a generalization of a crisp relation in [31].

Nel [27] introduced the notion of a topological universe which implies concrete quasitopos [1]. Every topological universe satisfies all the properties of a topos except one condition on the subobject classifier. The notion of a topological universe has already been put to effective use in several areas of mathematics in [24, 25, 28]. In 1980, Cerruti [8] introduced the category of L-fuzzy relations and investigated some of its properties. After that time, Hur [14] introduced the category **Rel**(H) of the fuzzy relational spaces with a complete Heyting algebra H as a codomain and he studied the category **Rel**(H) in the sense of a topological universe.

In 1983, Atanassov [2] introduced the concept of an intuitionistic fuzzy set as the generalization of fuzzy sets and he also investigated many properties of intuitionistic fuzzy sets (cf. [3]). After that time, Banerjee and Basnet [4], Biswas [6], and Hur and his colleagues [15, 16, 17, 20] applied the concept of intuitionistic fuzzy sets to algebra. Also, Çoker [9], Hur and his colleagues [21], and S. J. Lee and E. P. Lee [26] applied one to topology. In particular, Hur and his colleagues [18] applied the notion of intuitionistic fuzzy sets to topological group.

In this paper, we introduce the category **IRel**(H) of intuitionistic H-fuzzy relational spaces and study the category **IRel**(H) in a topological universe viewpoint. In particular, we show that **IRel**(H) satisfies all the conditions of a topological universe over **Set** except

International Journal of Mathematics and Mathematical Sciences 2005:17 (2005) 2723–2734
DOI: 10.1155/IJMMS.2005.2723

the terminal separator property. Also **IRel**(H) is shown to be cartesian closed over **Set**. For general categorical background, we refer to Herrlich and Strecker [12].

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results which are needed in the next sections.

Let X be a set, let $(X_i)_{i\in I}$ be a family of sets indexed by a class I, and let f_i be a mapping with domain X for each $i \in I$. Then a pair $(X,(f_i)_I)$ (simply, $(f_i)_I$) is called a *source of mappings*. A *sink of mappings* is the dual notion of a source of mappings.

Definition 2.1 [12]. Let **A** be a concrete category and let I be a class.

(1) A *source in* **A** is a pair $(X,(f_i)_I)$ (simply, (X,f_i) or $(f_i)_I$), where X is an **A**-object and $(f_i : X \to X_i)_I$ is a family of **A**-morphisms each with domain X. In this case, X is called the *domain of the source* and the family $(X_i)_I$ is called the *codomain of the source*.

(2) A source (X,f_i) is called a *monosource* provided that the f_i can be simutaneously canceled from the left; that is, provided that for any pair $Y \underset{s}{\overset{r}{\rightrightarrows}} X$ of morphisms such that $f_i \circ r = f_i \circ s$ for each $i \in I$, it follows that $r = s$.

Dual notions: *sink in* **A** and *episink*.

Definition 2.2 [23]. Let **A** be a concrete category and let $((Y_i,\xi_i))_I$ be a family of objects in **A** indexed by a class I. For any set X, let $(f_i : X \to Y_i)_I$ be a source of mappings indexed by I. An **A**-structure ξ on X is said to be *initial with respect to* $(X,(f_i),((Y_i,\xi_i)))$ provided that the following conditions hold.

(1) For each $i \in I$, $f_i : (X,\xi) \to (Y_i,\xi_i)$ is an **A**-morphism.

(2) If (Z,ρ) is an **A**-object and $g : Z \to X$ is mapping such that for each $i \in Z$, the mapping $f_i \circ g : (Z,\rho) \to (Y_i,\xi_i)$ is an **A**-morphism, then $g : (Z,\rho) \to (X,\xi)$ is an **A**-morphism. In this case, $(f_i : (X,\xi) \to (Y_i,\xi_i))_I$ is called an *initial source in* **A**.

Dual notions: *final structure* and *final sink*.

Definition 2.3 [23]. A concrete category **A** is said to be *topological over* ***Set*** provided that for each set X, for any family $((Y_i,\xi_i))_I$ of **A**-objects, and for any source $(f_i : X \to Y_i)_I$ of mappings, there exists a unique **A**-structure ξ on X which is initial with respect to $(X,(f_i),((Y_i,\xi_i)))$.

Dual notions: *cotopological category*.

Result 2.4 [23, Theorem 1.5]. *A concrete category* **A** *is topological if and only if* **A** *is cotopological.*

Result 2.5 [23, Theorem 1.6]. *Let* **A** *be a topological category over* ***Set***. *Then* **A** *is complete and cocomplete.*

Definition 2.6 [11]. A category **A** is called *cartesian closed* provided that the following conditions hold.

(1) For any **A**-objects A and B, there exists a product $A \times B$ in **A**.

(2) Exponential exists in **A**, that is, for any **A**-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, that is, for any **A**-object B, there exists an **A**-object B^A and an **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the *evaluation*) such that for any **A**-object C

and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\overline{f} : C \to B^A$ such that the diagram

$$\begin{array}{ccc} A \times B^A & \xrightarrow{e_{A,B}} & B \\ & \nwarrow^{\exists 1_A \times \overline{f}} \quad \nearrow_{f} & \\ & A \times C & \end{array} \tag{2.1}$$

commutes.

Definition 2.7 [23]. Let **A** be a concrete category.

(1) The **A**-*fiber* of a set X is the class of all **A**-structures on X.

(2) **A** is called *properly fibered over **Set*** provided that the following conditions hold.

(i) *Fiber-smallness*. For each set X, the **A**-fiber of X is a set.

(ii) *Terminal separator property*. For each singleton set X, the **A**-fiber of X has precisely one element.

(iii) If ξ and η are **A**-structures on a set X such that $1_X : (X,\xi) \to (X,\eta)$ and $1_X : (X,\eta) \to (X,\xi)$ are **A**-morphisms, then $\xi = \eta$.

Definition 2.8 [27]. A category **A** is called a *topological universe over **Set*** provided that the following conditions hold.

(1) **A** is well structured over **Set**, that is, (i) **A** is a concrete category; (ii) **A** has the fiber-smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, that is, for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Definition 2.9 [29]. A category **A** is called a *topos* provided that the following conditions hold.

(1) There is a terminal object U in **A**, that is, for each **A**-object A, there exists one and only one **A**-morphism from A to U.

(2) **A** has equalizers, that is, for any **A**-objects A and B and **A**-morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \tag{2.2}$$

there exist an **A**-object C and an **A**-morphism $h : C \to A$ such that

(a) $f \circ h = g \circ h$,

(b) for each **A**-object C' and **A**-morphism $h' : C' \to A$ with $f \circ h' = g \circ h'$, there exists a unique **A**-morphism $\overline{h'} : C' \to C$ such that $h' = h \circ \overline{h'}$, that is, the diagram

$$\begin{array}{ccccc} C & \xrightarrow{h} & A & \underset{g}{\overset{f}{\rightrightarrows}} & B \\ {\scriptstyle \exists \overline{h'}} \uparrow & \nearrow_{h'} & & & \\ C' & & & & \end{array} \tag{2.3}$$

commutes;

(3) **A** is cartesian closed;
(4) there is a subobject classifier in **A**, that is, there is an **A**-object Ω and **A**-morphism $v : U \to \Omega$ such that for each **A**-monomorphism $m : A' \to A$, there exists a unique **A**-morphism $\phi_m : A \to \Omega$ such that the following diagram is a pullback:

$$\begin{array}{ccc} A' & \longrightarrow & U \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ A & \underset{\phi_m}{\longrightarrow} & \Omega \end{array} \tag{2.4}$$

Remark 2.10. Let **A** be any category with a subobject classifier. If f is any bimorphism in **A**, then f is an isomorphism in **A** (cf. [7]).

3. The category IRel(H)

First we will list some concepts and one result which are needed in this section and the next section. Next, we introduce the category **IRel(H)** of intuitionistic H-fuzzy relational spaces and show that it has similar structures as those of **ISet(H)**.

Definition 3.1 [5, 22]. A lattice H is called a *complete Heyting algebra* if H satisfies the following conditions:
(1) H is a complete lattice;
(2) for any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has a greatest element denoted by $a \to b$ (called *pseudocomplement of a and b*), that is, $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.
In particular, for each $a \in H$, $N(a) = a \to 0$ is called the *negation* or the *pseudocomplement* of a.

Result 3.2 [5, Example 6, page 46]. *Let H be a complete Heyting algebra and let $a, b \in H$. Then*
(1) *if $a \leq b$, then $N(b) \leq N(a)$, that is, $N : H \to H$ is an involutive order-reversing operation in $(H, \leq)$;*
(2) *$a \leq NN(a)$;*
(3) *$N(a) = NNN(a)$;*
(4) *$N(a \vee b) = N(a) \wedge N(b)$ and $N(a \wedge b) = N(a) \wedge N(b)$.*

Throughout this paper, we use H as a complete Heyting algebra.

Definition 3.3 [19]. Let X be a set. A triple (X, μ, ν) is called an *intuitionistic H-fuzzy set* (in short, *IHFS*) on X if the following conditions holds:
(i) $\mu, \nu \in H^X$, that is, μ and ν are H-fuzzy sets;
(ii) $\mu \leq N(\nu)$, that is, $\mu(x) \leq N(\nu(x))$ for each $x \in X$, where $N : H \to H$ is an involutive order-reversing operation in $(H, \leq)$.

Definition 3.4 [19]. Let (X, μ_X, ν_X) and (Y, μ_Y, ν_Y) be IHFSs. A mapping $f : X \to Y$ is called a *morphism* if $\mu_X \leq \mu_Y \circ f$ and $\nu_X \geq \nu_Y \circ f$.

From Definitions 3.3 and 3.4, we can form a concrete category **ISet**(H) consisting of all IHFSs and morphisms between them. In this case, each **ISet**(H)-morphism will be called an **ISet**(H)-*mapping*.

It is clear that if $f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y)$ is an **ISet**(H)-mapping, then $f : (X, \mu_X) \to (Y, \mu_Y)$ is a **Set** (H)-mapping (cf. [13]).

Definition 3.5 [14]. (1) Let X be a set. R is called an *H-fuzzy relation* (or simply, a *fuzzy relation*) on X if $\mu_R : X \times X \to H$ is a mapping. In this case, (X, R) is called an *H-fuzzy relational space* (or simply, a *fuzzy relational space*).

(2) Let (X, R_X) and (Y, R_Y) be any fuzzy relational spaces. A map $f : X \to Y$ is called a *relation-preserving map* provided that $\mu_R \leq \mu_R \circ f^2$, where $f^2 = f \times f$.

From Definition 3.5, we can form a concrete category **Rel**(H) consisting of all relational spaces and relation preserving mappings between them. Every **Rel**(H)-morphism will be called a ***Rel*** (H)*-mapping*.

Definition 3.6. Let X be a set. A pair $R = (\mu_R, \nu_R)$ is called an *intuitionistic H-fuzzy relation* (in short, *IHFR*) on X if it satisfies the following conditions:

(i) $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ are mappings, where μ_R and ν_R denote the *degree of membership* (namely, $\mu_R(x, y)$) and the *degree of nonmembership* (namely, $\nu_R(x, y)$) of each $(x, y) \in X \times X$ to R;

(ii) $\mu_R \leq N(\nu_R)$, that is, $\mu_R(x, y) \leq N(\nu_R(x, y))$ for each $(x, y) \in X \times X$.

In this case, (X, R) or (X, μ_R, ν_R) is called an *intuitionistic H-fuzzy relatinal space* (in short, *IHFRS*).

Definition 3.7. Let (X, R_X) and (Y, R_Y) be an IHFRSs. A mapping $f : X \to Y$ is called a *relation-preserving mapping* if $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$.

The following is the immediate result of Definition 3.7.

Proposition 3.8. *Let (X, R_X), (Y, R_Y), and (Z, R_Z) be IHFRSs.*

(1) $1_X : (X, R_X) \to (X, R_X)$ *is a relation-preserving mapping.*

(2) *If $f : (X, R_X) \to (Y, R_Y)$ and $g : (Y, R_Y) \to (Z, R_Z)$ are relation-preserving mappings, then $g \circ f : (X, R_X) \to (Z, R_Z)$ is a relation-preserving mapping.*

From Definitions 3.6 and 3.7, and Proposition 3.8, we can form a concrete category **IRel**(H) consisting of all IHFRSs and relation-preserving mappings between them. Every **IRel**(H)-morphism will be called an **IRel**(H)*-mapping*. Moreover, it is clear that if $f : (X, R_X) \to (Y, R_Y)$ is an **IRel**(H)-mapping, then $f : (X, \mu_{R_X}) \to (Y, \mu_{R_Y})$ is a **Rel** (H)-mapping.

Theorem 3.9. **IRel**(H) *is topological over* ***Set***.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs indexed by a class Γ. Let $(f_\alpha : X \to X_\alpha)_\Gamma$ be any source of mappings. We define two mappings $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ by $\mu_R = \bigwedge_\Gamma \mu_{R_\alpha} \circ f_\alpha^2$ and $\nu_R = \bigvee_\Gamma \nu_{R_\alpha} \circ f_\alpha^2$. Then, by the definition of $R = (\mu_R, \nu_R)$, $\mu_R \leq N(\nu_R)$. Thus $(X, R) \in$ **IRel**(H). Moreover, $f_\alpha : (X, R) \to (X_\alpha, R_\alpha)$ is an **IRel**(H)-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in$ **IRel**(H), let $g : Y \to X$ be any mapping for which $f_\alpha \circ g : (Y, R_Y) \to (X_\alpha, R_\alpha)$ is an **IRel**(H)-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (Y, R_Y) \to (X, R)$ is an **IRel**(H)-mapping. Hence $R = (\mu_R, \nu_R)$ is the initial structure on X with respect to $(X, (f_\alpha), ((X_\alpha, R_\alpha)))$. This completes the proof. □

Example 3.10. (1) *Inverse image of an IHFR*. Let X be a set, let (Y, R_Y) be an IHFRS, and let $f : X \to Y$ be any mapping. Then there exists the initial IHFR R on X for which $f : (X,R) \to (Y,R_Y)$ is an **IRel**(H)-mapping. In this case, R is called the *inverse image* of R_Y under f. In particular, if $X \subset Y$ and $f : X \to Y$ is the canonical mapping, then (X,R) is called an *intuitionistic H-fuzzy relational subspace* of (Y,R_Y), where $R = (\mu_R, \nu_R)$ is the inverse image of R_Y under f. In fact, $\mu_R = \mu_{R_Y}|_{X\times X}$ and $\nu_R = \nu_{R_Y}|_{X\times X}$.

(2) *Intuitionistic fuzzy product structure*. Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs and let $X = \prod X_\alpha$ be the product set of $(X_\alpha)_\Gamma$. Then there exists the initial IHFR R on X for which each projection $\pi_\alpha : (X,R) \to (X_\alpha, R_\alpha)$ is an **IRel**(H)-mapping. In this case, R is called the *product* of $(R_\alpha)_\Gamma$ and is denoted by $R = \prod R_\alpha$ and $(\prod X_\alpha, \prod R_\alpha)$ is called the *intuitionistic H-fuzzy product relational space* of $((X_\alpha, R_\alpha))_\Gamma$. In fact, $\mu_{\Pi R} = \bigwedge_\Gamma \mu_{R_\alpha} \circ \pi_\alpha^2$ and $\nu_{\Pi R} = \bigvee_\Gamma \nu_{R_\alpha} \circ \pi_\alpha^2$.

In particular, if $H = \{1,2\}$, then $\mu_{R_1\times R_2}((x_1,y_1),(x_2,y_2)) = \mu_{R_1}(x_1,x_2) \wedge \mu_{R_2}(y_1,y_2)$ and $\nu_{R_1\times R_2}((x_1,y_1),(x_2,y_2)) = \nu_{R_1}(x_1,x_2) \vee \nu_{R_2}(y_1,y_2)$ for any $(x_1,y_1),(x_2,y_2) \in X_1 \times X_2$.

Corollary 3.11. **IRel**(H) *is complete and cocomplete. Moreover, by definition, it is easy to show that* **IRel**(H) *is well powered and co-well-powered.*

From Result 2.4 and Theorem 3.9, it is clear that **IRel**(H) *is cotopological. However, we show directly that* **IRel**(H) *is cotopological.*

Theorem 3.12. **IRel**(H) *is cotopological over* ***Set***.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs indexed by a class Γ. Let $(f_\alpha : X_\alpha \to X)_\Gamma$ be any sink of mappings. We define two mappings $\mu_R : X\times X \to H$ and $\nu_R : X \times X \to H$ by, for each $(x,y) \in X \times X$,

$$\begin{aligned} \mu_R(x,y) &= \bigvee_{\Gamma} \bigvee_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} \mu_{R_\alpha}(x_\alpha,y_\alpha), \\ \nu_R(x,y) &= \bigwedge_{\Gamma} \bigwedge_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} \nu_{R_\alpha}(x_\alpha,y_\alpha), \end{aligned} \tag{3.1}$$

where $f_\alpha^{-1^2} = f_\alpha^{-1} \times f_\alpha^{-1}$. Then clearly $(X,R) \in$ **IRel**(H). Moreover, $f_\alpha : (X_\alpha, R_\alpha) \to (X,R)$ is an **IRel**(H)-mapping for each $\alpha \in \Gamma$.

For any $(Y,R_Y) \in$ **IRel**(H), let $g : X \to Y$ be any mapping for which $g \circ f_\alpha : (X_\alpha, R_\alpha) \to (Y,R_Y)$ is an **IRel**(H)-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (X,R) \to (Y,R_Y)$ is an **IRel**(H)-mapping. Hence $R = (\mu_R, \nu_R)$ is the final structure on X with respect to $(((X_\alpha, R_\alpha)), (f_\alpha), X)$. This completes the proof. □

Example 3.13. (1) *Intuitionistic H-fuzzy quotient relation*. Let $(X,R) \in$ **IRel**(H), let $\sim$ be an equivalence relation on X, and let $\varphi : X \to X/R$ the canonical mapping. Then there exists the final intuitionistic H-fuzzy relation $(\mu_{X/\sim}, \nu_{X/\sim})$ on $X/\sim$ for which $\varphi : (X,R) \to (X/\sim, \mu_{X/\sim}, \nu_{X/\sim})$ is an **IRel**(H)-mapping. In this case, $(\mu_{X/\sim}, \nu_{X/\sim})$ is called the *intuitionistic H-fuzzy quotient relation* of X by R.

(2) *Sum of intuitionistic H-fuzzy relations*. Let $((X_\alpha, R_\alpha))_\Gamma$ be a family of IHFRSs, let X be the sum of $(X_\alpha)_\Gamma$ and let $j_\alpha : X_\alpha \to X$ be the canonical (injection) mapping for

each $\alpha \in \Gamma$. Then there exists the final IHFR R on X. In fact, for each $((x_\alpha,\alpha),(y_\beta,\beta)) \in X \times X$, $\mu_R((x_\alpha,\alpha),(y_\beta,\beta)) = \bigvee_\Gamma \mu_{R_\alpha}(x,y)$ and $\nu_R((x_\alpha,\alpha),(y_\beta,\beta)) = \bigwedge_\Gamma \nu_{R_\alpha}(x,y)$. In this case, R is called the *sum* of $(R_\alpha)_\Gamma$ and (X,R) is called the *sum* of $((X_\alpha,R_\alpha))_\Gamma$.

Theorem 3.14. *Final episinks in* **IRel**(H) *are preserved by pullbacks.*

Proof. Let $(g_\alpha : (X_\alpha,R_\alpha) \to (Y,R_Y))_\Gamma$ be any final episink in **IRel**(H) and let $f : (W,R_W) \to (Y,R_Y)$ be any **IRel**(H)-mapping. For each $\alpha \in \Gamma$, let $U_\alpha = \{(w,x_\alpha) \in W \times X_\alpha : f(w) = g_\alpha(x_\alpha)\}$ and let us define two mappings $\mu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \to H$ and $\nu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \to H$ by for each $((w,x_\alpha),(w',x'_\alpha)) \in U_\alpha \times U_\alpha$,

$$\begin{aligned} \mu_{R_{U_\alpha}}((w,x_\alpha),(w',x'_\alpha)) &= \mu_{R_W}(w,w') \wedge \mu_{R_\alpha}(x_\alpha,x'_\alpha), \\ \nu_{R_{U_\alpha}}((w,x_\alpha),(w',x'_\alpha)) &= \nu_{R_W}(w,w') \vee \nu_{R_\alpha}(x_\alpha,x'_\alpha). \end{aligned} \tag{3.2}$$

Let $e_\alpha : U_\alpha \to W$ and $p_\alpha : U_\alpha \to X_\alpha$ denote the usual projections of U_α. Then clearly $(U_\alpha, R_{U_\alpha}) \in$ **IRel**(H) for each $\alpha \in \Gamma$. Moreover, $e_\alpha : (U_\alpha,R_{U_\alpha}) \to (W,R_W)$ and $p_\alpha : (U_\alpha,R_{U_\alpha}) \to (X_\alpha,R_\alpha)$ are **IRel**(H)-mappings for each $\alpha \in \Gamma$. And the following diagram is a pullback square in **IRel**(H):

$$\begin{array}{ccc} (U_\alpha,R_{\mu_\alpha}) & \xrightarrow{p_\alpha} & (X_\alpha,R_\alpha) \\ {\scriptstyle e_\alpha}\downarrow & & \downarrow{\scriptstyle g_\alpha} \\ (W,R_W) & \xrightarrow[f]{} & (Y,R_Y) \end{array} \tag{3.3}$$

We will show that $(e_\alpha : (U_\alpha,R_{U_\alpha}) \to (W,R_w))_\Gamma$ is a final episink in **IRel**(H). By the process of the proof of [14, Theorem 2.5], $(e_\alpha)_\Gamma$ is an episink in **IRel**(H). Suppose $R = (\mu_R,\nu_R)$ is another final IHFR on W with respect to $(e_\alpha)_\Gamma$. By the process of the proof of [14, Theorem 2.5], $\mu_R = \mu_{R_W}$. Thus it is sufficient to show that $\nu_R = \nu_{R_W}$. Let $(w,w') \in W \times W$. Then

$$\begin{aligned} \nu_{R_W}(w,w') &= \nu_{R_W}(w,w') \vee \nu_{R_W}(w,w') \\ &\geq \nu_{R_W}(w,w') \vee [\nu_{R_Y} \circ f^2(w,w')] \\ &\qquad (\text{since } f : (W,R_w) \longrightarrow (Y,R_Y) \text{is an } \mathbf{IRel}(H)\text{-mapping}) \\ &= \nu_{R_W}(w,w') \vee \nu_{R_Y}(f(w),f(w')) \\ &= \nu_{R_W}(w,w') \vee \Bigg[\bigwedge_\Gamma \bigwedge_{(x_\alpha,x'_\alpha)\in g_\alpha^{-1^2}(f(w),f(w'))} \nu_{R_\alpha}(x_\alpha,x'_\alpha)\Bigg] \\ &\qquad (\text{since } (g_\alpha)_\Gamma \text{ is final}) \\ &= \bigwedge_\Gamma \bigwedge_{(x_\alpha,x'_\alpha)\in g_\alpha^{-1^2}(f(w),f(w'))} [\nu_{R_W}(w,w') \vee \nu_{R_\alpha}(x_\alpha,x'_\alpha)] \\ &= \bigwedge_\Gamma \bigwedge_{((w,x_\alpha),(w',x'_\alpha))\in e_\alpha^{-1}(w,w')} \nu_{R_{U_\alpha}}((w,x_\alpha),(w',x'_\alpha)). \end{aligned} \tag{3.4}$$

Thus $\nu_{R_W}(w,w') \geq \nu_R(w,w')$ for each $(w,w') \in W \times W$. So $\nu_{R_W} \geq \nu_R$. On the other hand, since $(e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W,R))_\Gamma$ is final, $1_W : (W,R) \to (W,R_W)$ is an **IRel**(H)-mapping. Thus $\nu_R \geq \nu_{R_W}$. So $\nu_R = \nu_{R_W}$. Hence $R = R_W$. This completes the proof. □

For any singleton set $\{a\}$, since the IHFR R on $\{a\}$ is not unique, the category **IRel**(H) is not properly fibered over **Set**. Hence, by Theorems 3.12 and 3.14, we obtain the following result.

THEOREM 3.15. **IRel**(H) *satisfies all the conditions of a topological universe over* ***Set*** *except the terminal separator property.*

THEOREM 3.16. **IRel**(H) *is cartesian closed over* ***Set***.

Proof. It is clear that **IRel**(H) has products by Corollary 3.11. We will show that **IRel**(H) has exponential objects.

For any IHFRSs $\mathbf{X} = (X, R_X)$ and $\mathbf{Y} = (Y, R_Y)$, let Y^X be the set of all mappings from X into Y. We define two mappings $\mu_R : Y^X \times Y^X \to H$ and $\nu_R : Y^X \times Y^X \to H$ as follows: for each $(f,g) \in Y^X \times Y^X$,

$$\begin{aligned} \mu_R(f,g) &= \bigwedge \{h \in H : \mu_{R_X}(x,y) \wedge h \leq \mu_{R_Y}(f(x),g(y)) \text{ for each } (x,y) \in X \times X\}, \\ \nu_R(f,g) &= \bigvee \{h \in H : \nu_{R_X}(x,y) \vee h \geq \nu_{R_Y}(f(x),g(y)) \text{ for each } (x,y) \in X \times X\}. \end{aligned} \tag{3.5}$$

Then clearly $(Y^X, R) \in$ **IRel**(H). Let $\mathbf{Y}^{\mathbf{X}} = (Y^X, R)$. Then, by the definition of R,

$$\begin{aligned} \mu_{R_X}(x,y) \wedge \mu_R(f,g) &\leq \mu_{R_Y}(f(x),g(y)), \\ \nu_{R_X}(x,y) \vee \nu_R(f,g) &\geq \nu_{R_Y}(f(x),g(y)) \end{aligned} \tag{3.6}$$

for each $(f,g) \in Y^X$ and $(x,y) \in X \times X$.

Define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x,f) = f(x)$ for each $(x,f) \in X \times Y^X$. Let $((x,f),(y,g)) \in (X \times Y^X) \times (X \times Y^X)$. Then, by the process of the proof of [14, Theorem 2.7], $\mu_{R_X \times R}((x,f),(y,g)) \leq \mu_{R_Y} \circ e^2_{X,Y}((x,f),(y,g))$. So $\mu_{R_X \times R} \leq \mu_{R_Y} \circ e^2_{X,Y}$. On the other hand,

$$\begin{aligned} \nu_{R_X \times R}((x,f),(y,g)) &= \nu_{R_X}(x,y) \vee \nu_R(f,g) \\ &\geq \nu_{R_Y}(f(x),g(y)) \\ &= \nu_{R_Y}(e_{X,Y}(x,f), e_{X,Y}(y,g)) \\ &= \nu_{R_Y} \circ e^2_{X,Y}((x,f),(y,g)). \end{aligned} \tag{3.7}$$

Thus $\nu_{R_X \times R} \geq \nu_{R_Y} \circ e^2_{X,Y}$. Hence $e_{X,Y} : \mathbf{X} \times \mathbf{Y}^{\mathbf{X}} \to \mathbf{Y}$ is an **IRel**(H)-mapping.

For any $\mathbf{Z} = (Z, R_Z) \in$ **IRel**(H), let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an **IRel**(H)-mapping. We define $\overline{h} : Z \to Y^X$ by $[\overline{h}(z)](x) = h(x,z)$ for each $z \in Z$ and each $x \in X$. Let $z, z' \in Z$ and let $x,x' \in X$. Then, by the process of the proof of [14, Theorem 2.7], $\mu_{R_Z}(z,z') \leq \mu_R \circ \overline{h}^2(z,z')$. So $\mu_{R_Z} \leq \mu_R \circ \overline{h}^2$. On the other hand,

$$\begin{aligned}\nu_{R_X \times R_Z}((x,z),(x',z')) &= \nu_{R_X}(x,x') \vee \nu_{R_Z}(z,z') \\ &\geq \nu_{R_Y} \circ h^2((x,z),(x',z')) \\ &\qquad (\text{since } h: \mathbf{X} \times \mathbf{Z} \longrightarrow \mathbf{Y} \text{ is an } \mathbf{IRel}(\mathrm{H})\text{-mapping}) \\ &= \nu_{R_Y}(h(x,z),h(x',z')) \\ &= \nu_{R_Y}([\overline{h}(z)](x),[\overline{h}(z')](x')).\end{aligned} \tag{3.8}$$

Thus, by the definition of R, $\nu_{R_Z}(z,z') \geq \nu_R(\overline{h}(z),\overline{h}(z')) = \nu_{R_Y} \circ \overline{h}^2(z,z')$. So $\nu_{R_Z} \geq \nu_R \circ \overline{h}^2$. Hence $\overline{h}: \mathbf{Z} \to \mathbf{Y}^{\mathbf{X}}$ is an $\mathbf{IRel}(H)$-mapping. Moreover, $\overline{h}$ is the unique $\mathbf{IRel}(H)$-mapping such that $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. □

Remark 3.17. $\mathbf{IRel}(H)$ has no subobject classifier. Hence $\mathbf{IRel}(H)$ is not topos.

Example 3.18. Let $H = \{0,1\}$ be the two points chain and let $X = \{a\}$. Let R_1 and R_2 be the IHFRs on X given by $\mu_{R_1}(a,a) = 0$, $\nu_{R_1}(a,a) = 1$ and $\mu_{R_2}(a,a) = 1$, $\nu_{R_2}(a,a) = 0$. Let $1_X : (X,R_1) \to (X,R_2)$ be the identity mapping. Then clearly, 1_X is both a monomorphism and an epimorphism in $\mathbf{IRel}(H)$. But, 1_X is not an isomorphism in $\mathbf{IRel}(H)$. Hence $\mathbf{IRel}(H)$ has no subobject classifier (see [7]).

4. The relations between IRel(H) and Rel (H)

LEMMA 4.1. *Define* $G_1, G_2 : \mathbf{IRel}(H) \to \mathbf{Rel}(H)$ *by*

$$\begin{aligned} G_1(X,\mu_R,\nu_R) &= (X,\mu_R), \\ G_2(X,\mu_R,\nu_R) &= (X,N(\nu_R)), \\ G_1(f) = G_2(f) &= f. \end{aligned} \tag{4.1}$$

Then G_1 *and* G_2 *are functors.*

Proof. Clearly $G_1(X,\mu_{R_X},\nu_{R_X}) = (X,\mu_{R_X}) \in \mathbf{Rel}(H)$ for each $(X,\mu_R,\nu_R) \in \mathbf{IRel}(H)$. Let $(X,\mu_{R_X},\nu_{R_X}),(Y,\mu_{R_Y},\nu_{R_Y}) \in \mathbf{IRel}(H)$ and let $f : (X,\mu_{R_X},\nu_{R_X}) \to (Y,\mu_{R_Y},\nu_{R_Y})$ be an $\mathbf{IRel}(H)$-mapping. Then $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$. Thus $G_1(f) = f : (X,\mu_{R_X}) \to (Y,\mu_{R_Y})$ is a $\mathbf{Rel}(H)$-mapping. Hence $G_1 : \mathbf{IRel}(H) \to \mathbf{Rel}(H)$ is a functor. Also $G_2(X,\mu_{R_X},\nu_{R_X}) = (X,N(\nu_{R_X})) \in \mathbf{Rel}(H)$ for each $(X,\mu_{R_X},\nu_{R_X}) \in \mathbf{IRel}(H)$. Now let $(X,\mu_{R_X},\nu_{R_X}),(Y,\mu_{R_Y},\nu_{R_Y}) \in \mathbf{IRel}(H)$ and let $f : (X,\mu_{R_X},\nu_{R_X}) \to (Y,\mu_{R_Y},\nu_{R_Y})$ be an $\mathbf{IRel}(H)$-mapping. Then $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$. Thus $N(\nu_{R_X}) \leq N(\nu_{R_Y}) \circ f^2$. So $G_2(f) = f : (X,N(\nu_{R_X})) \to (Y,N(\nu_{R_Y}))$ is a $\mathbf{Rel}(H)$-mapping. Hence $G_2 : \mathbf{IRel}(H) \to \mathbf{Rel}(H)$ is a functor. □

LEMMA 4.2. *Define* $F_1 : \boldsymbol{Rel}(H) \to \boldsymbol{IRel}(H)$ *by* $F_1(X,\mu_R) = (X,\mu_R,N(\mu_R))$ *and* $F_1(f) = f$. *Then* F_1 *is a functor.*

Proof. For each $(X,\mu_{R_X}) \in \mathbf{Rel}(H)$, $\mu \leq NN(\mu_{R_X})$. Thus $F_1(X,\mu_{R_X}) = (X,\mu_{R_X},N(\mu_{R_X})) \in \mathbf{IRel}(H)$. Let $(X,\mu_{R_X}),(Y,\mu_{R_Y}) \in \mathbf{Rel}(H)$ and let $f:(X,\mu_{R_X}) \to (Y,\mu_{R_Y})$ be an $\mathbf{Rel}(H)$-mapping. Then $\mu_{R_X} \leq \mu_{R_Y} \circ f$. Consider the mapping $F_1(f) = f:(X,\mu_{R_X},N(\mu_{R_X})) \to (Y,\mu_{R_Y},N(\mu_{R_Y}))$. Since $\mu_{R_X} \leq \mu_{R_Y} \circ f$, $N(\mu_{R_X}) \geq N(\mu_{R_Y}) \circ f$. So $f:(X,\mu_{R_X},N(\mu_{R_X})) \to (Y,\mu_{R_Y},N(\mu_{R_Y}))$ is an $\mathbf{IRel}(H)$-mapping. Hence F_1 is a functor. □

Lemma 4.3. *Define $F_2:$ **Rel**$(H) \to$ **IRel**(H) by $F_2(X,\mu_R) = (X,NN(\mu_R),N(\mu_R))$ and $F_2(f) = f$. Then F_2 is a functor.*

Proof. It is clear that $F_2(X,\mu_{R_X}) \in \mathbf{IRel}(H)$ for each $(X,\mu_{R_X}) \in \mathbf{Rel}(H)$. Let (X,μ_{R_X}), $(Y,\mu_{R_Y}) \in \mathbf{Rel}(H)$ and let $f:(X,\mu_{R_X}) \to (Y,\mu_{R_Y})$ be an $\mathbf{Rel}(H)$-mapping. Consider the mapping $F_2(f) = f : F_2(X,\mu_{R_X}) \to (Y,NN(\mu_{R_Y}),N(\mu_{R_Y}))$, where $F_2(X,\mu_{R_X}) = (X, NN(\mu_{R_X}),N(\mu_{R_X}))$ and $F_2(Y,\mu_{R_Y}) = (Y,NN(\mu_{R_Y}),N(\mu_{R_Y}))$. Since $f:(X,\mu_{R_X}) \to (Y,\mu_{R_Y})$ is a $\mathbf{Rel}(H)$-mapping, $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$. Thus $NN(\mu_{R_X}) \leq NN(\mu_{R_Y}) \circ f^2$. Moreover $N(\mu_{R_X}) \geq N(\mu_{R_Y}) \circ f^2$. So $F_2(f) = f : F_2(X,\mu_{R_X}) \to F_2(Y,\mu_{R_Y})$ is an $\mathbf{IRel}(H)$-mapping. Hence F_2 is a functor. □

Theorem 4.4. *The functor $F_1:$ **Rel**$(H) \to$ **IRel**(H) is a left adjoint of the functor $G_1:$ **IRel**$(H) \to$ **Rel**(H).*

Proof. For each $(X,\mu_R) \in \mathbf{Rel}(H)$, $1_X : (X,\mu_R) \to G_1F_1(X,\mu_R) = (X,\mu_R)$ is a $\mathbf{Rel}(H)$-mapping. Let $(Y,\mu_{R_Y},\nu_{R_Y}) \in \mathbf{IRel}(H)$ and let $f:(X,\mu_R) \to G_1(Y,\mu_{R_Y},\nu_{R_Y})$ be an $\mathbf{IRel}(H)$-mapping. We will show that $f:F_1(X,\mu_R) = (X,\mu_R,N(\mu_R)) \to (Y,\mu_{R_Y},\nu_{R_Y})$ is an $\mathbf{IRel}(H)$-mapping. Since $f:(X,\mu_R) = G_1(Y,\mu_{R_Y},\mu_{R_Y}) \to (Y,\mu_{R_Y})$ is a $\mathbf{Rel}(H)$-mapping, $\mu_R \leq \mu_{R_Y} \circ f^2$. Then $N(\mu_R) \geq N(\mu_{R_Y}) \circ f^2$. Since $\mu_{R_Y} \leq N(\nu_{R_Y})$, $\nu_{R_Y} \leq NN(\nu_{R_Y}) \leq N(\mu_{R_Y})$. Thus $N(\mu_R) \geq \nu_{R_Y} \circ f^2$. So $f:F_1(X,\mu_R) = (Y,\mu_{R_Y},\nu_{R_Y})$ is an $\mathbf{IRel}(H)$-mapping. Hence 1_X is a G_1-universal map for (X,μ_R) in $\mathbf{Rel}(H)$. This completes the proof. □

For each $(X,\mu_R) \in \mathbf{Rel}(H)$, $F_1(X,\mu_R) = (X,\mu_R,N(\mu_R))$ is called an *intuitionistic H-fuzzy set in X induced* by (X,μ_R). Let us denote the category of all induced intuitionistic H-fuzzy sets and $\mathbf{IRel}(H)$-mappings as $\mathbf{IRel}^*(H)$. Then it is clear $\mathbf{IRel}^*(H)$ is a full subcategory of $\mathbf{IRel}(H)$.

Theorem 4.5. *Two categories* $\mathbf{Rel}(H)$ *and* **IRel**$^*(H)$ *are isomorphic.*

Proof. It is clear that $F_1:\mathbf{Rel}(H) \to \mathbf{IRel}^*(H)$ is a functor by Lemma 4.2. Consider the restriction $G_1:\mathbf{IRel}^*(H) \to \mathbf{Rel}(H)$ of the functor G_1 in Lemma 4.1. Let $(X,\mu_R) \in \mathbf{Rel}(H)$. Then, by Lemma 4.2, $F_1(X,\mu_R) = (X,\mu_R,N(\mu_R))$. Thus $G_1F_1(X,\mu_R) = G_1(X,\mu_R,N(\mu_R)) = (X,\mu_R)$. So $G_1 \circ F = 1_{\mathbf{Rel}(H)}$. Now let $(X,\mu_R,N(\mu_R)) \in \mathbf{ISet}^*(H)$. Then, by Lemma 4.1, $G_1(X,\mu_R,N(\mu_R)) = (X,\mu_R)$. Thus $FG_1(X,\mu_R,N(\mu_R)) = (X,\mu_R,N(\mu_R))$. So $F \circ G_1 = 1_{\mathbf{ISet}^*(H)}$. Hence $F:\mathbf{Rel}(H) \to \mathbf{ISet}^*(H)$ is an isomorphism. This completes the proof. □

Remark 4.6. We are going to investigate "intuitionistic H-fuzzy reflexive relations," "some subcategories of the category IRelk (H)," and "intuitionistic H-fuzzy relations on intuitionistic H-fuzzy sets" in the viewpoint of topological universe.

Acknowledgment

The authors are highly grateful to the referee for his valuable comments and suggestions for improving the paper.

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