ON WEAK CONVERGENCE OF ITERATES IN QUANTUM L_p-SPACES ($p \geq 1$)

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Equivalent conditions are obtained for weak convergence of iterates of positive contractions in the L_1-spaces for general von Neumann algebra and general JBW algebras, as well as for Segal-Dixmier L_p-spaces ($1 \leq p < \infty$) affiliated to semifinite von Neumann algebras and semifinite JBW algebras without direct summands of type I_2.

1. Introduction and preliminaries

This paper is devoted to a presentation of some results concerning ergodic-type properties of weak convergence of iterates of operators acting in L_1-space for general von Neumann algebras and JBW algebras, as well as Segal-Dixmier L_p-spaces ($1 \leq p < \infty$) of operators affiliated with semifinite von Neumann algebras and semifinite JBW algebras.

The first results in the field of noncommutative ergodic theory were obtained independently by Sinaĭ and Anšelevič [21] and Lance [15]. Developments of the subject are reflected in the monographs of Jajte [13] and Krengel [14] (see also [8, 9, 10, 18]).

We will use facts and the terminology from the general theory of von Neumann algebras (see [5, 7, 17, 19, 22]), the general theory of Jordan and real operator algebras (see [2, 3, 11, 16]), and the theory of noncommutative integration (see [20, 23, 24]).

Let M be a von Neumann algebra, acting on a separable Hilbert space H, M_* is a predual space of M, which always exists according to the Sakai theorem [19]. It is well known that M_* could be identified with L_1-space for M.

Spaces L_1 and L_2 of the operators affiliated with the semifinite von Neumann algebra M with semifinite faithful trace τ were introduced by Segal (see [20]). This result was extended to L_p-space of operators affiliated with von Neumann algebras M, τ, and integrated with pth power by Dixmier (see [6]). For an alternative exposition of building L_p based on Grothendieck's idea of using rearrangements of functions, see also [24]. The theory of L_p-spaces was extended further to the von Neumann algebras with faithful normal weight ρ. However, these spaces lack some of the properties, for example, in general, these spaces do not intersect.

Recall some standard terminology (see [8, 9, 10, 14]).

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Definition 1.1. A linear mapping T from M_* in itself is called a *contraction* if its norm is not greater than one.

Definition 1.2. A contraction T is said to be *positive* if

$$TM_{*+} \subset M_{*+}. \tag{1.1}$$

We will consider the two topologies on the space M_*: the *weak topology*, or the $\sigma(M_*, M)$ topology, and the *strong topology* of the M_*-space norm convergence.

Definition 1.3. A matrix $(a_{n,i})$, $i,n = 1,2,\dots$, of real numbers is called *uniformly regular* if

$$\sup_n \sum_{i=1}^{\infty} |a_{n,i}| \le C < \infty; \qquad \lim_{n\to\infty} \sup_i |a_{n,i}| = 0; \qquad \lim_{n\to\infty} \sum_i a_{n,i} = 1. \tag{1.2}$$

2. Main result: the case of quantum L_1-spaces

2.1. The case of noncommutative L_1-spaces. The following theorem is valid.

Theorem 2.1. *The following conditions for a positive contraction T in the predual space of a complex von Neumann algebras M are equivalent.*

(i) *The sequence $\{T^i\}_{i=1,2,\dots}$ converges weakly.*

(ii) *For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,\dots}$,*

$$n^{-1}\sum_{i<n} T^{k_i} \tag{2.1}$$

converges strongly.

(iii) *For any uniformly regular matrix $(a_{n,i})$, the sequence $\{A_n(T)\}_{n=1,2,\dots}$,*

$$A_n(T) = \sum_i a_{n,i}T^i, \tag{2.2}$$

converges strongly.

Proof of Theorem 2.1. We first prove the following lemma.

Lemma 2.2. *Let there exist a uniformly regular matrix $(a_{n,i})$ such that for each strictly increasing sequence $\{k_i\}_{i=1,2,\dots}$ of natural numbers,*

$$B_n = \sum_i a_{n,i}T^{k_i} \tag{2.3}$$

converges strongly. Then the sequence $\{T^i\}_{i=1,2,\dots}$ converges weakly.

Proof. Let $(a_{n,i})$ be a matrix with the aforementioned properties. Then the limit B_n is not dependant upon the choice of the sequence $\{k_i\}_{i=1,2,\dots}$. In fact, let $\{k_i\}_{i=1,2,\dots}$ and $\{l_i\}_{i=1,2,\dots}$ be the sequences for which the limits B_n are different. This means that for some $x \in M_*$,

$$\sum_i a_{n,i}T^{k_i}x \longrightarrow x_1, \qquad \sum_i a_{n,i}T^{l_i}x \longrightarrow x_2, \tag{2.4}$$

for $n \to \infty$. For a matrix $(a_{n,i})$, we build increasing sequences $\{i_j\}_{j=1,2,\dots}$ and $\{n_j\}_{j=1,2,\dots}$, such that

$$\lim_{j\to\infty}\left(\sum_{i<i_{j-1}}|a_{n_j,i}|+\sum_{i>i_j}|a_{n_j,i}|\right)=0. \tag{2.5}$$

Let

$$m_i=k_i \quad \text{for } i\in[i_{2j-1},i_{2j}), \qquad m_i=l_i \quad \text{for } i\in[i_{2j},i_{2j+1}),\ j=1,2,\dots. \tag{2.6}$$

Then

$$\lim_j\left\|\sum_i a_{n_{2j+1},i}T^{m_i}x-x_1\right\|=0; \qquad \lim_j\left\|\sum_i a_{n_{2j},i}T^{m_i}x-x_2\right\|=0, \tag{2.7}$$

which contradict (2.3), and therefore $x_1=x_2$. Let now $y\in M$ such that

$$(T^nx-x_1,y)\longrightarrow 0, \tag{2.8}$$

when $n\to\infty$. We choose a subsequence $\{k_i\}$ such that

$$(T^{k_i}x-x_1,y)\longrightarrow \gamma\neq 0, \tag{2.9}$$

where γ is a real number. Then, from the uniform regularity of the matrix $(a_{n,i})$, it follows that

$$\lim_n\left(\sum_i a_{n,i}T^{k_i}x-x_1,y\right)=\gamma, \tag{2.10}$$

which contradicts the choice of the matrix $(a_{n,i})$. □

The implication (iii)⇒(ii) is trivial because the matrix $(a_{n,i})$,

$$a_{n,i}=\frac{1}{n}\sum_{i<n}\delta_{j,k_i}, \tag{2.11}$$

is uniformly regular. Applying Lemma 2.2 to the matrix

$$a_{n,i}=\frac{1}{n}, \tag{2.12}$$

$i\le n$ and $a_{n,i}=0$ for $i>n$, we get the implication (ii)⇒(i).

To prove the implication (i)⇒(iii), we would need the following lemma.

Lemma 2.3. *Let Q be a contraction in the Hilbert space H. Then the weak convergence of Q^nx in H, where $x\in H$, implies the strong convergence of*

$$\sum_i a_{n,i}Q^ix \tag{2.13}$$

for any uniformly regular matrix $(a_{n,i})$.

Proof. If the weak limit $Q^n x$ exists and is equal to x_1, then

$$Qx_1 = Q\Big(\lim_{n\to\infty} Q^n x\Big) = x_1, \tag{2.14}$$

where the limit is considered in the weak topology, that is, x_1 is Q-invariant. Replacing x on $x - x_1$ (if necessary), we may suppose that $Q^n x$ converges weakly to $\mathbf{0}$, and hence

$$(Q^n x, x) \longrightarrow 0. \tag{2.15}$$

We are going to show that

$$\sum_n a_{i,n} Q^n x \xrightarrow{\|\cdot\|} \mathbf{0}, \tag{2.16}$$

where $(a_{i,n})$ is uniformly regular matrix. One can see that

$$\Big\|\sum_i a_{N,i} Q^i x\Big\|^2 \le \sum_i \sum_j a_{N,i} a_{N,j} (Q^i x, Q^j x) \le \sum_i \sum_j |a_{N,i} a_{N,j} (Q^i x, Q^j x)|. \tag{2.17}$$

We fix $\varepsilon > 0$. Because Q is a contraction, the limit $\|Q^n x\|$ does exist. Now, we can find $K > 0$, such that for $k > K$ and $j \ge 0$,

$$||Q^k x|| - ||Q^{k+j} x|| \le \varepsilon^2, \qquad |(Q^k x, x)| \le \varepsilon. \tag{2.18}$$

Then,

$$\begin{aligned}
&|(Q^k x, x) - (Q^{k+j} x, Q^j x)| \\
&\quad = |(Q^k x, x) - (Q^{*j} Q^{k+j} x, x)| \\
&\quad \le ||Q^k x - Q^{*j} Q^{k+j} x|| \cdot \|x\| = \big(||Q^k x - Q^{*j} Q^{k+j}||^2\big)^{1/2} \cdot \|x\| \\
&\quad = \big(||Q^k x||^2 - 2||Q^{k+j} x||^2 + ||Q^{*j} Q^{k+j} x||^2\big)^{1/2} \cdot \|x\| \\
&\quad \le \big(||Q^k x||^2 - ||Q^{k+j} x||^2\big) \cdot \|x\| \le \varepsilon \cdot \|x\|,
\end{aligned} \tag{2.19}$$

and therefore

$$|(Q^{k+j} x, Q^j x)| \le \varepsilon \cdot (1 + \|x\|) \tag{2.20}$$

for all $k > K$ and $j \ge 0$, or for $|i - j| \ge k$, the inequality

$$|(Q^i x, Q^j x)| \le \varepsilon \cdot (1 + \|x\|) \tag{2.21}$$

is valid. We will fix $\eta > 0$, and let N be a natural number such that

$$\max_i |a_{n,i}| < \eta, \tag{2.22}$$

for $n \geq N$. Then the expression (1) for $n \geq N$ could be estimated in the following way:

$$\begin{aligned}
&\sum_i \sum_j |a_{N,i}a_{N,j}(Q^i x, Q^j x)| \\
&\quad = \sum_{|i-j|\leq k} |a_{n,i}a_{n,j}(Q^i x, Q^j x)| + \sum_{|i-j|>k} |a_{n,i}a_{n,j}(Q^i x, Q^j x)| \\
&\quad \leq \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + \sum_i \sum_j |a_{n,i}a_{n,j}| \cdot \varepsilon \cdot (1+\|x\|) \\
&\quad \leq C \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + C^2 \cdot \varepsilon \cdot (1+\|x\|).
\end{aligned} \tag{2.23}$$

From the arbitrarity of the values of ε and η, it follows that the strong convergence is present and the lemma is proven. □

We prove the implication (i)⇒(iii). Let $x \in M_{*+}$ and the sequence $\{T^i x\}_{i=1,2,\dots}$ converges weakly. Without the loss of generality, we can consider $\|x\| \leq 1$, and let

$$\overline{x} = \lim_{n\to\infty} T^n x, \tag{2.24}$$

where the limit is understood in the weak sense. We consider

$$y = \sum_{n=0}^{\infty} 2^{-n} T^n x. \tag{2.25}$$

The series that defines y is convergent in the norm of the space M_*. From the positivity of x and the properties of the operator T, it follows that

$$Ty \leq 2y, \tag{2.26}$$

and, therefore, for all $k = 1,2,\dots$,

$$s(T^k y) \leq s(y), \tag{2.27}$$

where we denote by $s(z)$ the support of the normal functional z.

Lemma 2.4. *Let $u \in M_{*+}$ and $s(u) \leq s(y)$. Then $s(\overline{u}) \leq s(\overline{x})$, where*

$$\overline{u} = \lim_{n\to\infty} T^n u. \tag{2.28}$$

Proof. In fact, We fix $\varepsilon > 0$. From the density of the set

$$\mathfrak{L}_y = \{w \in M_{*+}, w \leq \lambda y, \text{ for some } \lambda > 0\} \tag{2.29}$$

in the set

$$\mathfrak{S} = \{w \in M_{*+}, s(w) \leq s(y)\} \tag{2.30}$$

in the norm of the space M_*, it follows that there are $\lambda > 0$ and $w \in \mathfrak{L}_y$ such that

$$\|w-u\| \leq \varepsilon, \quad w \leq \lambda y. \tag{2.31}$$

Let

$$\overline{w} = \lim_{n\to\infty} T^n w. \tag{2.32}$$

Then

$$\begin{aligned}\overline{w}(\mathbf{1}-s(\overline{x})) &= \lim_{n\to\infty}(T^n(w))(\mathbf{1}-s(\overline{x}))\\ &\le \lambda\cdot\lim_{n\to\infty}(T^n y)(\mathbf{1}-s(\overline{x}))\\ &\le \lambda\cdot\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}2^{-k}\cdot(T^{n+k}x)(\mathbf{1}-s(\overline{x}))\right)\\ &= \lambda\cdot\sum_{k=0}^{\infty}2^{-k}\lim_{n\to\infty}(T^{n+k}x)(\mathbf{1}-s(\overline{x})) = 0.\end{aligned} \tag{2.33}$$

Because the operator T does not increase the norm of the functionals from M_*, we get that

$$\overline{u}(\mathbf{1}-s(\overline{x})) = \lim_{n\to\infty}(T^n u)(\mathbf{1}-s(\overline{x})) \le \lim_{n\to\infty}(T^n w)(\mathbf{1}-s(\overline{x})) + \lim_{n\to\infty}\|T^n(w-u)\| \le \varepsilon. \tag{2.34}$$

The needed inequality follows from the arbitrarity of ε. □

We introduce the following notion. For $\mu \in M_*$, we will denote by $\mu\cdot E$, where E is a projection from the algebra M, the functional

$$(\mu\cdot E)(A) = \mu(EAE), \tag{2.35}$$

where $A \in M$.

We fix $\varepsilon > 0$. We will find a number N, such that

$$(T^n x)(\mathbf{1}-s(\overline{x})) < \varepsilon^2 \tag{2.36}$$

for $n > N$.

Then,

$$\begin{aligned}&\|T^N x\cdot s(\overline{x}) - T^N x\|\\ &\quad= \sup_{\substack{A\in M\\ \|A\|_\infty\le 1}} \big|(T^N x)((\mathbf{1}-s(\overline{x}))A(\mathbf{1}-s(\overline{x})))\\ &\qquad\qquad + (T^N x)((s(\overline{x}))A(\mathbf{1}-s(\overline{x}))) + (T^N x)((\mathbf{1}-s(\overline{x}))A(s(\overline{x})))\big|\\ &\quad\le \varepsilon\cdot(\varepsilon + 2\|x\|^{1/2}),\end{aligned} \tag{2.37}$$

because

$$|\mu(AB)|^2 \le \mu(A^*A)\cdot\mu(B^*B), \tag{2.38}$$

where $\mu \in M_{*+}$ and $A, B \in M$.

Let $w \in \mathfrak{L}_{\overline{y}}$ be such that

$$w \leq \lambda \overline{x} \tag{2.39}$$

for some $\lambda > 0$ and

$$||T^N x \cdot s(\overline{x}) - w|| \leq \varepsilon. \tag{2.40}$$

Then, for $n > N$, the following is valid:

$$||T^n x - T^{n-N} w|| \leq ||T^{n-N}(T^N x - T^N x \cdot s(\overline{x}))|| + ||T^{n-N}(T^N x \cdot s(\overline{x}) - w)|| \leq 4 \cdot \varepsilon. \tag{2.41}$$

By taking the weak limit in the inequality (2.37) and because the unit ball of M_* is closed weakly, we will get

$$||\overline{x} - \overline{w}|| \leq 4 \cdot \varepsilon, \tag{2.42}$$

where

$$\overline{w} = \lim_{n \to \infty} T^n w. \tag{2.43}$$

We now consider the algebra $M_{s(x)}$. The functional $\overline{x}$ is faithful on the algebra $M_{s(x)}$. We will consider the representation $\pi_{\overline{x}}$ of the algebra $M_{s(x)}$ constructed using the functional x [7]. Because the functional $\overline{x}$ is faithful, we can conclude that the representation $\pi_{\overline{x}}$ is faithful on the algebra $M_{s(\overline{x})}$, and therefore $\pi_{\overline{x}}$ is an isomorphism of the algebra $M_{s(\overline{x})}$ and some algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is a von Neumann algebra, and its preconjugate space $\mathfrak{A}_*$ is isomorphic to the space $M_* \cdot s(\overline{x})$ ([19]). We note now that

$$TM_* \cdot s(\overline{x}) \subset M_* \cdot s(\overline{x}). \tag{2.44}$$

In fact,

$$T\mathfrak{L}_y \subset \mathfrak{L}_y, \tag{2.45}$$

and therefore, by taking the norm closure, we get

$$TS \subset S; \tag{2.46}$$

by taking now the linear span, we get

$$TM_* \cdot s(\overline{x}) \subset M_* \cdot s(\overline{x}). \tag{2.47}$$

We denote by $\overline{T}$ the isomorphic image of the operator T, acting on the space $\mathfrak{A}_*$. Let

$$u \in \mathfrak{A}_{*+}, \quad u \leq \lambda \overline{x}, \tag{2.48}$$

for some $\lambda > 0$. Then there exists the operator $B \in \mathfrak{A}'$, where $\mathfrak{A}'$ is a commutant of $\mathfrak{A}$, such that

$$(AB\Omega,\Omega) = u(A) \tag{2.49}$$

for all $A \in \mathfrak{A}$. Note, that from Lemma 2.3,

$$(\overline{T}u)(A) = u((\overline{T})^* A) = (((\overline{T})^* A)B\Omega,\Omega) = (A((\overline{T}^*)' B)\Omega,\Omega). \tag{2.50}$$

Also, from

$$\overline{T}\mathfrak{A}_{*+} \subset \mathfrak{A}_{*+}, \qquad \|\overline{T}u\| \le \|u\|, \qquad \overline{T}\overline{x} = \overline{x}, \tag{2.51}$$

it follows that

$$(\overline{T})^* \mathfrak{A}_+; \qquad (\overline{T}^*)\mathbf{1} \le \mathbf{1}, \qquad ||(\overline{T})^* A||_\infty \le ||A||_\infty, \tag{2.52}$$

for all $A \in \mathfrak{A}$. Based on the lemma, we now conclude that

$$||(\overline{T}^* B)||_\infty \le \|B\|_\infty; \qquad \overline{T}^{*\prime}\mathfrak{A}'_+ \subset \mathfrak{A}'_+; \qquad \overline{T}^{*\prime}\mathbf{1} \le \mathbf{1}, \tag{2.53}$$

for all $B \in \mathfrak{A}'$.

The space $\mathfrak{A}'_{\text{sa}}$ is a pre-Hilbert space of the selfadjoint operators from $\mathfrak{A}'$ with the scalar product

$$(B,C)_{\overline{x}} = (CB\Omega,\Omega), \tag{2.54}$$

and using the Kadison inequality [5], we have

$$((\overline{T}^{*\prime}B)(\overline{T}^{*\prime}B)\Omega,\Omega) \le (\overline{T}^{*\prime}(B^2)\Omega,\Omega) \le (B\Omega,B\Omega), \tag{2.55}$$

that is, the operator $\overline{T}^{*\prime}$ is a contraction in the pre-Hilbert space $(\mathfrak{A}'_{\text{sa}},(\cdot,\cdot)_{\overline{x}})$.

We will identify $M_* \cdot s(\overline{x})$ and $\mathfrak{A}_*$. Because $w \in \mathfrak{L}$, that is,

$$w \le \lambda \overline{x} \tag{2.56}$$

for some $\lambda > 0$, then

$$\overline{w} \le \lambda \overline{x} \tag{2.57}$$

as well. Let

$$w(A) = (BA\Omega,\Omega), \qquad \overline{w}(A) = (\overline{B}A\Omega,\Omega), \tag{2.58}$$

for all $A \in \mathfrak{A}$, where $B, \overline{B} \in \mathfrak{A}'$.

Let now $(a_{n,i})$ be a uniformly regular matrix. Using Lemma 2.3, we will find $k \in \mathbb{N}$ so that

$$\begin{aligned}
&\left\|\sum_i a'_{k,i}T^i w - \overline{w}\right\| \\
&\quad = \sup_{\substack{A\in\mathfrak{A}\\ \|A\|_\infty=1}} \left|\left(\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})A\Omega,\Omega\right)\right| \\
&\quad \le \left(\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})\Omega,\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})\Omega\right)^{1/2}\cdot \sup_{\substack{A\in\mathfrak{A}\\ \|A\|_\infty\le 1}}(A\Omega,A\Omega)^{1/2} \\
&\quad \le (\overline{x}(\mathbf{1}))^{1/2}\cdot\left\|\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})\right\|_{(\cdot,\cdot)_{\overline{x}}} < \varepsilon
\end{aligned} \tag{2.59}$$

for $k > K$, where by $(a'_{n,i})$, we will denote a matrix with the elements

$$a'_{n,i} = \left(\sum_{i>N} a_{n,j}\right)^{-1} a_{n,j+N}. \tag{2.60}$$

It is easy to see that the matrix $(a'_{n,i})$ will be uniformly regular as well.

Then, for a big enough $k > K$, we will have

$$\begin{aligned}
\left\|\sum_i a_{k,i}T^i x - \overline{x}\right\| &\le \sum_{i\le N} |a_{k,i}|\,\|T^i x - \overline{x}\| + \sum_{i>N} |a_{k,i}|\,\|T^i x - T^{i-N}w\| \\
&\quad + \sum_{i>N} |a_{k,i}| \left|1 - \left(\sum_{i>N} a_{k,i}\right)^{-1}\right| \|T^{i-N}w\| \\
&\quad + \left\|\sum_{j=1}^{\infty} a_{k,j+N}\cdot\left(\sum_{i>N} a_{k,i}\right)^{-1} T^j w - \overline{w}\right\| \\
&\quad + \left\|\left(\sum_{i\le N} a_{k,i}\right)\cdot\overline{w}\right\| + \left|\sum_{i>N} a_{k,i}\right| \|\overline{w} - \overline{x}\| \\
&\le \sum_{i\le N} 2\cdot\frac{\varepsilon}{N} + \sum_{i>N} |a_{k,i}|\cdot 4\varepsilon + \sum_{i>N} |a_{k,i}| \left(1-(1+\varepsilon)^{-1}\right)\cdot 2 \\
&\quad + \sum_{i\le N} 2\cdot\frac{\varepsilon}{N} + (1+\varepsilon)\cdot 4\varepsilon \\
&\le 2\varepsilon + (1+\varepsilon)\cdot 4\varepsilon + \varepsilon\cdot 2\cdot(1+\varepsilon) + \varepsilon + 2\varepsilon + (1+\varepsilon)\cdot 4\varepsilon \le 25\varepsilon.
\end{aligned} \tag{2.61}$$

The arbitrarity of ε proves the needed statement. The proof of the theorem is now completed. □

2.2. The case of L_1-spaces for JBW **algebras.** The L_1-spaces for semifinite JBW algebras were considered by [4] (see also [1, 12]), where it has been proven that they do coincide with predual spaces. A semifinite JBW algebra A is always represented as

$$A = A_{\text{sp}} \dotplus A_{\text{ex}}, \tag{2.62}$$

where A_{sp} is isometrically isomorphic to operator JW algebra, and A_{ex} is isometrically isomorphic to the space $C(X, M_3^8)$ of all continuous mappings from a Hyperstonean compact topological space X onto the exceptional Jordan algebra M_3^8 (see [11]). In this case, when A does not have direct summands of type I_2, it is going to be a selfadjoint part of a real von Neumann algebra $R(A_{\text{sp}})$, whose complexification

$$R(A_{\text{sp}}) \dotplus iR(A_{\text{sp}}) = M, \tag{2.63}$$

where M is the enveloping von Neumann algebra of A_{sp}, and the predual space of A, and the space

$$A_* = (A_{\text{sp}})_* \dotplus (A_{\text{ex}})_*, \tag{2.64}$$

where $(A_{\text{sp}})_*$ is the predual space of A_{sp}, and $(A_{\text{ex}})_*$ is the predual space of A_{ex} (see, e.g., [2, 11]). The main result for the summand A_{ex} follows immediately from the result for $C(X)$, and the fact that the algebra M_3^8 is finite dimentional. So, without the loss of generality, we are interested in the operator case only. But in the operator case, the space $(A_{\text{sp}})_*$ is a selfadjoint part of $R_* = (R(A_{\text{sp}}))_*$, and

$$M_* = R_* \dotplus iR_* \tag{2.65}$$

(see [2, 16] for details). So, the main result for R_* thus follows from the complex case by restriction of scalars, and we obtain the main result for L_1-spaces affiliated to semifinite JBW algebras without direct type I_2 summand.

3. Main result: the case of quantum L_p-spaces ($1 < p < \infty$)

In the case of a noncommutative L_p-space for a semifinite von Neumann algebra, the main result is discussed in [25].

We will discuss here the nonassociative case.

In this section, A denotes a semifinite JBW algebra without direct summands of type I_2, with a faithful normal trace τ. By L_p, we denote the space of operators affiliated to A, and integrated with pth power ($p > 1$, see, e.g., [1, 2, 12]). Space L_q (here $q = p/(p-1)$) is a dual as Banach space to L_p (see [1, 12]). The following theorem is valid.

Theorem 3.1. *The following conditions for a positive contraction T in the L_p are equivalent.*

(i) *The sequence $\{T^i x\}_{i=1,2,\dots}$ converges in $\sigma(L_p, L_q)$ topology for $x \in L_p$.*

(ii) *For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,\dots}$,*

$$n^{-1} \sum_{i<n} T^{k_i} x \tag{3.1}$$

converges in norm of L_p for all $x \in L_p$.

(iii) *For any uniformly regular matrix* $(a_{n,i})$, *the sequence* $\{A_n(T)x\}_{n=1,2,\dots}$,

$$A_n(T)x = \sum_i a_{n,i}T^i x, \tag{3.2}$$

converges in norm of L_p *for all* $x \in L_p$.

For the sake of completeness, we give the following definitions (see, e.g., [25]) and sketch of the proof. Let ϕ be a gauge function

$$\phi : \mathbb{R}^+ \longmapsto \mathbb{R}^+, \tag{3.3}$$

with

$$\phi(0) = 0, \qquad \lim_{t\to\infty}\phi(t) = \infty. \tag{3.4}$$

Hahn-Banach theorem implies for strictly convex Banach spaces E with conjugate E' that there exists a duality map

$$\Phi : E \longmapsto E', \tag{3.5}$$

associated with ϕ such that

$$\langle x, \Phi(x)\rangle = \|x\|\,\|\Phi(x)\|, \quad ||\Phi(x)|| = \phi(x). \tag{3.6}$$

Definition 3.2. Map Φ is said to satisfy property (S) uniformly if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that for any $x, y \in E$,

$$|\langle x, \Phi(y)\rangle| < \delta(\epsilon) \tag{3.7}$$

implies that

$$|\langle y, \Phi(x)\rangle| < \epsilon. \tag{3.8}$$

Proof. From [12, Section 4], it follows that the duality map defined as

$$\Phi(a) = s|a|^{p-1}, \tag{3.9}$$

for

$$a = s|a| \in A \tag{3.10}$$

(where $a = s|a|$ is a polar decomposition of element a) satisfies the property (S) uniformly. Hence, the statement of the theorem follows from [25, Theorem 3.1]. □

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