EXISTENCE AND ALGORITHM OF SOLUTIONS FOR GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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We introduce and study a new class of generalized nonlinear variational-like inequalities. Under suitable conditions, we prove the existence of solutions for the class of generalized nonlinear variational-like inequalities. A new iterative algorithm for finding the approximate solutions of the generalized nonlinear variational-like inequality is given and the convergence of the algorithm is also proved. The results presented in this paper improve and generalize some results in recent literature.

1. Introduction

Variational-like inequalities are a useful and important generalization of variational inequalities [3, 8, 26]. They have potential and significant applications in optimization theory, structural analysis, and economics, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Some mixed variational-like inequalities have been studied by Parida and Sen [26], Tian [27], and Yao [29] by using the Berge maximum theorem in finite- and infinite-dimensional spaces. Huang and Deng [10] extended the auxiliary principle technique to study the existence of solutions for a class of generalized strongly nonlinear mixed variational-like inequalities. By using the minimax inequality technique, Ding [5, 6] studied some classes of nonlinear variational-like inequalities in reflexive Banach spaces.

The purpose of this paper is to introduce and study a new class of generalized nonlinear variational-like inequalities, which includes several kinds of variational-like inequalities as special cases. A few existence results of solutions for the generalized nonlinear variational-like inequality are established. We construct an iterative algorithm for finding the approximate solutions of the generalized nonlinear variational-like inequality and obtain the convergence of the algorithm under certain conditions.

2. Preliminaries

Let H be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of H, let $A, C, F : K \to H$, $N : H \times H \to H$, and $\eta : K \times K \to H$ be mappings, and let $f : K \to (-\infty, \infty]$ be a real functional.

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Suppose that $a : H \times H \to (-\infty, \infty)$ is a coercive continuous bilinear form, that is, there exist positive constants c and d such that

(C1) $a(v,v) \geq c\|v\|^2$, for all $v \in H$;

(C2) $a(u,v) \leq d\|u\|\|v\|$, for all $u,v \in H$.

Clearly, $c \leq d$.

We consider the following generalized nonlinear variational-like inequality problem.

Find $u \in K$ such that

$$a(u,v-u) + f(v) - f(u) \geq \langle N(Au,Cu) + Fu, \eta(v,u)\rangle, \quad \forall v \in K. \tag{2.1}$$

Special cases. (A) If $N(Au,Cu) = Au - Cu$, $a(u,v) = 0$ and $Fu = 0$ for all $u,v \in K$, then the generalized nonlinear variational-like inequality problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, \eta(v,u)\rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.2}$$

which was introduced and studied by Ding [5].

(B) If $N(Au,Cu) = Au - Cu$, $a(u,v) = 0$ and $\eta(u,v) = gu - gv$ for all $u,v \in K$, then the generalized nonlinear variational-like inequality problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, gv - gu\rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.3}$$

which was studied by Yao [29].

Definition 2.1. Let $A, C : K \to H$, $N : H \times H \to H$ and $\eta : K \times K \to H$ be mappings.

(1) A is said to be *Lipschitz continuous* with constant α if there exists a constant $\alpha > 0$ such that

$$\|Au - Av\| \leq \alpha\|u - v\|, \quad \forall u,v \in K. \tag{2.4}$$

(2) N is said to be *Lipschitz continuous* with constant β in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u,w) - N(v,w)\| \leq \beta\|u - v\|, \quad \forall u,v,w \in H. \tag{2.5}$$

(3) N is said to be *η-antimonotone* with respect to A in the first argument if

$$\langle N(Au,w) - N(Av,w), \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K,\ w \in H. \tag{2.6}$$

(4) N is said to be *η-relaxed Lipschitz* with constant γ with respect to C in the second argument if there exists a constant $\gamma > 0$ such that

$$\langle N(w,Cu) - N(w,Cv), \eta(u,v)\rangle \leq -\gamma\|u - v\|^2, \quad \forall u,v \in K,\ w \in H. \tag{2.7}$$

(5) η is said to be *Lipschitz continuous* with constant δ if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leq \delta\|u - v\|, \quad \forall u,v \in K. \tag{2.8}$$

Similarly, we can define the Lipschitz continuity of N in the second argument.

Definition 2.2. Let K be a nonempty closed convex subset of a Hilbert space H and $f : K \to (-\infty, \infty]$ be a real functional.

(1) f is said to be *convex* if for any $u, v \in K$ and for any $\alpha \in [0,1]$,

$$f(\alpha u + (1-\alpha)v) \le \alpha f(u) + (1-\alpha) f(v). \tag{2.9}$$

(2) f is said to be *lower semicontinuous* on K if for each $\alpha \in (-\infty, \infty]$, the set $\{u \in K : f(u) \le \alpha\}$ is closed in K.

Lemma 2.3 [1, 2]. *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and let $\phi, \psi : K \times K \to R$ be mappings satisfying the following conditions:*

(a) *$\psi(x,y) \le \phi(x,y)$, for all $x, y \in X$, and $\psi(x,x) \ge 0$, for all $x \in X$;*

(b) *for each $x \in X$, $\phi(x,y)$ is upper semicontinuous with respect to y;*

(c) *for each $y \in X$, the set $\{x \in X : \psi(x,y) < 0\}$ is a convex set;*

(d) *there exists a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0, y) < 0$, for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \ge 0$, for all $x \in X$.

3. Existence theorems

In this section, we give four existence theorems of solutions for the generalized nonlinear variational-like inequality (2.1).

Theorem 3.1. *Let K be a nonempty closed convex subset of a Hilbert space H. Let $a : H \times H \to (-\infty, \infty)$ be a coercive continuous bilinear form with (C1) and (C2) and let $f : K \to (-\infty, \infty]$ be a proper convex lower semicontinuous functional with $\operatorname{int}(\operatorname{dom} f) \cap K \ne \emptyset$. Suppose that $A, C : K \to H$ and $N : H \times H \to H$ are continuous mappings, $\eta : K \times K \to H$ is Lipschitz continuous with constant δ, for each $v \in K$, $\eta(\cdot, v)$ is continuous and $\eta(v,u) = -\eta(u,v)$ for all $u, v \in K$. Assume that N is η-antimonotone with respect to A in the first argument and η-relaxed Lipschitz with constant ξ with respect to C in the second argument. Suppose that for given $x, y \in H$ and $v \in K$, the mapping $u \mapsto \langle N(x,y), \eta(u,v)\rangle$ is concave and upper semicontinuous. If $F : K \to H$ is completely continuous, then the generalized nonlinear variational-like inequality (2.1) has a solution $u \in K$.*

Proof. We first prove that for each fixed $\hat{u} \in K$, there exists a unique $\hat{w} \in K$ such that

$$a(\hat{w}, v - \hat{w}) + f(v) - f(\hat{w}) \ge \langle N(A\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w})\rangle, \quad \forall v \in K. \tag{3.1}$$

Let $\hat{u}$ be in K. Define the functionals ϕ and $\psi : K \times K \to R$ by

$$\begin{aligned} \phi(v,w) &= a(v, v-w) + f(v) - f(w) - \langle N(Av, Cv) + F\hat{u}, \eta(v,w)\rangle, \\ \psi(v,w) &= a(w, v-w) + f(v) - f(w) - \langle N(Aw, Cw) + F\hat{u}, \eta(v,w)\rangle \end{aligned} \tag{3.2}$$

for all $v, w \in K$.

We check that the functionals ϕ and ψ satisfy all the conditions of Lemma 2.3 in the weak topology. It follows from the definitions of ϕ and ψ that for all $v,w \in K$,

$$\begin{aligned}\phi(v,w)-\psi(v,w) &= a(v-w,v-w)-\langle N(Av,Cv)-N(Aw,Cv),\eta(v,w)\rangle \\ &\quad -\langle N(Aw,Cv)-N(Aw,Cw),\eta(v,w)\rangle \\ &\geq (c+\xi)\|v-w\|^2 \geq 0,\end{aligned} \tag{3.3}$$

which means that ϕ and ψ satisfy the condition (a) of Lemma 2.3. Notice that f is a convex lower semicontinuous functional and for given $x,y \in H$, $v \in K$, the mapping $u \mapsto \langle N(x,y),\eta(u,v)\rangle$ is concave and upper semicontinuous. It follows that $\phi(v,w)$ is weakly upper semicontinuous with respect to w and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. Therefore, the conditions (b) and (c) of Lemma 2.3 hold. Since f is proper convex lower semicontinuous, for each $v \in \operatorname{int}(\operatorname{dom} f)$, $\partial f(v) \neq \emptyset$, see Ekeland and Temam [9]. Let v^* be in $\operatorname{int}(\operatorname{dom} f) \cap K$. It follows that

$$f(u) \geq f(v^*)+\langle r,u-v^*\rangle, \quad \forall r \in \partial f(v^*),\ u \in K. \tag{3.4}$$

Put

$$\begin{gathered}D=(c+\xi)^{-1}\left(\|r\|+\delta\|N(Av^*,Cv^*)\|+\delta\|F\hat{u}\|\right), \\ T=\{w \in K : \|w-v^*\| \leq D\}.\end{gathered} \tag{3.5}$$

Obviously, T is a weakly compact subset of K and for any $w \in K \setminus T$,

$$\begin{aligned}\psi(v^*,w) &= a(w-v^*,v^*-w)+f(v^*)-f(w)-\langle N(Aw,Cw)+F\hat{u},\eta(v^*,w)\rangle \\ &\leq -a(w-v^*,w-v^*)-\langle r,w-v^*\rangle \\ &\quad +\langle N(Aw,Cw)-N(Av^*,Cw),\eta(w,v^*)\rangle \\ &\quad +\langle N(Av^*,Cw)-N(Av^*,Cv^*),\eta(w,v^*)\rangle \\ &\quad +\langle N(Av^*,Cv^*),\eta(w,v^*)\rangle+\langle F\hat{u},\eta(w,v^*)\rangle \\ &\leq -\|w-v^*\|\left[(c+\xi)\|w-v^*\|-\|r\|-\delta\|N(Av^*,Cv^*)\|-\delta\|F\hat{u}\|\right]<0,\end{aligned} \tag{3.6}$$

which yields that the condition (d) of Lemma 2.3 holds. Thus Lemma 2.3 ensures that there exists a $\hat{w} \in K$ such that $\phi(v,\hat{w}) \geq 0$ for all $v \in K$, that is,

$$a(v,v-\hat{w})+f(v)-f(\hat{w}) \geq \langle N(Av,Cv)+F\hat{u},\eta(v,\hat{w})\rangle, \quad \forall v \in K. \tag{3.7}$$

Let t be in $(0,1]$ and v be in K. Replacing v by $v_t = tv+(1-t)\hat{w}$ in (3.7), we see that

$$a(v_t,t(v-\hat{w}))+f(v_t)-f(\hat{w}) \geq \langle N(Av_t,Cv_t)+F\hat{u},\eta(v_t,\hat{w})\rangle, \quad \forall v \in K. \tag{3.8}$$

Note that a is bilinear and f is convex. From (3.8) we deduce that

$$t\left[a(v_t,v-\hat{w})+f(v)-f(\hat{w})\right] \geq t\langle N(Av_t,Cv_t)+F\hat{u},\eta(v,\hat{w})\rangle, \quad \forall v \in K, \tag{3.9}$$

which implies that

$$a(v_t, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(Av_t, Cv_t) + F\hat{u}, \eta(v, \hat{w})\rangle, \quad \forall v \in K. \tag{3.10}$$

Letting $t \to 0^+$ in the above inequality, we conclude that

$$a(\hat{w}, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(A\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w})\rangle, \quad \forall v \in K. \tag{3.11}$$

That is, $\hat{w}$ is a solution of (3.1). Now we prove the uniqueness. For any two solutions $w_1, w_2 \in K$ of (3.1), we know that

$$\begin{aligned} a(w_1, w_2 - w_1) + f(w_2) - f(w_1) &\geq \langle N(Aw_1, Cw_1) + F\hat{u}, \eta(w_2, w_1)\rangle, \\ a(w_2, w_1 - w_2) + f(w_1) - f(w_2) &\geq \langle N(Aw_2, Cw_2) + F\hat{u}, \eta(w_1, w_2)\rangle. \end{aligned} \tag{3.12}$$

Adding these inequalities, we deduce that

$$\begin{aligned} c\|w_1 - w_2\|^2 &\leq a(w_1 - w_2, w_1 - w_2) \\ &\leq \langle N(Aw_1, Cw_1) - N(Aw_2, Cw_1), \eta(w_1, w_2)\rangle \\ &\quad + \langle N(Aw_2, Cw_1) - N(Aw_2, Cw_2), \eta(w_1, w_2)\rangle \\ &\leq -\xi\|w_1 - w_2\|^2, \end{aligned} \tag{3.13}$$

which yields that $w_1 = w_2$. That is, $\hat{w}$ is a unique solution of (3.1). This means that there exists a mapping $G : K \to K$ satisfying $G(\hat{u}) = \hat{w}$, where $\hat{w}$ is the unique solution of (3.1) for each $\hat{u} \in K$.

Next we show that G is a completely continuous mapping. Let u_1 and u_2 be arbitrary elements in K. Using (3.1), we get that

$$\begin{aligned} a(Gu_1, Gu_2 - Gu_1) + f(Gu_2) - f(Gu_1) &\geq \langle N(A(Gu_1), C(Gu_1)) + Fu_1, \eta(Gu_2, Gu_1)\rangle, \\ a(Gu_2, Gu_1 - Gu_2) + f(Gu_1) - f(Gu_2) &\geq \langle N(A(Gu_2), C(Gu_2)) + Fu_2, \eta(Gu_1, Gu_2)\rangle. \end{aligned} \tag{3.14}$$

Adding (3.14), we arrive at

$$\begin{aligned} c\|Gu_1 - Gu_2\|^2 &\leq a(Gu_1 - Gu_2, Gu_1 - Gu_2) \\ &\leq \langle N(A(Gu_1), C(Gu_1)) - N(A(Gu_2), C(Gu_1)), \eta(Gu_1, Gu_2)\rangle \\ &\quad + \langle N(A(Gu_2), C(Gu_1)) - N(A(Gu_2), C(Gu_2)), \eta(Gu_1, Gu_2)\rangle \\ &\quad + \langle Fu_1 - Fu_2, \eta(Gu_1, Gu_2)\rangle \\ &\leq -\xi\|Gu_1 - Gu_2\|^2 + \delta\|Fu_1 - Fu_2\|\|Gu_1 - Gu_2\|, \end{aligned} \tag{3.15}$$

that is,

$$\|Gu_1 - Gu_2\| \leq \frac{\delta}{c + \xi}\|Fu_1 - Fu_2\|. \tag{3.16}$$

Since F is completely continuous, it follows from (3.16) that $G : K \to K$ is a completely continuous mapping. Hence the Schauder fixed point theorem guarantees that G has a fixed point $u \in K$, which means that u is a solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. □

Theorem 3.2. *Let a, f, C, N, F, and η be as in Theorem 3.1 and let N be Lipschitz continuous with constant ζ in the first argument. Suppose that $A : K \to H$ is Lipschitz continuous with constant ρ. If $c + \xi > \delta\zeta\rho$, then the generalized nonlinear variational-like inequality (2.1) has a solution $u \in K$.*

Proof. Put

$$\begin{gathered} D = (c+\xi-\delta\zeta\rho)^{-1}\left(\|r\| + \delta\|N(Av^*, Cv^*)\| + \delta\|F\hat{u}\|\right), \\ T = \{w \in K : \|w - v^*\| \le D\}. \end{gathered} \tag{3.17}$$

As in the proof of Theorem 3.1, we conclude that

$$\begin{aligned} \psi(v^*, w) \le & -a(w - v^*, w - v^*) - \langle r, w - v^* \rangle \\ & + \langle N(Aw, Cw) - N(Av^*, Cw), \eta(w, v^*) \rangle \\ & + \langle N(Av^*, Cw) - N(Av^*, Cv^*), \eta(w, v^*) \rangle \\ & + \langle N(Av^*, Cv^*), \eta(w, v^*) \rangle + \langle F\hat{u}, \eta(w, v^*) \rangle \\ \le & -\|w - v^*\| \big[(c+\xi-\delta\zeta\rho)\|w - v^*\| \\ & - \|r\| - \delta\|N(Av^*, Cv^*)\| - \delta\|F\hat{u}\| \big] < 0 \end{aligned} \tag{3.18}$$

for any $w \in K \setminus T$. The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and therefore is omitted. □

Theorem 3.3. *Let a, f, A, C, N, and η be as in Theorem 3.1. Suppose that $F : K \to H$ is Lipschitz continuous with constant l. If $\delta l/(c + \xi) < 1$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

Proof. Let u_1 and u_2 be arbitrary elements in K. As in the proof of Theorem 3.1, we deduce that

$$\|Gu_1 - Gu_2\| \le \frac{\delta}{c+\xi}\|Fu_1 - Fu_2\| \le \frac{\delta l}{c+\xi}\|u_1 - u_2\|, \quad \forall u_1, u_2 \in K, \tag{3.19}$$

which yields that $G : K \to K$ is a contraction mapping and hence it has a unique fixed point $u \in K$, which is a unique solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. □

The following theorem follows from the arguments of Theorems 3.1, 3.2 and, 3.3.

Theorem 3.4. *Let a, f, A, C, N, and η be as in Theorem 3.2. Suppose that $F : K \to H$ is Lipschitz continuous with constant l. If $0 < \delta l/(c + \xi - \delta\zeta\rho) < 1$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

4. Algorithm and convergence theorems

Based on Theorem 3.1, we suggest the following iterative algorithm.

ALGORITHM 4.1. *Let $A, C, F : K \to H$, $N : H \times H \to H$, and $\eta : K \times K \to H$ be mappings, and let $f : K \to (-\infty, \infty]$ be a real functional. For any given $u_0 \in K$, compute sequence $\{u_n\}_{n\geq 0}$ by the iterative scheme*

$$a(u_{n+1}, v - u_{n+1}) + f(v) - f(u_{n+1}) \geq \langle N(Au_{n+1}, Cu_{n+1}) + Fu_n, \eta(v, u_{n+1})\rangle, \tag{4.1}$$

for all $v \in K$ and $n \geq 0$.

THEOREM 4.2. *Let a, f, F, N, A, C, and η be as in Theorem 3.3. If $\delta l/(c + \xi) < 1$, then the generalized nonlinear variational-like inequality (2.1) possesses a unique solution and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converges strongly to the unique solution.*

Proof. Using Algorithm 4.1, we obtain that

$$\begin{aligned} a(u_{n+1}, u_n - u_{n+1}) + f(u_n) - f(u_{n+1}) &\geq \langle N(Au_{n+1}, Cu_{n+1}) + Fu_n, \eta(u_n, u_{n+1})\rangle, \\ a(u_n, u_{n+1} - u_n) + f(u_{n+1}) - f(u_n) &\geq \langle N(Au_n, Cu_n) + Fu_{n-1}, \eta(u_{n+1}, u_n)\rangle, \end{aligned} \tag{4.2}$$

for all $n \geq 1$. Adding (4.2), we get that

$$\begin{aligned} c\|u_{n+1} - u_n\|^2 &\leq a(u_{n+1} - u_n, u_{n+1} - u_n) \\ &\leq \langle N(Au_{n+1}, Cu_{n+1}) - N(Au_n, Cu_{n+1}), \eta(u_{n+1}, u_n)\rangle \\ &\quad + \langle N(Au_n, Cu_{n+1}) - N(Au_n, Cu_n), \eta(u_{n+1}, u_n)\rangle \\ &\quad + \langle Fu_n - Fu_{n-1}, \eta(u_{n+1}, u_n)\rangle \\ &\leq -\xi\|u_{n+1} - u_n\|^2 + \delta l\|u_n - u_{n-1}\|\|u_{n+1} - u_n\|, \end{aligned} \tag{4.3}$$

that is,

$$\|u_{n+1} - u_n\| \leq \frac{\delta l}{c + \xi}\|u_n - u_{n-1}\|, \quad \forall n \geq 1, \tag{4.4}$$

which yields that $\{u_n\}_{n\geq 0}$ is a Cauchy sequence by $\delta l/(c + \xi) < 1$. Consequently, $\{u_n\}_{n\geq 0}$ converges to some element u in K. Letting $n \to \infty$ in (4.1), we infer that

$$a(u, v - u) + f(v) - f(u) \geq \langle N(Au, Cu) + Fu, \eta(v, u)\rangle, \quad \forall v \in K. \tag{4.5}$$

Hence u is a solution of the generalized nonlinear variational-like inequality (2.1). It follows from Theorem 3.3 that u is the unique solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. □

Similarly we have the following result.

THEOREM 4.3. *Let a, f, F, N, A, C, and* η *be as in Theorem 3.4. If* $0 < \delta l/(c + \xi - \delta\zeta\rho) < 1$, *then the generalized nonlinear variational-like inequality (2.1) possesses a unique solution and the iterative sequence* $\{u_n\}_{n\geq 0}$ *generated by Algorithm 4.1 converges strongly to the unique solution.*

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References

[1] S. S. Chang, *Variational Inequalitity and Complementarity Theory with Applications*, Shanghai Scientific Technology and Literature Press, Shanghai, 1991.
[2] ———, *On the existence of solutions for a class of quasi-bilinear variational inequalities*, J. Systems Sci. Math. Sci. **16** (1996), 136–140 (Chinese).
[3] M. S. R. Chowdhury and K.-K. Tan, *Generalization of Ky Fan's minimax inequality with applications to generalized variational inequalities for pseudo-monotone operators and fixed point theorems*, J. Math. Anal. Appl. **204** (1996), no. 3, 910–929.
[4] P. Cubiotti, *Existence of solutions for lower semicontinuous quasi-equilibrium problems*, Comput. Math. Appl. **30** (1995), no. 12, 11–22.
[5] X. P. Ding, *Algorithm of solutions for mixed-nonlinear variational-like inequalities in reflexive Banach space*, Appl. Math. Mech. **19** (1998), no. 6, 489–496 (Chinese).
[6] ———, *Existence and algorithm of solutions for nonlinear mixed variational-like inequalities in Banach spaces*, J. Comput. Appl. Math. **157** (2003), no. 2, 419–434.
[7] X. P. Ding and K.-K. Tan, *A minimax inequality with applications to existence of equilibrium point and fixed point theorems*, Colloq. Math. **63** (1992), no. 2, 233–247.
[8] X. P. Ding and E. Tarafdar, *Generalized variational-like inequalities with pseudomonotone set-valued mappings*, Arch. Math. (Basel) **74** (2000), no. 4, 302–313.
[9] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
[10] N.-J. Huang and C.-X. Deng, *Auxiliary principle and iterative algorithms for generalized set-valued strongly nonlinear mixed variational-like inequalities*, J. Math. Anal. Appl. **256** (2001), no. 2, 345–359.
[11] Z. Liu, L. Debnath, S. M. Kang, and J. S. Ume, *Completely generalized multivalued nonlinear quasi-variational inclusions*, Int. J. Math. Math. Sci. **30** (2002), no. 10, 593–604.
[12] ———, *On the generalized nonlinear quasivariational inclusions*, Acta Math. Inform. Univ. Ostraviensis **11** (2003), no. 1, 81–90.
[13] ———, *Sensitivity analysis for parametric completely generalized nonlinear implicit quasivariational inclusions*, J. Math. Anal. Appl. **277** (2003), no. 1, 142–154.
[14] ———, *Generalized mixed quasivariational inclusions and generalized mixed resolvent equations for fuzzy mappings*, Appl. Math. Comput. **149** (2004), no. 3, 879–891.
[15] Z. Liu and S. M. Kang, *Generalized multivalued nonlinear quasivariational inclusions*, Math. Nachr. **253** (2003), no. 1, 45–54.

[16] ______, *Convergence and stability of perturbed three-step iterative algorithm for completely generalized nonlinear quasivariational inequalities*, Appl. Math. Comput. **149** (2004), no. 1, 245–258.
[17] Z. Liu, S. M. Kang, and J. S. Ume, *On general variational inclusions with noncompact valued mappings*, Adv. Nonlinear Var. Inequal. **5** (2002), no. 2, 11–25.
[18] ______, *Completely generalized multivalued strongly quasivariational inequalities*, Publ. Math. Debrecen **62** (2003), no. 1-2, 187–204.
[19] ______, *Generalized variational inclusions for fuzzy mappings*, Adv. Nonlinear Var. Inequal. **6** (2003), no. 1, 31–40.
[20] ______, *The solvability of a class of quasivariational inequalities*, Adv. Nonlinear Var. Inequal. **6** (2003), no. 2, 69–78.
[21] Z. Liu, J. S. Ume, and S. M. Kang, *General strongly nonlinear quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings*, J. Optim. Theory Appl. **114** (2002), no. 3, 639–656.
[22] ______, *Resolvent equations technique for general variational inclusions*, Proc. Japan Acad. Ser. A Math. Sci. **78** (2002), no. 10, 188–193.
[23] ______, *Nonlinear variational inequalities on reflexive Banach spaces and topological vector spaces*, Int. J. Math. Math. Sci. **2003** (2003), no. 4, 199–207.
[24] ______, *Completely generalized quasivariational inequalities*, Adv. Nonlinear Var. Inequal. **7** (2004), no. 1, 35–46.
[25] P. D. Panagiotopoulos and G. E. Stavroulakis, *New types of variational principles based on the notion of quasidifferentiability*, Acta Mech. **94** (1992), no. 3-4, 171–194.
[26] J. Parida and A. Sen, *A variational-like inequality for multifunctions with applications*, J. Math. Anal. Appl. **124** (1987), no. 1, 73–81.
[27] G. Q. Tian, *Generalized quasi-variational-like inequality problem*, Math. Oper. Res. **18** (1993), no. 3, 752–764.
[28] J. C. Yao, *The generalized quasi-variational inequality problem with applications*, J. Math. Anal. Appl. **158** (1991), no. 1, 139–160.
[29] ______, *Existence of generalized variational inequalities*, Oper. Res. Lett. **15** (1994), no. 1, 35–40.

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