SEMIDISCRETIZATION FOR A NONLOCAL PARABOLIC PROBLEM

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A time discretization technique by Euler forward scheme is proposed to deal with a nonlocal parabolic problem. Existence and uniqueness of the approximate solution are proved.

1. Introduction

In this work, we study the time discretization by Euler forward scheme of the nonlocal initial boundary value problem

$$\begin{gathered} \frac{\partial u}{\partial t} - \triangle u = \lambda \frac{f(u)}{\left(\int_\Omega f(u)dx\right)^2} \quad \text{in } \Omega \times]0;T[, \\ u = 0 \quad \text{on } \partial\Omega \times]0;T[, \\ u(0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ a bounded regular domain and λ a positive parameter. The hypotheses we will assume on f are the same as in [6]. We recall first that (1.1) arises by reducing the following system of two equations modeling the thermistor problem:

$$\begin{gathered} u_t = \nabla \cdot (k(u)\nabla u) + \sigma(u) \left| \nabla \varphi \right|^2, \\ \nabla (\sigma(u)\nabla\varphi) = 0, \end{gathered} \tag{1.2}$$

where u represents the temperature generated by the electric current flowing through a conductor, φ the electric potential, $\sigma(u)$ and $k(u)$ are, respectively, the electric and thermal conductivities. For more description, we refer to [5, 6, 7, 8, 11] among others.

We recall also that the Euler forward method was used by several authors to treat semidiscretization of nonlinear parabolic problems, see [3, 4]. Concerning problem (1.1), results of existence and uniqueness of solutions are known under particular forms of f, we refer to [2] and the references therein. On the other hand, little is known about

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the solutions to the discrete problem

$$\begin{aligned} U^n - \tau\triangle U^n &= U^{n-1} + \lambda\tau\frac{f(U^n)}{\left(\int_\Omega f(U^n)\,dx\right)^2} \quad \text{in } \Omega,\\ U^n &= 0 \quad \text{on } \partial\Omega,\\ U^0 &= u_0 \quad \text{in } \Omega. \end{aligned} \tag{1.3}$$

Whereas, semidiscretization has been involved for the equations of the thermistor problem in [1, 9]. Our aim here is to continue the study of problem (1.1) initiated in [6], where an a priori L^∞-estimate is derived. In addition to habitual existence and uniqueness questions concerning the solutions of (1.3), we will prove some results of stability and proceed to error estimates analysis. In [1], the authors derived an L^2 and H^1-norm error by requiring more regularity on the solution u, for instance u, u_t in $H^2(\Omega)\cap W^{1,\infty}(\Omega)$. Unfortunately, such smoothness is not always possible since the function f is nonlinear.

2. The semidiscrete problem

2.1. Existence and uniqueness. We consider the Euler scheme (1.3), with $N\tau = T$, $T > 0$ fixed, and $1 \le n \le N$, under the following hypotheses.

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian function.

(H2) There exist positive constants σ, c_1, c_2, and α such that $\alpha < 4/(d-2)$ and for all $\xi \in \mathbb{R}$,

$$\sigma \le f(\xi) \le c_1|\xi|^{\alpha+1} + c_2. \tag{2.1}$$

In the sequel, we will denote the norms in the spaces $L^\infty(\Omega)$, $L^k(\Omega)$ by $|\cdot|_{L^\infty(\Omega)}$ and $|\cdot|_k$, respectively, $(\cdot,\cdot)$ will denote the associated inner product in $L^2(\Omega)$ or the duality product between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$.

Theorem 2.1. *Let* (H1)-(H2) *be satisfied. Then, for each n, there exists a unique solution U^n of (1.3) in $H_0^1(\Omega)\cap L^\infty(\Omega)$ provided that τ is small enough.*

Proof. For simplicity, we write $U = U^n$, $h(x) = U^{n-1}$. Then (1.3) becomes

$$\begin{aligned} U - \tau\triangle U &= h(x) + \lambda\frac{f(U)}{\left(\int_\Omega f(U)\,dx\right)^2} \quad \text{in } \Omega,\\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Existence. Define the map $S(\mu,\cdot)$ by $U = S(\mu, v)$, $\mu \in [0,1]$ if and only if

$$\begin{aligned} U - \tau\triangle U &= \mu g(x,v) \quad \text{in } \Omega,\\ U &= 0 \quad \text{on } \partial\Omega,\\ U^0 &= \mu u_0, \end{aligned} \tag{2.3}$$

where $g(x,v) = h(x) + \lambda(f(v)/(\int_\Omega f(v)dx)^2)$. For a fixed $v \in H_0^1(\Omega)$, (2.3) has a unique solution $U \in H_0^1(\Omega)$. Then, for each $\mu \in [0,1]$, the operator $S(\mu,\cdot)$ is well defined. Moreover, $S(\mu,\cdot)$ is compact from $H_0^1(\Omega)$ into it self. Indeed, using (H2), we have the estimate

$$|U|_2^2 + \tau|\nabla U|_2^2 \le c_3. \tag{2.4}$$

We can easily see that $\mu \to S(\mu,v)$ is continuous and that $S(0,v) = U$, for any v, if and only if $U = 0$. From Leray-Schauder fixed point theorem, there exists therefore a fixed point U of $S(\mu,\cdot)$. □

Now, we derive an a priori estimate.

LEMMA 2.2. *If $u_0 \in L^\infty(\Omega)$, then for all $n \in \{1,\dots,N\}$, $U^n \in L^\infty(\Omega)$.*

Proof. The proof is similar to the one used by De Thélin in [10] concerning a very different problem and we will give here only a sketch. Suppose that $d \ge 2$ and define

$$\delta = \begin{cases} \dfrac{2d}{d-2} & \text{if } 2 < d, \\ 2(\alpha+2) & \text{if } d = 2. \end{cases} \tag{2.5}$$

For each $k \in \mathbb{N}*$, we consider the number

$$q_k = \left\{\left(\frac{\delta}{2}\right)^{k-1}(\delta-\gamma) - (2-\gamma)\right\}\frac{\delta}{\delta-2}, \quad k \ge 2, \qquad q_1 = \delta, \tag{2.6}$$

we have

$$q_{k+1} = (q_k + 2 - \gamma)\frac{\delta}{2} \quad \text{with } \gamma = \alpha + 2,\ \forall k \in \mathbb{N}^*. \tag{2.7}$$

□

LEMMA 2.3. *For all $k \in \mathbb{N}^*$, $U^n \in L^{q_k}(\Omega)$, and moreover*

$$|U^n|_\infty = \overline{\lim}|U^n|_{q_k} < +\infty. \tag{2.8}$$

Proof. We prove by recurrence that $U \in L^{q_k}$. The property is true for $k = 1$, since $H_0^1(\Omega) \subset L^\delta(\Omega)$. We show now that $U \in L^{q_{k+1}}$. Let $m \in \mathbb{N}$, $1 \le m \le k$. Multiplying (2.2) by $|U|^{q_m-\gamma}U$, using (H2), and Young's inequality, we get

$$(q_m - \gamma + 1)\int_\Omega |\nabla U|^2|U|^{q_m-\gamma}dx \le c_4|U|_{q_m}^{q_m} + c_5. \tag{2.9}$$

On the other hand, we have

$$|U|_{q_{m+1}}^{q_m+2-\gamma} \le c_6\left(1 + \frac{q_m-\gamma}{2}\right)^2\int_\Omega |\nabla U|^2|U|^{q_m-\gamma}dx. \tag{2.10}$$

Therefore, we obtain

$$|U|_{q_{m+1}}^{q_m+2-\gamma} \leq \left(c_7 + c_8 |U|_{q_m}^{q_m}\right)\left(q_m + 2 - \gamma\right). \tag{2.11}$$

Thus,

$$\left(|U|_{q_{k+1}}^{q_{k+1}}\right)^{2/\delta} \leq \left(c_7 + c_8 |U|_{q_k}^{q_k}\right)\left(q_k + 2 - \gamma\right). \tag{2.12}$$

The rest of the proof follows the same lines as in [10, pages 383-384].

Uniqueness. Consider U and V two different solutions of (2.2) and define $w = U - V$. Then, we have

$$\begin{aligned} w - \tau \triangle w = {} & \frac{\lambda\tau}{\left(\int_\Omega f(U)dx\right)^2}\left(f(U) - f(V)\right) \\ & + \lambda\tau \frac{\left(\int_\Omega f(U) - f(V)dx\right)\left(\int_\Omega f(V) + f(U)dx\right)}{\left(\int_\Omega f(U)dx\right)^2\left(\int_\Omega f(V)dx\right)^2} f(V). \end{aligned} \tag{2.13}$$

Multiplying (2.13) by w, integrating on Ω, and using the L^∞-estimate obtained in Lemma 2.2, we get

$$|w|_2^2 + \tau |\nabla w|_2^2 \leq c_9 \tau |w|_2^2. \tag{2.14}$$

Therefore, $w = 0$ if $\tau \leq 1/c_9$. □

We address now the question of stability.

3. Stability

Theorem 3.1. *Assume* (H1)-(H2) *hold. Then, there exists* $c(T,u_0) > 0$ *depending on data but not on* N *such that for any* $n \in \{1,\dots,N\}$,

(a) $|U^n|_{L^\infty(\Omega)} \leq c(T,u_0)$;
(b) $|U^n|_2^2 + \tau \sum_{k=1}^n |\nabla U^k|_2^2 \leq c(T,u_0)$;
(c) $\sum_{k=1}^n |U^k - U^{k-1}|_2^2 \leq c(T,u_0)$.

Proof. (i) Multiplying (1.3) by $|U^k|^m U^k$ for some integer $m \geq 1$, using Lemma 2.2, and Hölder's inequality, we obtain after simplification

$$\left|U^k\right|_{m+2} \leq \left|U^{k-1}\right|_{m+2} + c_{10}\tau. \tag{3.1}$$

By induction and taking the limit in the resulting inequality as $m \to +\infty$, we get

$$\left|U^k\right|_{L^\infty(\Omega)} \leq c(T,u_0). \tag{3.2}$$

(ii) Multiplying the first equation of (1.3) by U^k and using the hypotheses on f, one easily has

$$\left(U^k - U^{k-1}, U^k\right) + \tau\left|\nabla U^k\right|_2^2 \leq c_{11}\tau\left|U^k\right|_1. \tag{3.3}$$

Using the elementary identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and summing from $k = 1$ to n, we obtain

$$|U^n|_2^2 + \sum_{k=1}^{n} |U^k - U^{k-1}|_2^2 + \tau \sum_{k=1}^{n} |\nabla U^k|_2^2 \le |u_0|_2^2 + \tau c_{12} \sum_{k=1}^{n} |U^k|_1. \tag{3.4}$$

Then, the inequalities (b)-(c) hold by using the uniform bound of U^n in L^∞ which is established in part (a). □

4. Error estimates for solutions

We will adopt the following notations concerning the time discretization for problem (1.1). We denote the time step $\tau = T/N$, $t^n = n\tau$, and $I_n = (t^n, t^{n-1})$ for $n = 1, \dots, N$. If z is a continuous function (resp., summable), defined in $(0,T)$ with values in $H^{-1}(\Omega)$ or $L^2(\Omega)$ or $H_0^1(\Omega)$, we define $z^n = z(t^n, \cdot)$, $\overline{z}^n = (1/\tau)\int_{I_n} z(t,\cdot)dt$, $\overline{z}^0 = z^0 = z(0,\cdot)$; the error $e_n = u(t) - U^n$ for all $t \in I_n$ and the local errors e_u^n and e^n defined by $e_u^n = \overline{u}^n(t) - U^n$, $e^n = u^n - U^n$.

We have the following theorem.

Theorem 4.1. *Let* (H1)-(H2) *hold. Then, the following error bounds are satisfied:*

(1) $\|e_n\|^2_{L^\infty(0,T,H^{-1}(\Omega))} + \int_0^T |e_n|^2 dt \le c_{13}\tau$,

(2) $\|e^m\|_{H^{-1}(\Omega)} \le c_{14}\tau^{1/2}$,

(3) $|\nabla \int_0^T e_n(t)dt|_2 \le c_{15}\tau^{1/4}$.

Proof. For the proof, we consider the following variational formulation of discrete problem (1.3):

$$(U^n - U^{n-1}, \varphi) + \tau(\nabla U^n, \nabla\varphi) = \frac{\lambda\tau}{\left(\int_\Omega f(U^n)dx\right)^2}(f(U^n), \varphi), \quad \forall \varphi \in H_0^1(\Omega). \tag{4.1}$$

Integrating the continuous problem (1.1) over I_n, we get

$$(u^n - u^{n-1}, \varphi) + \tau(\nabla \overline{u}^n, \nabla\varphi) = \lambda\tau \frac{\left(\overline{f(u^n)}, \varphi\right)}{\left(\int_\Omega f(u^n)dx\right)^2}, \quad \forall \varphi \in H_0^1(\Omega). \tag{4.2}$$

Substracting (4.2) from (4.1) and adding from $n = 1$ to m with $m \le N$, we obtain

$$\begin{aligned} &\sum_{n=1}^{m} (e^n - e^{n-1}, \varphi) + \tau \sum_{n=1}^{m} (\nabla e_u^n, \nabla\varphi) \\ &\quad \le c_{16}\tau \left| \sum_{n=1}^{m} \left(\overline{f(u)}^n - f(U^n), \varphi\right) \right| + c_{17}\tau \left| \sum_{n=1}^{m} (f(U^n), \varphi) \right|. \end{aligned} \tag{4.3}$$

Let $(-\triangle)^{-1}$ be the green operator satisfying

$$(\nabla(-\triangle)^{-1}\psi,\nabla\varphi) = (\psi,\varphi)_{H^{-1}(\Omega),H_0^1(\Omega)} \tag{4.4}$$

for all $\psi \in H^{-1}(\Omega)$, $\varphi \in H_0^1(\Omega)$. Choosing $\varphi = (-\triangle)^{-1}(e^n)$ as test function, we then obtain

$$I_1 + I_2 \le I_3 + I_4, \tag{4.5}$$

where

$$\begin{aligned} I_1 &= \sum_{n=1}^{m} \left(e^n - e^{n-1}, (-\triangle)^{-1}(e^n)\right), \\ I_2 &= \tau \sum_{n=1}^{m} \left(e_u^n, e^n\right), \\ I_3 &\le c_{16}\tau \left| \sum_{n=1}^{m} \left(\overline{f(u)}^n - f(U^n), (-\triangle)^{-1}(e^n)\right) \right|, \\ I_4 &= c_{17}\tau \left| \sum_{n=1}^{m} \left(f(U^n), (-\triangle)^{-1}(e^n)\right) \right|. \end{aligned} \tag{4.6}$$

With the aid of the elementary identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and the property of $(-\triangle)^{-1}$, I_1 reduces after straightforward calculations to

$$I_1 = \frac{1}{2}||e^m||_{H^{-1}(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^{m} ||e^n - e^{n-1}||_{H^{-1}(\Omega)}^2. \tag{4.7}$$

On the other hand,

$$\begin{aligned} I_2 &= \tau \sum_{n=1}^{m} \left(e_u^n, e^n\right) \\ &= \sum_{n=1}^{m} \int_{I_n} \left(u(t) - U^n, u(t) - U^n\right) dt + \sum_{n=1}^{m} \int_{I_n} \left(u(t) - U^n, u^n - u(t)\right) dt \\ &= I_{21} + I_{22}, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} I_{21} &= \sum_{n=1}^{m} \int_{I_n} \left(u(t) - U^n, u(t) - U^n\right) dt = \sum_{n=1}^{m} \int_{I_n} |\, e_n \,|_2^2 dt, \\ I_{22} &= \sum_{n=1}^{m} \int_{I_n} \left(u(t), u^n - u(t)\right) dt - \sum_{n=1}^{m} \int_{I_n} \left(U^n, u^n - u(t)\right) dt \\ &= I_{22}^1 + I_{22}^2. \end{aligned} \tag{4.9}$$

We now estimate I_{22}^1. Using the boundedness of $\partial u/\partial s$ (see [6]), we have

$$\begin{aligned} |I_{22}^1| &= \left| \sum_{n=1}^m \int_{I_n} \left(u(t), \int_t^{t^n} \frac{\partial u}{\partial s} ds \right) dt \right| \\ &\le \sum_{n=1}^m \int_{I_n} \left(\int_t^{t^n} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}(\Omega)} ds \right) \|u(t)\|_{H_0^1(\Omega)} dt \\ &\le \tau \left\| \frac{\partial u}{\partial s} \right\|_{L^2(0,t^m,H^{-1}(\Omega))} \|u\|_{L^2(0,t^m,H_0^1(\Omega))} \\ &\le c_{18}\tau. \end{aligned} \tag{4.10}$$

In the same manner, we have

$$\begin{aligned} |I_{22}^2| &\le \tau \left\| \frac{\partial u}{\partial s} \right\|_{L^2(0,t^m,H^{-1}(\Omega))} \left(\tau \sum_{n=1}^m \|U^n\|_{H_0^1(\Omega)}^2 \right)^{1/2} \\ &\le c_{18}\tau. \end{aligned} \tag{4.11}$$

Next, we estimate the first term on the right-hand side of (4.5) by using Hölder's and Young's inequalities and (H1),

$$\begin{aligned} |I_3| &\le \left| \sum_{n=1}^m \left(\int_{I_n} [f(u) - f(U^n)] dt, (-\triangle)^{-1}(e^n) \right) \right| \\ &\le c_{20}\tau^{1/2} \sum_{n=1}^m \left(\int_{I_n} |f(u) - f(U^n)|_2^2 dt \right)^{1/2} \|e^n\|_{H^{-1}(\Omega)} \\ &\le \eta \sum_{n=1}^m \left(\int_{I_n} |f(u) - f(U^n)|_2^2 dt \right) + \frac{c_{21}}{\eta} \tau \sum_{n=1}^m \|e^n\|_{H^{-1}(\Omega)}^2 \\ &\le c_{22}\eta \sum_{n=1}^m \left(\int_{I_n} |e_n|_2^2 dt \right) + \frac{c_{21}}{\eta} \tau \sum_{n=1}^m \|e^n\|_{H^{-1}(\Omega)}^2. \end{aligned} \tag{4.12}$$

Moreover, we have

$$|I_4| \le c_{23}\tau + c_{24}\tau \sum_{n=1}^m \|e^n\|_{H^{-1}(\Omega)}^2. \tag{4.13}$$

Choosing suitable η, we conclude that

$$\begin{aligned} &\|e^m\|_{H^{-1}(\Omega)}^2 + \sum_{n=1}^m \|e^n - e^{n-1}\|_{H^{-1}(\Omega)}^2 + \sum_{n=1}^m \int_{I_n} |e_n|_2^2 dt \\ &\quad \le c_{25}\tau + c_{26}\tau \sum_{n=1}^m \|e^n\|_{H^{-1}(\Omega)}^2. \end{aligned} \tag{4.14}$$

On the other hand, setting $y^m = \sum_{n=1}^m \|e^n\|_{H^{-1}(\Omega)}^2$, then from (4.14), we get

$$y^m - y^{m-1} \le c_{25}\tau + c_{26}\tau y^m. \tag{4.15}$$

By applying the discrete Gronwall inequality, we deduce that $y^m \leq c(T)$. Therefore, we have

$$||e^m||_{H^{-1}(\Omega)} \leq c_{27}\tau^{1/2}. \tag{4.16}$$

On the other hand, we have

$$\sup_{t\in(0,t_m)} ||e_n(t)||_{H^{-1}(\Omega)} - c_{27}\tau^{1/2} \leq \max_{1\leq n\leq m} ||e_n(t^n)||_{H^{-1}(\Omega)} = \max_{1\leq n\leq m} ||e^n||_{H^{-1}(\Omega)}. \tag{4.17}$$

Thus, we get

$$||e_n||_{L^\infty(0,T,H^{-1}(\Omega))} - c_{27}\tau^{1/2} \leq \max_{1\leq n\leq m} ||e^n||_{H^{-1}(\Omega)}. \tag{4.18}$$

From the last inequality, we obtain

$$\begin{gathered} ||e_n||^2_{L^\infty(0,T,H^{-1}(\Omega))} + \int_0^T |e_n|_2^2 dt \leq c_{29}\tau, \\ \sum_{n=1}^m ||e^n - e^{n-1}||^2_{H^{-1}(\Omega)} \leq c_{29}\tau. \end{gathered} \tag{4.19}$$

Choosing $\varphi = \tau\sum_{n=1}^m(\overline{u}^n - U^n)$ in (4.3), we get

$$\begin{aligned} \tau\int_\Omega (u^m - U^m)\left(\sum_{n=1}^m (\overline{u}^n - U^n)dx\right) + \tau^2\left|\sum_{n=1}^m \nabla(\overline{u}^n - U^n)\right|_2^2 \\ \leq c_{30}\tau^2\left|\int_\Omega \sum_{n=1}^m \left(\overline{f(u)}^n - f(U^n)\right)\left(\sum_{n=1}^m (\overline{u}^n - U^n)\right)dx\right| \\ + c_{31}\tau^2\left|\sum_{n=1}^m \left(f(U^n), \sum_{n=1}^m (\overline{u}^n - U^n)\right)\right|. \end{aligned} \tag{4.20}$$

Thus,

$$\begin{aligned} \tau^2\left|\sum_{n=1}^m \nabla(\overline{u}^n - U^n)\right|_2^2 = \left|\nabla\int_0^{t^m} e_n dt\right|_2^2 &\leq \tau\left|\int_\Omega (u^m - U^m)\left(\sum_{n=1}^m (\overline{u}^n - U^n)dx\right)\right| \\ &+ c_{30}\tau^2\left|\int_\Omega \sum_{n=1}^m \left(\overline{f(u)}^n - f(U^n)\right)\left(\sum_{n=1}^m (\overline{u}^n - U^n)\right)dx\right| \\ &+ c_{31}\tau^2\left|\sum_{n=1}^m \left(f(U^n), \sum_{n=1}^m (\overline{u}^n - U^n)\right)\right| \\ &\leq I + II + III. \end{aligned} \tag{4.21}$$

Clearly,

$$\begin{aligned} I &\leq \|e^m\|_{H^{-1}(\Omega)}\left(\sum_{n=1}^{m}\int_{I_n}\|u(t)\|_{H_0^1(\Omega)}dt+\tau\sum_{n=1}^{m}\|U^n\|_{H_0^1(\Omega)}\right) \\ &\leq c_{32}\|e^m\|_{H^{-1}(\Omega)} \\ &\leq c_{33}\tau^{1/2}. \end{aligned} \tag{4.22}$$

We also get

$$\begin{aligned} II &\leq \left(\int_\Omega\left(\sum_{n=1}^{m}\int_{I_n}(f(u)-f(U^n))dt\right)^2dx\right)^{1/2}\times\left(\int_\Omega\left(\sum_{n=1}^{m}\int_{I_n}(u(t)-U^n)dt\right)^2dx\right)^{1/2} \\ &\leq T^2\left(\sum_{n=1}^{m}\int_{I_n}|f(u)-f(U^n)|_2^2dt\right)^{1/2}\times\left(\sum_{n=1}^{m}\int_{I_n}|u(t)-U^n|_2^2dt\right)^{1/2} \\ &\leq T^2\left(\sum_{n=1}^{m}\int_{I_n}|f(u)-f(U^n)|_2^2dt\right)^{1/2}\times\left(2\|u\|^2_{L^2(0,T,H_0^1(\Omega))}+2\tau\sum_{n=1}^{m}|U^n|_2^2\right)^{1/2} \\ &\leq c_{34}\tau^{1/2}. \end{aligned} \tag{4.23}$$

The last inequality follows by using simultaneously the L^∞-estimate of $u(t)$ (see [6]), U^n, and the error bound given in (a). Arguing exactly as in the previous estimate, we get

$$III \leq T^2\left(\sum_{n=1}^{m}\int_{I_n}|f(U^n)|_2^2dt\right)^{1/2}\times\left(2\|u\|^2_{L^2(0,T,H_0^1(\Omega))}+2\tau\sum_{n=1}^{m}|U^n|_2^2\right)^{1/2}. \tag{4.24}$$

Using again the hypothesis (H1) and the estimates above, we obtain

$$III \leq c_{35}\tau^{1/2}. \tag{4.25}$$

Finally collecting these results, it follows that

$$\left|\nabla\int_0^T e_n dt\right|_2^2 \leq c_{36}\tau^{1/2}. \tag{4.26}$$

This completes the proof. □

Corollary 4.2. *Under hypotheses* (H1)-(H2), *problem (1.3) generates a continuous semigroup* S_τ *defined by* $S_\tau U^{n-1} = U^n$.

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