

COMPACTNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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We introduce fuzzy almost continuous mapping, fuzzy weakly continuous mapping, fuzzy compactness, fuzzy almost compactness, and fuzzy near compactness in intuitionistic fuzzy topological space in view of the definition of Šostak, and study some of their properties. Also, we investigate the behavior of fuzzy compactness under several types of fuzzy continuous mappings.

1. Introduction and preliminaries

The concept of a fuzzy set was introduced by Zadeh [13], and later Chang [3] defined fuzzy topological spaces. These spaces and their generalizations are later studied by several authors, one of which, developed by Šostak [11, 12], used the idea of degree of openness. This type of generalization of fuzzy topological spaces was later rephrased by Chattopadhyay et al. [4], and by Ramadan [10].

In 1983, Atanassov introduced the concept of “Intuitionistic fuzzy set” [1, 2]. Using this type of generalized fuzzy set, Çoker [5, 8] defined “Intuitionistic fuzzy topological spaces.”

In 1996, Çoker and Demirci [7] introduced the basic definitions and properties of intuitionistic fuzzy topological spaces in Šostak’s sense, which is a generalized form of “fuzzy topological spaces” developed by Šostak [11, 12].

In this paper, we introduce the following concepts: fuzzy almost continuous mapping, fuzzy weakly continuous mapping, fuzzy compactness, fuzzy almost compactness, and fuzzy near compactness in intuitionistic fuzzy topological spaces in view of the definition of Šostak.

Definition 1.1 [1]. Let X be a nonempty fixed set and I the closed unit interval $[0,1]$. An intuitionistic fuzzy set (IFS) A is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \}, \quad (1.1)$$

where the mappings $\mu_A : X \rightarrow I$ and $\nu_A : X \rightarrow I$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\nu_A(x)$) of each element $x \in X$ to

the set A , respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. The complement of the IFS A , is $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x) \rangle : x \in X\}$. Obviously, every fuzzy set A on a nonempty set X is an IFS having the form

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\}. \quad (1.2)$$

For a given nonempty set X , denote the family of all IFSs in X by the symbol ζ^X .

Definition 1.2 [6]. Let X be a nonempty set and $x \in X$ a fixed element in X . If $r \in I_0$, $s \in I_1$ are fixed real numbers such that $r + s \leq 1$, then the IFS $x_{r,s} = \langle y, x_r, 1 - x_{1-s} \rangle$ is called an intuitionistic fuzzy point (IFP) in X , where r denotes the degree of membership of $x_{r,s}$, s the degree of nonmembership of $x_{r,s}$, and $x \in X$ the support of $x_{r,s}$. The IFP $x_{r,s}$ is contained in the IFS A ($x_{r,s} \in A$) if and only if $r < \mu_A(x)$, $s > \nu_A(x)$.

Definition 1.3 [6]. (i) An IFP $x_{r,s}$ in X is said to be quasicoincident with the IFS A , denoted by $x_{r,s}qA$, if and only if $r > \gamma_A(x)$ or $s < \mu_A(x)$. $x_{r,s}qA$ if and only if $x_{r,s} \notin \bar{A}$.

(ii) The IFSs A and B are said to be quasicoincident, denoted by AqB if and only if there exists an element $x \in X$ such that $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$. If A is not quasicoincident with A , denote $A\bar{q}B$. $A\bar{q}B$ if and only if $A \subseteq \bar{B}$.

Definition 1.4 [8]. Let a and b be two real numbers in $[0,1]$ satisfying the inequality $a + b \leq 1$. Then the pair $\langle a, b \rangle$ is called an intuitionistic fuzzy pair.

Let $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle$ be two intuitionistic fuzzy pairs. Then define

- (i) $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$ if and only if $a_1 \leq a_2$ and $b_1 \geq b_2$;
- (ii) $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$ if and only if $a_1 = a_2$ and $b_1 = b_2$;
- (iii) if $\{\langle a_i, b_i \rangle : i \in J\}$ is a family of intuitionistic fuzzy pairs, then $\vee \langle a_i, b_i \rangle = \langle \vee a_i, \wedge b_i \rangle$ and $\wedge \langle a_i, b_i \rangle = \langle \wedge a_i, \vee b_i \rangle$;
- (iv) the complement of an intuitionistic fuzzy pair $\langle a, b \rangle$ is the intuitionistic fuzzy pair defined by $\overline{\langle a, b \rangle} = \langle b, a \rangle$;
- (v) $1^\sim = \langle 1, 0 \rangle$ and $0^\sim = \langle 0, 1 \rangle$.

Definition 1.5 [5]. An intuitionistic fuzzy topology (IFT) in Chang's sense on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

- (T₁) $0^\sim, 1^\sim \in \tau$, where $0^\sim = \{\langle x, 0, 1 \rangle : x \in X\}$ and $1^\sim = \{\langle x, 1, 0 \rangle : x \in X\}$;
- (T₂) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;
- (T₃) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called Chang intuitionistic fuzzy topological space and each IFS in τ is known as intuitionistic fuzzy open set in X .

Definition 1.6 [8]. An IFS ξ on the set ζ^X is called an intuitionistic fuzzy family (IFF) on X . In symbols, denote such an IFF in form $\xi = \langle \mu_\xi, \nu_\xi \rangle$.

Let ξ be an IFF on X . Then the complemented IFF of ξ on X is defined by $\xi^* = \langle \mu_{\xi^*}, \nu_{\xi^*} \rangle$, where $\mu_{\xi^*}(A) = \mu_\xi(\bar{A})$ and $\nu_{\xi^*}(A) = \nu_\xi(\bar{A})$, for each $A \in \zeta^X$. If τ is an IFF on X , then for any $A \in \zeta^X$, construct the intuitionistic fuzzy pair $\langle \mu_\tau(A), \nu_\tau(A) \rangle$ and use the symbol $\tau(A) = \langle \mu_\tau(A), \nu_\tau(A) \rangle$.

Definition 1.7 [7]. An IFT in Šostak's sense on a nonempty set X is an IFF τ on X satisfying the following axioms:

- (T₁) $\tau(0_{\sim}) = \tau(1_{\sim}) = 1_{\sim}$;
- (T₂) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for any $A, B \in \zeta^X$;
- (T₃) $\tau(\cup A_i) \geq \wedge \tau(A_i)$ for any $\{A_i : i \in J\} \subseteq \zeta^X$.

In this case, the pair (X, τ) is called an intuitionistic fuzzy topological space in Šostak's sense (IFTS). For any $A \in \zeta^X$, the number $\mu_{\tau}(A)$ is called the openness degree of A , while $\nu_{\tau}(A)$ is called the nonopenness degree of A .

Example 1.8. Let $X = \{a, b\}$. Define a mapping $\tau : \zeta^X \rightarrow I \times I$

$$\tau(A) = \begin{cases} 1_{\sim} & \text{if } A \in \{0_{\sim}, 1_{\sim}\}, \\ \langle \min(\mu_A(a), \mu_A(b)), \max(\nu_A(a), \nu_A(b)) \rangle & \text{otherwise.} \end{cases} \quad (1.3)$$

Then, τ is an IFT in the sense of Šostak and neither a Chang fuzzy topology nor a Chang IFT.

Definition 1.9 [7]. Let (X, τ) be an IFTS on X . Then the IFF τ^* is defined by $\tau^*(A) = \tau(\overline{A})$. The number $\mu_{\tau^*}(A) = \mu_{\tau}(\overline{A})$ is called the closedness degree of A , while $\nu_{\tau^*}(A) = \nu_{\tau}(A)$ is called the nonclosedness degree of A .

THEOREM 1.10 [7]. *The IFF τ^* on X satisfies the following properties:*

- (C₁) $\tau^*(0_{\sim}) = \tau^*(1_{\sim}) = 1_{\sim}$;
- (C₂) $\tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$ for any $A, B \in \zeta^X$;
- (C₃) $\tau^*(\cap A_i) \geq \wedge \tau^*(A_i)$ for any $\{A_i : i \in J\} \subseteq \zeta^X$.

Definition 1.11 [7]. Let (X, τ) be an IFTS and A be an IFS in X . Then the fuzzy closure and fuzzy interior of A are defined by

$$\begin{aligned} \text{cl}_{\alpha, \beta}(A) &= \cap \{K \in \zeta^X : A \subseteq K, \tau^*(K) \geq \langle \alpha, \beta \rangle\}, \\ \text{int}_{\alpha, \beta}(A) &= \cup \{G \in \zeta^X : G \subseteq A, \tau(G) \geq \langle \alpha, \beta \rangle\}, \end{aligned} \quad (1.4)$$

where $\alpha \in I_0 = (0, 1]$, $\beta \in I_1 = [0, 1)$ with $\alpha + \beta \leq 1$.

THEOREM 1.12 [7]. *The closure and interior operator satisfy the following properties:*

- (i) $A \subseteq \text{cl}_{\alpha, \beta}(A)$;
- (ii) $\text{int}_{\alpha, \beta}(A) \subseteq A$;
- (iii) $A \subseteq B$ and $\langle \alpha, \beta \rangle \leq \langle r, s \rangle$ implies $\text{cl}_{\alpha, \beta}(A) \subseteq \text{cl}_{r, s}(B)$;
- (iv) $A \subseteq B$ and $\langle \alpha, \beta \rangle \leq \langle r, s \rangle$ implies $\text{int}_{\alpha, \beta}(A) \subseteq \text{int}_{r, s}(B)$;
- (v) $\text{cl}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(A)) = \text{cl}_{\alpha, \beta}(A)$;
- (vi) $\text{int}_{\alpha, \beta}(\text{int}_{\alpha, \beta}(A)) = \text{int}_{\alpha, \beta}(A)$;
- (vii) $\text{cl}_{\alpha, \beta}(A \cup B) = \text{cl}_{\alpha, \beta}(A) \cup \text{cl}_{\alpha, \beta}(B)$;
- (viii) $\text{int}_{\alpha, \beta}(A \cap B) = \text{int}_{\alpha, \beta}(A) \cap \text{int}_{\alpha, \beta}(B)$;

- (ix) $\overline{\text{cl}_{\alpha,\beta}(A)} = \text{int}_{\alpha,\beta}(\overline{A})$;
- (x) $\overline{\text{int}_{\alpha,\beta}(A)} = \text{cl}_{\alpha,\beta}(\overline{A})$.

Definition 1.13 [7]. Let (X, τ_1) and (Y, τ_2) be two IFTSs and $f : X \rightarrow Y$ be a mapping. Then f is said to be

- (i) intuitionistic fuzzy continuous if and only if $\tau_1(f^{-1}(B)) \geq \tau_2(B)$, for each $B \in \zeta^Y$;
- (ii) intuitionistic fuzzy open if and only if $\tau_2(f(A)) \geq \tau_1(A)$, for each $A \in \zeta^X$.

2. Intuitionistic fuzzy almost continuous and intuitionistic fuzzy weakly continuous mapping

Definition 2.1. Let A be an IFS in an IFTS (X, τ) . For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, A is called

- (i) (α, β) -intuitionistic fuzzy regular open $((\alpha, \beta)$ -IFRO) set of X if $\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} A) = A$;
- (ii) (α, β) -intuitionistic fuzzy regular closed $((\alpha, \beta)$ -IFRC) set of X if $\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta} A) = A$.

THEOREM 2.2. Let A be an IFS in an IFTS (X, τ) . Then, for $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.

- (i) If A is (α, β) -IFRO (resp., (α, β) -IFRC), set then $\tau(A) \geq \langle \alpha, \beta \rangle$ (resp., $\tau^*(A) \geq \langle \alpha, \beta \rangle$).
- (ii) A is (α, β) -IFRO set if and only if \overline{A} is (α, β) -IFRC set.

Proof. We will prove (ii) only:

$$\begin{aligned}
 A \text{ is } (\alpha, \beta)\text{-IFRO} &\iff \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} A) = A \\
 &\iff \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta} \overline{A}) = \overline{A} \\
 &\iff \overline{A} \text{ is } (\alpha, \beta)\text{-IFRC}.
 \end{aligned} \tag{2.1}$$

THEOREM 2.3. Let (X, τ) be an IFTS. Then,

- (i) the union of two (α, β) -IFRC sets is (α, β) -IFRC set,
- (ii) the intersection of two (α, β) -IFRO sets is (α, β) -IFRO set.

Proof. (i) Let A, B be any two (α, β) -IFRC sets. By Theorem 2.2, we have $\tau^*(A) \geq \langle \alpha, \beta \rangle$, $\tau^*(B) \geq \langle \alpha, \beta \rangle$ then, $\tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B) \geq \langle \alpha, \beta \rangle$, but $\text{int}_{\alpha,\beta}(A \cup B) \subseteq A \cup B$, this implies that $\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A \cup B)) \subseteq \text{cl}_{\alpha,\beta}(A \cup B) = A \cup B$. Now, $A = \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A)) \subseteq \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A \cup B))$ and $B = \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(B)) \subseteq \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A \cup B))$. Then, $A \cup B \subseteq \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A \cup B))$. So, $\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A \cup B)) = A \cup B$. Hence, $A \cup B$ is (α, β) -IFRC set.

(ii) It can be proved by the same manner. □

THEOREM 2.4. Let (X, τ) be an IFTS. Then,

- (i) if $A \in \zeta^X$ such that $\tau^*(A) \geq \langle \alpha, \beta \rangle$, then $\text{int}_{\alpha,\beta}(A)$ is (α, β) -IFRO set,
- (ii) if $B \in \zeta^X$ such that $\tau(B) \geq \langle \alpha, \beta \rangle$, then $\text{cl}_{\alpha,\beta}(B)$ is (α, β) -IFRC set.

Proof. (i) Let $A \in \zeta^X$ such that $\tau^*(A) \geq \langle \alpha, \beta \rangle$. Clearly,

$$\text{int}_{\alpha,\beta}(A) \subseteq \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(A)); \tag{2.2}$$

this implies that

$$\text{int}_{\alpha,\beta}(A) \subseteq \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A))). \tag{2.3}$$

Now, since $\tau^*(A) \geq \langle \alpha, \beta \rangle$, then $\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A)) \subseteq A$; this implies that

$$\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A))) \subseteq \text{int}_{\alpha,\beta}(A). \tag{2.4}$$

Thus, $\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(A))) = \text{int}_{\alpha,\beta}(A)$. Hence, $\text{int}_{\alpha,\beta}(A)$ is (α, β) -IFRO set.

(ii) It can be proved by the same manner. □

Definition 2.5. A mapping $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ from an IFTS (X, τ_1) to another IFTS (Y, τ_2) is called

- (i) intuitionistic fuzzy strong continuous if and only if $\tau_1(f^{-1}(A)) = \tau_2(A)$, for each $A \in \zeta^Y$,
- (ii) (α, β) -intuitionistic fuzzy almost continuous if and only if $\tau_1(f^{-1}(A)) \geq \langle \alpha, \beta \rangle$, for each (α, β) -IFRO set A of Y ,
- (iii) (α, β) -intuitionistic fuzzy weakly continuous if and only if $\tau_2(A) \geq \langle \alpha, \beta \rangle$ implies $\tau_1(f^{-1}(A)) \geq \langle \alpha, \beta \rangle$, for each $A \in \zeta^Y$.

Remark 2.6. From the above definition, it is clear that the following implications are true for $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$:

$$\begin{array}{c}
 (\alpha, \beta)\text{-intuitionistic fuzzy almost continuous mapping} \\
 \uparrow\uparrow \\
 \text{intuitionistic fuzzy strong continuous} \Rightarrow \text{intuitionistic fuzzy continuous mapping} \\
 \downarrow\downarrow \\
 (\alpha, \beta)\text{-intuitionistic fuzzy weakly continuous mapping}
 \end{array}
 \tag{2.5}$$

But, the reciprocal implications are not true in general, as shown by the following examples.

Example 2.7. Let $X = \{a, b, c\}$ and G_1, G_2 be IFSs in X defined as follows:

$$\begin{aligned}
 G_1 &= \{ \langle a, 0.4, 0.1 \rangle, \langle b, 0.6, 0.2 \rangle, \langle c, 0.5, 0.3 \rangle \}, \\
 G_2 &= \{ \langle a, 0.4, 0.4 \rangle, \langle b, 0.4, 0.4 \rangle, \langle c, 0.4, 0.4 \rangle \}.
 \end{aligned}
 \tag{2.6}$$

We define an IFTs $\tau_1, \tau_2 : \zeta^X \rightarrow I \times I$ as follows:

$$\tau_1(A) = \begin{cases} 1^\sim & \text{if } A \in \{0^\sim, 1^\sim\}, \\ \langle 0.5, 0.2 \rangle & \text{if } A = G_1, \\ \langle 0.5, 0.3 \rangle & \text{if } A = G_2, \\ 0^\sim & \text{otherwise,} \end{cases} \quad (2.7)$$

$$\tau_2(A) = \begin{cases} 1^\sim & \text{if } A \in \{0^\sim, 1^\sim\}, \\ \langle 0.6, 0.2 \rangle & \text{if } A = G_2, \\ 0^\sim & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.4, \beta = 0.5$. Then, the identity mapping $\text{id}_X : (X, \tau_1) \rightarrow (X, \tau_2)$ is (α, β) -intuitionistic fuzzy almost continuous, but not intuitionistic fuzzy continuous.

Example 2.8. Let $X = \{a, b\}, Y = \{1, 2\}$. Let G_1 be an IFS of X and G_2 be an IFS of Y , defined as follows:

$$\begin{aligned} G_1 &= \{\langle a, 0.4, 0.4 \rangle, \langle b, 0.4, 0.4 \rangle\}, \\ G_2 &= \{\langle 1, 0.4, 0.4 \rangle, \langle 2, 0.5, 0.4 \rangle\}. \end{aligned} \quad (2.8)$$

We define an IFTs $\tau_1 : \zeta^X \rightarrow I \times I$ and $\tau_2 : \zeta^Y \rightarrow I \times I$ as follows:

$$\tau_1(A) = \begin{cases} 1^\sim & \text{if } A \in \{0^\sim, 1^\sim\}, \\ \langle 0.7, 0.1 \rangle & \text{if } A = G_1, \\ 0^\sim & \text{otherwise,} \end{cases} \quad (2.9)$$

$$\tau_2(A) = \begin{cases} 1^\sim & \text{if } A \in \{0^\sim, 1^\sim\}, \\ \langle 0.8, 0.1 \rangle & \text{if } A = G_2, \\ 0^\sim & \text{otherwise.} \end{cases}$$

Consider the mapping $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ defined by

$$f(a) = 1, \quad f(b) = 1. \quad (2.10)$$

Let $\alpha = 0.6, \beta = 0.3$. Then, f is (α, β) -intuitionistic fuzzy weakly continuous, but not intuitionistic fuzzy continuous.

Example 2.9. In the above example, if

$$\tau_1(A) = \begin{cases} 1^\sim & \text{if } A \in \{0^\sim, 1^\sim\}, \\ \langle 0.8, 0.2 \rangle & \text{if } A = G_1, \\ 0^\sim & \text{otherwise,} \end{cases} \quad (2.11)$$

then f is intuitionistic fuzzy continuous, but not intuitionistic fuzzy strong continuous.

THEOREM 2.10. *Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a mapping from an IFTS (X, τ_1) to another IFTS (Y, τ_2) . Then, the following statements are equivalent:*

- (i) f is (α, β) -intuitionistic fuzzy almost continuous;
- (ii) $\tau_1^*(f^{-1}(B)) \geq \langle \alpha, \beta \rangle$, for each (α, β) -IFRC set B of Y ;
- (iii) $f^{-1}(B) \subseteq \text{int}_{\alpha, \beta} f^{-1}(\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(B)))$, for each $B \in \zeta^Y$ such that $\tau_2(B) \geq \langle \alpha, \beta \rangle$;
- (iv) $\text{cl}_{\alpha, \beta} f^{-1}(\text{cl}_{\alpha, \beta}(\text{int}_{\alpha, \beta}(B))) \subseteq f^{-1}(B)$, for each $B \in \zeta^Y$ such that $\tau_2(B) \geq \langle \alpha, \beta \rangle$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.

Proof. (i) \Rightarrow (ii). Let B be (α, β) -IFRC set of Y . Then, by Theorem 2.2, \bar{B} is (α, β) -IFRO set. By, (i), we have $\tau_1(f^{-1}(\bar{B})) = \tau_1(\overline{f^{-1}(B)}) = \tau_1^*(f^{-1}(B)) \geq \langle \alpha, \beta \rangle$.

(ii) \Rightarrow (i). It is analogous to the proof of (ii) \Rightarrow (i).

(i) \Rightarrow (iii). Since $\tau_2(B) \geq \langle \alpha, \beta \rangle$, then $B = \text{int}_{\alpha, \beta}(B) \subseteq \text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(B))$ and hence, $f^{-1}(B) \subseteq f^{-1}(\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(B)))$ since, $\tau_2^*(\text{cl}_{\alpha, \beta}(B)) \geq \langle \alpha, \beta \rangle$, then by Theorem 2.4, $\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(B))$ is (α, β) -IFRO set. So, $\tau_1(f^{-1}(\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(B)))) \geq \langle \alpha, \beta \rangle$. Then, $f^{-1}(B) \subseteq f^{-1}(\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta}(B))) = \text{int}_{\alpha, \beta}(f^{-1}(\text{cl}_{\alpha, \beta}(B)))$.

(iii) \Rightarrow (i). Let B be (α, β) -IFRO set of Y . Then, we have

$$f^{-1}(B) \subseteq \text{int}_{\alpha, \beta}(f^{-1}(\text{cl}_{\alpha, \beta}(B))) = \text{int}_{\alpha, \beta}(f^{-1}(B)); \tag{2.12}$$

this implies that $f^{-1}(B) = \text{int}_{\alpha, \beta}(f^{-1}(B))$, then

$$\tau_1(f^{-1}(B)) = \tau_1(\text{int}_{\alpha, \beta}(f^{-1}(B))) \geq \langle \alpha, \beta \rangle. \tag{2.13}$$

Hence, f is (α, β) -intuitionistic fuzzy almost continuous.

(ii) \Leftrightarrow (iv). Can similarly be proved. □

THEOREM 2.11. *Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a mapping from an IFTS (X, τ_1) to another IFTS (Y, τ_2) . Then, the following are equivalent:*

- (i) f is (α, β) -intuitionistic fuzzy weakly continuous;
- (ii) $f(\text{cl}_{\alpha, \beta}(A)) \subseteq \text{cl}_{\alpha, \beta}(f(A))$ for each $A \in \zeta^X$.

Proof. (i) \Rightarrow (ii). Let $A \in \zeta^X$. Then,

$$\begin{aligned} & f^{-1}(\text{cl}_{\alpha, \beta}(f(A))) \\ &= f^{-1}[\cap \{k \in \zeta^Y : \tau_2^*(k) \geq \langle \alpha, \beta \rangle, k \supseteq f(A)\}] \\ &= f^{-1}[\cap \{k \in \zeta^Y : \tau_2(\bar{k}) \geq \langle \alpha, \beta \rangle, k \supseteq f(A)\}] \\ &\supseteq f^{-1}[\cap \{k \in \zeta^Y : \tau_1^*(f^{-1}(k)) = \tau_1(\overline{f^{-1}(k)}) \geq \langle \alpha, \beta \rangle, k \supseteq f(A)\}] \\ &\supseteq \cap \{f^{-1}(k) : k \in \zeta^Y : \tau_1^*(f^{-1}(k)) \geq \langle \alpha, \beta \rangle, f^{-1}(k) \supseteq A\} \\ &\supseteq \cap \{G \in \zeta^X : \tau_1^*(G) \geq \langle \alpha, \beta \rangle, G \supseteq A\} = \text{cl}_{\alpha, \beta}(A). \end{aligned} \tag{2.14}$$

Then, $f(\text{cl}_{\alpha, \beta}(A)) \subseteq f(f^{-1}(\text{cl}_{\alpha, \beta}(f(A)))) \subseteq \text{cl}_{\alpha, \beta}(f(A))$.

(ii) \Rightarrow (i). Let $B \in \zeta^Y$ such that $\tau_2(B) \geq \langle \alpha, \beta \rangle$. Then, $\tau_2^*(\bar{B}) = \tau_2(B) \geq \langle \alpha, \beta \rangle$. So, we have $\text{cl}_{\alpha, \beta}(\bar{B}) = \bar{B}$. Further, since $f(\text{cl}_{\alpha, \beta}(f^{-1}(\bar{B}))) \subseteq \text{cl}_{\alpha, \beta}(f(f^{-1}(\bar{B}))) \subseteq \text{cl}_{\alpha, \beta}(\bar{B}) = \bar{B}$, we have

$\text{cl}_{\alpha,\beta}(f^{-1}(\overline{B})) \subseteq f^{-1}(\overline{B})$. Then, $\text{cl}_{\alpha,\beta}(f^{-1}(\overline{B})) = f^{-1}(\overline{B})$. This implies that $\tau_1^*(f^{-1}(\overline{B})) \geq \langle \alpha, \beta \rangle$, therefore, $\tau_1^*(f^{-1}(\overline{B})) = \tau_1(f^{-1}(\overline{B})) \geq \langle \alpha, \beta \rangle$. Hence, f is (α, β) -intuitionistic fuzzy weakly continuous. \square

THEOREM 2.12. *Let $f : X \rightarrow Y$ be an intuitionistic fuzzy continuous mapping with respect to the IFTs τ_1 and τ_2 respectively. Then for every IFS A in X ,*

$$f(\text{cl}_{\alpha,\beta}(A)) \subseteq \text{cl}_{\alpha,\beta}(f(A)), \quad (2.15)$$

where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.

Proof. Let $f : X \rightarrow Y$ be an intuitionistic fuzzy continuous mapping with respect to τ_1 and τ_2 , and let $A \in \zeta^X$. Then,

$$\begin{aligned} f^{-1}(\text{cl}_{\alpha,\beta}(f(A))) &= f^{-1}(\cap \{K \in \zeta^Y, \tau_2^*(K) \geq \langle \alpha, \beta \rangle, f(A) \subseteq K\}) \\ &= \cap \{f^{-1}(K) : K \in \zeta^Y, \tau_2^*(K) \geq \langle \alpha, \beta \rangle, A \subseteq f^{-1}(K)\} \\ &\supseteq \cap \{f^{-1}(K) : K \in \zeta^Y, \tau_1^*(f^{-1}(K)) \geq \langle \alpha, \beta \rangle, A \subseteq f^{-1}(K)\} \\ &\supseteq \cap \{G \in \zeta^X : \tau_1^*(G) \geq \langle \alpha, \beta \rangle, A \subseteq G\} = \text{cl}_{\alpha,\beta}(A). \end{aligned} \quad (2.16)$$

This implies that $f(\text{cl}_{\alpha,\beta}(A)) \subseteq \text{cl}_{\alpha,\beta}(f(A))$. \square

THEOREM 2.13. *Let $f : X \rightarrow Y$ be an intuitionistic fuzzy continuous mapping with respect to the IFTs τ_1 and τ_2 , respectively. Then, for every IFS A in Y ,*

$$\text{cl}_{\alpha,\beta}(f^{-1}(A)) \subseteq f^{-1}(\text{cl}_{\alpha,\beta}(A)), \quad (2.17)$$

where, $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.

Proof. Let $A \in \zeta^Y$. We get from Theorem 2.12

$$\text{cl}_{\alpha,\beta}(f^{-1}(A)) \subseteq f^{-1}(f(\text{cl}_{\alpha,\beta}(f^{-1}(A)))) \subseteq f^{-1}(\text{cl}_{\alpha,\beta}(A)). \quad (2.18)$$

Hence, $\text{cl}_{\alpha,\beta}(f^{-1}(A)) \subseteq f^{-1}(\text{cl}_{\alpha,\beta}(A))$, for every $A \in \zeta^Y$. \square

3. Various cases of compactness in intuitionistic fuzzy topological spaces

Definition 3.1. An IFTS (X, τ) is called (α, β) -intuitionistic fuzzy compact (resp., (α, β) -intuitionistic fuzzy nearly compact and (α, β) -intuitionistic fuzzy almost compact) if and only if for every family $\{G_i : i \in J\}$ in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$ such that $\cup_{i \in J} G_i = 1_{\sim}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} G_i = 1_{\sim}$ (resp., $\cup_{i \in J_0} \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(G_i)) = 1_{\sim}$ and $\cup_{i \in J_0} \text{cl}_{\alpha,\beta}(G_i) = 1_{\sim}$).

Definition 3.2. Let (X, τ) be an IFTS and A an IFS in X . A is said to be (α, β) -intuitionistic fuzzy compact if and only if every family $\{G_i : i \in J\}$ in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$ such that $A \subseteq \cup_{i \in J} G_i$, there exists a finite subset J_0 of J such that $A \subseteq \cup_{i \in J_0} G_i$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.

Example 3.3. Let $X = I$ and consider the IFSs $\{G_n : n = 2, 3, 4, \dots\}$ as follows: first we define IFSs $G_n = \langle x, \mu_{G_n}, \nu_{G_n} \rangle$ and $G = \langle x, \mu_G, \nu_G \rangle$ by

$$\begin{aligned} \mu_{G_n}(x) &= \begin{cases} 0.8, & x = 0, \\ nx, & 0 < x \leq \frac{1}{n}, \\ 1, & \frac{1}{n} < x \leq 1, \end{cases} \\ \nu_{G_n}(x) &= \begin{cases} 0.1, & x = 0, \\ 1 - nx, & 0 < x \leq \frac{1}{n}, \\ 0, & \frac{1}{n} < x \leq 1, \end{cases} \\ \mu_G(x) &= \begin{cases} 0.8, & x = 0, \\ 1, & \text{otherwise,} \end{cases} \\ \nu_G(x) &= \begin{cases} 0.1, & x = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1}$$

Second, we define the IFT $\tau : \zeta^X \rightarrow I \times I$ as follows:

$$\tau(A) = \begin{cases} 1\sim & \text{if } A \in \{0\sim, 1\sim\}, \\ \left\langle \frac{1}{n}, \frac{1}{2n} \right\rangle & \text{if } A = G_n, \\ \langle 0.7, 0.2 \rangle & \text{if } A = G, \\ 0\sim & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $\alpha = 0.6, \beta = 0.2$. Then, the IFS $C_{0.85,0.15} = \{\langle x, 0.85, 0.15 \rangle : x \in X\}$ is (α, β) -intuitionistic fuzzy compact and the IFS $C_{0.75,0.15} = \{\langle x, 0.75, 0.15 \rangle : x \in X\}$ is not (α, β) -intuitionistic fuzzy compact.

THEOREM 3.4. *For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, (α, β) -intuitionistic fuzzy compactness implies (α, β) -intuitionistic fuzzy nearly compactness which implies (α, β) -intuitionistic fuzzy almost compactness.*

Proof. Let an IFTS (X, τ) be (α, β) -intuitionistic fuzzy compact. Then, for every family $\{G_i : i \in J\}$ in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\cup_{i \in J} G_i = 1\sim$, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} G_i = 1\sim$. Now, since $\tau(G_i) > \langle \alpha, \beta \rangle$ for each $i \in J$, then $G_i = \text{int}_{\alpha, \beta} G_i$ for each $i \in J$. Also, $G_i = \text{int}_{\alpha, \beta} G_i \subseteq \text{int}_{\alpha, \beta} (\text{cl}_{\alpha, \beta} G_i)$ for each $i \in J$. Then, $1\sim = \cup_{i \in J_0} G_i = \cup_{i \in J_0} \text{int}_{\alpha, \beta} G_i \subseteq \cup_{i \in J_0} \text{int}_{\alpha, \beta} (\text{cl}_{\alpha, \beta} G_i)$. Thus, $\cup_{i \in J_0} \text{int}_{\alpha, \beta} (\text{cl}_{\alpha, \beta} G_i) = 1\sim$. Hence, an IFTS (X, τ) is (α, β) -intuitionistic fuzzy nearly compact. \square

For the second implication, suppose that the IFTS (X, τ) is (α, β) -intuitionistic fuzzy nearly compact, then for every family $\{G_i : i \in J\}$ in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} \text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} G_i) = 1_{\sim}$ since, $G_i = \text{int}_{\alpha, \beta} G_i \subseteq \text{int}_{\alpha, \beta} \text{cl}_{\alpha, \beta} G_i \subseteq \text{cl}_{\alpha, \beta} G_i$ for each $i \in J$ then, $1_{\sim} = \cup_{i \in J_0} \text{int}_{\alpha, \beta} \text{cl}_{\alpha, \beta} G_i \subseteq \cup_{i \in J_0} \text{cl}_{\alpha, \beta} G_i$. Thus, $\cup_{i \in J_0} \text{cl}_{\alpha, \beta} G_i = 1_{\sim}$. Hence, the IFTS (X, τ) is (α, β) -intuitionistic fuzzy almost compact.

Remark 3.5. In IFTS in Chang's sense, the converse of these two implications are not valid for compactness, nearly compactness, and almost compactness [9], which are special cases of compactness, nearly compactness and almost compactness, respectively in IFTS, in Šostak's sense. Thus, the converse implications in Theorem 3.4 are not true in general.

Definition 3.6. A family $\{K_i : i \in J\}$ in $\{K : K \in \zeta^X, \tau^*(K) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ has the finite intersection property (FIP) if and only if for any finite subset J_0 of J , $\cap_{i \in J_0} K_i \neq 0_{\sim}$.

THEOREM 3.7. An IFTS (X, τ) is (α, β) -intuitionistic fuzzy compact, if and only if every family in $\{K : K \in \zeta^X, \tau^*(K) > \langle \alpha, \beta \rangle\}$, where, $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ having the FIP, has a nonempty intersection.

Proof. Let an IFTS (X, τ) be (α, β) -intuitionistic fuzzy compact, and consider the family $\{K_i : i \in J\}$ in $\{K : K \in \zeta^X, \tau^*(K) > \langle \alpha, \beta \rangle\}$ having the FIP. Now suppose that $\cap_{i \in J} K_i = 0_{\sim}$ then, $\cup_{i \in J} \overline{K_i} = 1_{\sim}$, from $\tau(\overline{K_i}) = \tau^*(K_i) > \langle \alpha, \beta \rangle$ and (X, τ) is (α, β) -intuitionistic fuzzy compact, we have $\cup_{i \in J_0} \overline{K_i} = 1_{\sim}$; this implies that $\cap_{i \in J_0} K_i = 0_{\sim}$, which is a contradiction.

Conversely, let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\cup_{i \in J} G_i = 1_{\sim}$. If $\cup_{i \in J_0} G_i \neq 1_{\sim}$ for every finite subset J_0 of J , then $\cap_{i \in J_0} \overline{G_i} \neq 0_{\sim}$ and the family $\{\overline{G_i} : i \in J\}$ has the FIP and hence from the given condition, we have $\cap_{i \in J} \overline{G_i} \neq 0_{\sim}$ so, $\cup_{i \in J} G_i \neq 1_{\sim}$, a contradiction. \square

Definition 3.8. An IFTS (X, τ) is called (α, β) -intuitionistic fuzzy regular if and only if for each IFS A in X such that $\tau(A) > \langle \alpha, \beta \rangle$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, can be written as $A = \cup \{B : B \in \zeta^X, \tau(B) \geq \tau(A), \text{cl}_{\alpha, \beta}(B) \subseteq A\}$.

THEOREM 3.9. Let (X, τ) be an IFTS. If (X, τ) is (α, β) -intuitionistic fuzzy almost compact and (α, β) -intuitionistic fuzzy regular, then it is (α, β) -intuitionistic fuzzy compact.

Proof. Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\cup_{i \in J} G_i = 1_{\sim}$. From the fuzzy regularity of (X, τ) , it follows that $G_i = \cup \{B_i : B_i \in \zeta^X, \tau(B_i) \geq \tau(G_i), \text{cl}_{\alpha, \beta}(B_i) \subseteq G_i\}$. Since $\cup_{i \in J} G_i = \cup_{i \in J} B_i = 1_{\sim}$, $\tau(B_i) \geq \tau(G_i) > \langle \alpha, \beta \rangle$ then from almost compactness of (X, τ) there exists a finite subset J_0 of J such that $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(B_i) = 1_{\sim}$. But $\text{cl}_{\alpha, \beta}(B_i) \subseteq G_i$, this implies that $\cup_{i \in J_0} G_i \supseteq \cup_{i \in J_0} \text{cl}_{\alpha, \beta}(B_i) = 1_{\sim}$ that implies $\cup_{i \in J_0} G_i = 1_{\sim}$. Hence, (X, τ) is (α, β) -intuitionistic fuzzy compact. \square

THEOREM 3.10. Let (X, τ) be an IFTS. If (X, τ) is (α, β) -intuitionistic fuzzy nearly compact and (α, β) -intuitionistic fuzzy regular, then it is (α, β) -intuitionistic fuzzy compact.

Proof. Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\cup_{i \in J} G_i = 1_{\sim}$.

From fuzzy regularity of (X, τ) it follows that $G_i = \cup \{B_i : B_i \in \zeta^X, \tau(B_i) \geq \tau(G_i), \text{cl}_{\alpha, \beta}(B_i) \subseteq G_i\}$ then, $1_{\sim} = \cup_{i \in J} G_i = \cup_{i \in J} B_i, \tau(B_i) \geq \tau(G_i) > \langle \alpha, \beta \rangle$. Since (X, τ) is (α, β) -intuitionistic fuzzy nearly compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} \text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} G_i) = 1_{\sim}$. But, $\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} B_i) \subseteq \text{cl}_{\alpha, \beta} B_i \subseteq G_i$ then, $\cup_{i \in J_0} G_i = 1_{\sim}$.

Hence, (X, τ) is (α, β) -intuitionistic fuzzy compact. □

THEOREM 3.11. *An IFTS (X, τ) is (α, β) -intuitionistic fuzzy almost compact if and only if every family $\{G_i : i \in J\}$ in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ having the FIP, $\cap_{i \in J} \text{cl}_{\alpha, \beta} G_i \neq 0_{\sim}$.*

Proof. Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ having the FIP. Suppose that $\cap_{i \in J} \text{cl}_{\alpha, \beta} G_i = 0_{\sim}$, then we have $\cup_{i \in J} \overline{\text{cl}_{\alpha, \beta} G_i} = \cup_{i \in J} \text{int}_{\alpha, \beta} \overline{G_i} = 1_{\sim}$. Since $\tau(\text{int}_{\alpha, \beta} \overline{G_i}) \geq \langle \alpha, \beta \rangle$ and X is (α, β) -intuitionistic fuzzy almost compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(\text{int}_{\alpha, \beta} \overline{G_i}) = 1_{\sim}$ this implies that $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(\text{int}_{\alpha, \beta} \overline{G_i}) = \cup_{i \in J_0} \text{cl}_{\alpha, \beta}(\overline{\text{cl}_{\alpha, \beta} G_i}) = \cup_{i \in J_0} \text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} G_i) = 1_{\sim}$. Thus, $\cap_{i \in J_0} \text{int}_{\alpha, \beta} \text{cl}_{\alpha, \beta} G_i = 0_{\sim}$, but from $G_i = \text{int}_{\alpha, \beta} G_i \subseteq \text{int}_{\alpha, \beta} \text{cl}_{\alpha, \beta} G_i$, we see that $\cap_{i \in J_0} G_i = 0_{\sim}$, which is a contradiction with the FIP of the family.

Conversely, let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\cup_{i \in J} G_i = 1_{\sim}$. Suppose that there exists no finite subset J_0 of J such that $\cup_{i \in J_0} \text{cl}_{\alpha, \beta} G_i = 1_{\sim}$. Since $\tau(\overline{\text{cl}_{\alpha, \beta} G_i}) = \tau^*(\text{cl}_{\alpha, \beta} G_i) \geq \langle \alpha, \beta \rangle$, then by our hypothesis, the family $\{\overline{\text{cl}_{\alpha, \beta} G_i} : i \in J\}$ has the FIP. So, we have

$$\begin{aligned} \cap_{i \in J} \overline{\text{cl}_{\alpha, \beta} (\overline{\text{cl}_{\alpha, \beta} G_i})} &\neq 0_{\sim} \\ \implies \cup_{i \in J} \overline{\text{cl}_{\alpha, \beta} (\overline{\text{cl}_{\alpha, \beta} G_i})} &\neq 1_{\sim} \\ \implies \cup_{i \in J} \text{int}_{\alpha, \beta} (\overline{\text{cl}_{\alpha, \beta} G_i}) &\neq 1_{\sim}, \end{aligned} \tag{3.3}$$

since $G_i \subseteq \text{int}_{\alpha, \beta}(\overline{\text{cl}_{\alpha, \beta} G_i})$, for each $i \in J$ then, $\cup_{i \in J} G_i \neq 1_{\sim}$, which is a contradiction with $\cup_{i \in J} G_i = 1_{\sim}$. □

LEMMA 3.12. *Let (X, τ) be an IFTS and $V \in \zeta^X$. Then, $x_{r,s} \in \text{cl}_{\alpha, \beta} V$ if and only if for each $U \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta \rangle$ and $x_{r,s} q U, U q V$, where $r, \alpha \in I_0, s, \beta \in I_1$ with $r + s, \alpha + \beta \leq 1$.*

Proof. Let $x_{r,s} \in \text{cl}_{\alpha, \beta} V$ and let U be any IFS in X such that $\tau(U) \geq \langle \alpha, \beta \rangle$ and $x_{r,s} q U$. Suppose for a contradiction that $V \not q U$. Then, we have $V \subseteq \overline{U}$. Since $x_{r,s} q U$, then $x_{r,s} \notin \overline{U} \supseteq V$, and since $\tau^*(\overline{U}) = \tau(U) \geq \langle \alpha, \beta \rangle$, then $x_{r,s} \notin \text{cl}_{\alpha, \beta} V$, which is a contradiction, then $V q U$. Conversely, suppose that for any $U \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta \rangle$ such that $x_{r,s} q U$, we have $U q V$. Suppose, for a contradiction, that $x_{r,s} \notin \text{cl}_{\alpha, \beta} V$. Then, there exists $B \in \zeta^X$ with $\tau^*(B) \geq \langle \alpha, \beta \rangle, B \supseteq V$ and $x_{r,s} \notin B$. Thus, $\tau(\overline{B}) = \tau^*(B) \geq \langle \alpha, \beta \rangle$ and $x_{r,s} q \overline{B}$. Then, from our hypotheses $V q \overline{B}$, which implies that $V \not\subseteq B$; this is a contradiction. Hence, $x_{r,s} \in \text{cl}_{\alpha, \beta} V$. □

LEMMA 3.13. *Let (X, τ) be an IFTS. For $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1, \text{cl}_{\alpha, \beta}(\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} A)) = \text{cl}_{\alpha, \beta} A$, for each $A \in \zeta^X$ with $\tau(A) \geq \langle \alpha, \beta \rangle$.*

Proof. Let $A \in \zeta^X$ with $\tau(A) \geq \langle \alpha, \beta \rangle$. $A \subseteq \text{cl}_{\alpha, \beta} A$ implies $A \subseteq \text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} A)$. Then,

$$\text{cl}_{\alpha, \beta} A \subseteq \text{cl}_{\alpha, \beta} (\text{int}_{\alpha, \beta} (\text{cl}_{\alpha, \beta} A)). \tag{3.4}$$

Let $x_{t,s} \notin \text{cl}_{\alpha,\beta} A$, where $r \in I_0, s \in I_1$ with $r + s \leq 1$. Then by Lemma 3.12, there exists $U \in \zeta^X$ with $\tau(U) \geq \langle \alpha, \beta \rangle$ such that $x_{t,s} q U$ and $A \bar{q} U$. From $A \bar{q} U$, it follows that $A \subseteq \bar{U}$, so using the fact that $\tau^*(\bar{U}) = \tau(U) \geq \langle \alpha, \beta \rangle$, we obtain that $\text{cl}_{\alpha,\beta} A \subseteq \bar{U}$. Thus $\text{cl}_{\alpha,\beta} A \bar{q} U$. This implies that $\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} A) \bar{q} U$. Since $\tau(U) \geq \langle \alpha, \beta \rangle$ and $x_{t,s} q U$, then by Lemma 3.12, we have $x_{t,s} \notin \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} A))$. Thus

$$\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} A)) \subseteq \text{cl}_{\alpha,\beta} A. \quad (3.5)$$

From (3.4) and (3.5) we have

$$\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} A)) = \text{cl}_{\alpha,\beta} A. \quad (3.6)$$

□

THEOREM 3.14. *In an IFTS (X, τ) the following conditions are equivalent.*

- (i) (X, τ) is (α, β) -intuitionistic fuzzy almost compact.
- (ii) For every family $G = \{G_i : i \in J\}$, where $G_i = \{\langle x, \mu_{G_i}, \nu_{G_i} \rangle : i \in J\}$ of (α, β) -IFRC sets such that $\bigcap_{i \in J} G_i = 0_{\sim}$, there exists a finite subset J_0 of J such that $\bigcap_{i \in J_0} \text{int}_{\alpha,\beta} G_i = 0_{\sim}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.
- (iii) $\bigcap_{i \in J} \text{cl}_{\alpha,\beta} G_i \neq 0_{\sim}$ holds for every family $\{G_i \in \zeta^X : i \in J\}$ of (α, β) -IFRO sets having the FIP, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.
- (iv) For every family $\{G_i \in \zeta^X : i \in J\}$ of (α, β) -IFRO sets such that $\bigcup_{i \in J} G_i = 1_{\sim}$, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} \text{cl}_{\alpha,\beta} G_i = 1_{\sim}$.

Proof. (i) \Rightarrow (ii). Let $G = \{G_i : i \in J\}$ be a family of (α, β) -IFRC sets in X with $\bigcap_{i \in J} G_i = 0_{\sim}$. Then, $\bigcup_{i \in J} \bar{G}_i = 1_{\sim}$. By, Theorem 2.2, \bar{G}_i is (α, β) -IFRO set then, $\bigcup_{i \in J} \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} \bar{G}_i) = 1_{\sim}$. Since $\tau(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} \bar{G}_i)) \geq \langle \alpha, \beta \rangle$ and (X, τ) is (α, β) -intuitionistic fuzzy almost compact then, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} \bar{G}_i)) = 1_{\sim}$, this implies that $\bigcup_{i \in J_0} \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} \bar{G}_i)) = \bigcap_{i \in J_0} \text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} \bar{G}_i)) = \bigcap_{i \in J_0} \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(\text{int}_{\alpha,\beta} \bar{G}_i)) = \bigcap_{i \in J_0} \text{int}_{\alpha,\beta} G_i = 0_{\sim}$.

(ii) \Rightarrow (iii). Let $\{G_i \in \zeta^X : i \in J\}$ be (α, β) -IFRO sets having the FIP and suppose that $\bigcap_{i \in J} \text{cl}_{\alpha,\beta} G_i = 0_{\sim}$. Since G_i is (α, β) -IFRO set, then by Theorem 2.2, $\tau(G_i) \geq \langle \alpha, \beta \rangle$ and by Theorem 2.4, we have $\{\text{cl}_{\alpha,\beta} G_i : i \in J\}$ is a family of (α, β) -IFRC sets then, by (ii), there exists a subset J_0 of J such that $\bigcap_{i \in J_0} \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} G_i) = \bigcap_{i \in J_0} G_i = 0_{\sim}$, which is a contradiction.

(iii) \Rightarrow (iv). Let $\{G_i \in \zeta^X : i \in J\}$ be a family of (α, β) -IFRO sets such that $\bigcup_{i \in J} G_i = 1_{\sim}$. Suppose that for every finite subset J_0 of J , $\bigcup_{i \in J_0} \text{cl}_{\alpha,\beta} G_i \neq 1_{\sim}$. Then, by Theorems 2.2 and 2.4, we have $\{\overline{\text{cl}_{\alpha,\beta}(G_i)} : i \in J\}$ is a family of (α, β) -IFRO sets having the FIP. Since, $\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(\overline{\text{cl}_{\alpha,\beta}(G_i)})) = \text{int}_{\alpha,\beta}(\overline{\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta}(G_i))}) = \overline{\text{int}_{\alpha,\beta} G_i} = \text{cl}_{\alpha,\beta} \bar{G}_i$, hence by (iii) $\bigcap_{i \in J} \text{cl}_{\alpha,\beta}(\overline{\text{cl}_{\alpha,\beta} G_i}) \neq 0_{\sim}$ implies $\bigcap_{i \in J} \overline{\text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} G_i)} \neq 0_{\sim}$ which implies $\bigcap_{i \in J} \bar{G}_i \neq 0_{\sim}$ which implies $\bigcup_{i \in J} G_i \neq 1_{\sim}$ which is a contradiction with $\bigcup_{i \in J} G_i = 1_{\sim}$.

(iv) \Rightarrow (i). Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\bigcup_{i \in J} G_i = 1_{\sim}$

$$G_i \subseteq \text{cl}_{\alpha,\beta} G_i \implies G_i \subseteq \text{int}_{\alpha,\beta}(\text{cl}_{\alpha,\beta} G_i). \quad (3.7)$$

Thus, $\cup_{i \in J} \text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} G_i) = 1_{\sim}$. Since $\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} G_i)$ is (α, β) -IFRO, then by (iv), there exists a finite subset J_0 of J such that $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(\text{int}_{\alpha, \beta}(\text{cl}_{\alpha, \beta} G_i)) = 1_{\sim}$. By using Lemma 3.13, we have, $\cup_{i \in J_0} \text{cl}_{\alpha, \beta} G_i = 1_{\sim}$. Hence, (X, τ) is (α, β) -intuitionistic fuzzy almost compact. \square

THEOREM 3.15. *Let (X, τ_1) , (Y, τ_2) be two IFTSs and $f : X \rightarrow Y$ an intuitionistic fuzzy continuous mapping. If A is (α, β) -intuitionistic fuzzy compact in (X, τ_1) then, so is $f(A)$ in (Y, τ_2) , where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.*

Proof. Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^Y, \tau_2(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $f(A) \subseteq \cup_{i \in J} G_i$. Then, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\cup_{i \in J} G_i) = \cup_{i \in J} f^{-1}(G_i)$. Since f is an intuitionistic fuzzy continuous, then $\tau_1(f^{-1}(G_i)) \geq \tau_2(G_i) > \langle \alpha, \beta \rangle$ and since A is (α, β) -intuitionistic fuzzy compact in (X, τ_1) , there exists a finite subset J_0 of J such that $A \subseteq \cup_{i \in J_0} f^{-1}(G_i)$; this implies that $f(A) \subseteq f(\cup_{i \in J_0} f^{-1}(G_i)) = \cup_{i \in J_0} f(f^{-1}(G_i)) \subseteq \cup_{i \in J_0} G_i$. Hence, $f(A)$ is (α, β) -intuitionistic fuzzy compact in (Y, τ_2) . \square

THEOREM 3.16. *Let (X, τ_1) , (Y, τ_2) be two IFTSs and $f : X \rightarrow Y$ a surjection intuitionistic fuzzy continuous. If (X, τ_1) is (α, β) -intuitionistic fuzzy compact, then so is (Y, τ_2) , where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.*

The proof is similar to that of Theorem 3.15.

THEOREM 3.17. *Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a surjective intuitionistic fuzzy continuous mapping with respect to τ_1 and τ_2 , respectively. If (X, τ_1) is (α, β) -intuitionistic fuzzy almost compact, then so is (Y, τ_2) , where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.*

Proof. Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^Y, \tau_2(G) > \langle \alpha, \beta \rangle\}$ such that $\cup_{i \in J} G_i = 1_{\sim}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$. Since f is intuitionistic fuzzy continuous, we have $\tau_1(f^{-1}(G_i)) \geq \tau_2(G_i) > \langle \alpha, \beta \rangle$ but, $\cup_{i \in J} f^{-1}(G_i) = f^{-1}(\cup_{i \in J} G_i) = 1_{\sim}$.

Then from the almost compactness of (X, τ_1) , there exists a subset J_0 of J such that $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(f^{-1}(G_i)) = 1_{\sim}$, this implies that

$$f(\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(f^{-1}(G_i))) = \cup_{i \in J_0} f(\text{cl}_{\alpha, \beta}(f^{-1}(G_i))) = 1_{\sim}. \tag{3.8}$$

But from Theorem 2.13, we have $f^{-1}(\text{cl}_{\alpha, \beta}(G_i)) \supseteq \text{cl}_{\alpha, \beta}(f^{-1}(G_i))$ and, from the surjectivity of f , we have $\text{cl}_{\alpha, \beta}(G_i) = f(f^{-1}(\text{cl}_{\alpha, \beta}(G_i))) \supseteq f(\text{cl}_{\alpha, \beta}(f^{-1}(G_i)))$. So, $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(G_i) \supseteq \cup_{i \in J_0} f(\text{cl}_{\alpha, \beta}(f^{-1}(G_i))) = 1_{\sim}$. Then, $\cup_{i \in J_0} \text{cl}_{\alpha, \beta}(G_i) = 1_{\sim}$. Hence, (Y, τ_2) is (α, β) -intuitionistic fuzzy almost compact. \square

THEOREM 3.18. *Let (X, τ_1) and (Y, τ_2) be IFTSs and let $f : X \rightarrow Y$ be (α, β) -intuitionistic fuzzy weakly continuous surjection mapping. If (X, τ_1) is (α, β) -intuitionistic fuzzy compact, then (Y, τ_2) is (α, β) -intuitionistic fuzzy almost compact.*

Proof. Let $\{G_i : i \in J\}$ be a family in $\{G : G \in \zeta^Y, \tau_2(G) > \langle \alpha, \beta \rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ such that $\cup_{i \in J} G_i = 1_{\sim}$. Since f is (α, β) -intuitionistic fuzzy weakly continuous, we have $\tau_1(f^{-1}(G_i)) > \langle \alpha, \beta \rangle$ then, $\cup_{i \in J} (f^{-1}(G_i)) = f^{-1}(\cup_{i \in J} G_i) = f^{-1}(1_{\sim}) = 1_{\sim}$. Since (X, τ_1) is (α, β) -intuitionistic fuzzy compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} f^{-1}(G_i) = 1_{\sim}$ but, $f^{-1}(\text{cl}_{\alpha, \beta} G_i) \supseteq f^{-1}(G_i)$, this implies that $\cup_{i \in J_0} f^{-1}(\text{cl}_{\alpha, \beta} G_i) \supseteq \cup_{i \in J_0} f^{-1}(G_i) = 1_{\sim}$.

Then, $\cup_{i \in J_0} f^{-1}(cl_{\alpha, \beta} G_i) = 1_{\sim}$. Since f is surjective,

$$f(\cup_{i \in J_0} f^{-1}(cl_{\alpha, \beta} G_i)) = \cup_{i \in J_0} f(f^{-1}(cl_{\alpha, \beta} G_i)) = \cup_{i \in J_0} cl_{\alpha, \beta} G_i = 1_{\sim}. \quad (3.9)$$

Thus, $\cup_{i \in J_0} cl_{\alpha, \beta} G_i = 1_{\sim}$. Hence, (Y, τ_2) is (α, β) -intuitionistic fuzzy almost compact. \square

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