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ON STARLIKENESS AND CLOSE-TO-CONVEXITY OF CERTAIN ANALYTIC FUNCTIONS

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Our purpose is to derive some sufficient conditions for starlikeness and close-to-convexity of order α of certain analytic functions in the open unit disk.

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1. Introduction. Let A_n be the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f \in A_n$ is said to be in the class $S_n^*(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

for some α $(0 \le \alpha < 1)$. A function in the class $S_n^*(\alpha)$ is starlike of order α in U. We also write $A_1 = A$ and $S_1^*(\alpha) = S^*(\alpha)$.

Let $C_n(\alpha)$ be the subclass of A_n consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\{f'(z)\} > \alpha \quad (z \in U) \tag{1.3}$$

for some α $(0 \le \alpha < 1)$. A function $f(z)$ in $C_n(\alpha)$ is close-to-convex of order α in U (cf. Duren [1]).

Let $f(z)$ and $g(z)$ be analytic in U. Then the function $f(z)$ is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ for $z \in U$. If $g(z)$ is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $H(p(z), zp'(z)) \prec h(z)$ be a first-order differential subordination. Then a univalent function $q(z)$ is called its dominant if $p(z) \prec q(z)$ for all analytic functions $p(z)$ that satisfy the differential subordination. A dominant $\bar{q}(z)$ is called the best dominant if $\bar{q}(z) \prec q(z)$ for all dominants $q(z)$. For the general theory of first-order differential subordination and its applications, we refer to [3].

Recently, Xu and Yang [5] obtained some results on starlikeness and close-to-convexity of certain meromorphic functions. In the present note, we investigate some sufficient conditions for starlikeness and close-to-convexity of order α of certain analytic functions in U by using the subordination principle, and obtain some useful corollaries as special cases. Furthermore, we extend the results given by Owa et al. [4].

2. Main results.

To derive our results, we need the following lemmas.

Lemma 2.1 [6]. *Let $g(z) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ $(n \in \mathbb{N})$ be analytic in U and let $h(z)$ be analytic and starlike (with respect to the origin), univalent in U with $h(0) = 0$. If $zg'(z) \prec h(z)$ $(z \in U)$, then*

$$g(z) \prec b_0 + \frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt. \tag{2.1}$$

Lemma 2.2 [3]. *Let $g(z)$ be analytic and univalent in U and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$. Set*

$$Q(z) = zg'(z)\varphi(g(z)), \qquad h(z) = \theta(g(z)) + Q(z) \tag{2.2}$$

and suppose that

(i) *$Q(z)$ is univalent and starlike in U;*

(ii) *$\operatorname{Re}\{zh'(z)/Q(z)\} = \operatorname{Re}\{\theta'(g(z))/\varphi(g(z)) + zQ'(z)/Q(z)\} > 0$ $(z \in U)$.*

If $p(z)$ is analytic in U, with $p(0) = g(0)$, $p(U) \subset D$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \tag{2.3}$$

then $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (2.3).

Lemma 2.3 [2]. *Let $g(z) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ $(n \in \mathbb{N})$ be analytic in U with $g(z) \not\equiv b_0$. If $0 < |z_0| < 1$ and $\operatorname{Re}\{g(z_0)\} = \min_{|z| \le |z_0|} \operatorname{Re}\{g(z)\}$, then*

$$z_0 g'(z_0) \le -\frac{n|b_0 - g(z_0)|^2}{2\operatorname{Re}\{b_0 - g(z_0)\}}. \tag{2.4}$$

Applying Lemma 2.1, we now derive the following.

Theorem 2.4. *Let $f \in A_n$ satisfy $f(z)f'(z) \neq 0$ for $z \in U\backslash\{0\}$ and*

$$-\alpha\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + \alpha \prec \frac{az}{1-bz} \quad (z \in U), \tag{2.5}$$

where α, a, and b are real numbers with $a \neq 0$ and $b \le 1$.

(i) *If $0 < a \le n$ and $0 < b \le 1$, then*

$$\operatorname{Re}\left\{\frac{z^\alpha f'(z)}{f^\alpha(z)}\right\} > \left(\frac{1}{1+b}\right)^{a/nb} \quad (z \in U). \tag{2.6}$$

(ii) *If $0 < a \le n$ and $b = 0$, then*

$$\operatorname{Re}\left\{\frac{z^\alpha f'(z)}{f^\alpha(z)}\right\} > e^{-a/n} \quad (z \in U). \tag{2.7}$$

(iii) *If* $a \neq 0$ *and* $0 < b \leq 1$, *then*

$$\left| \left(\frac{z^\alpha f'(z)}{f^\alpha(z)} \right)^{-nb/a} - 1 \right| < b \quad (z \in U). \tag{2.8}$$

(iv) *If* $a > 0$ *and* $b = 0$, *then*

$$\left| \frac{z^\alpha f'(z)}{f^\alpha(z)} - 1 \right| < e^{a/n} - 1 \quad (z \in U). \tag{2.9}$$

Proof. Let $f \in A_n$ with $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and define

$$g(z) = -\alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) + \frac{zf''(z)}{f'(z)}. \tag{2.10}$$

Then $g(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic in U and (2.5) can be rewritten as

$$g(z) \prec h(z), \tag{2.11}$$

where $h(z) = az/(1-bz)$ is analytic and starlike in U. Applying Lemma 2.1 to (2.11), we have

$$\int_0^z \frac{g(t)}{t} dt \prec \frac{1}{n} \int_0^z \frac{h(t)}{t} dt, \tag{2.12}$$

that is,

$$-\alpha \int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t} \right) dt + \int_0^z \frac{f''(t)}{f'(t)} dt \prec \frac{a}{n} \int_0^z \frac{dt}{1-bt}. \tag{2.13}$$

(i) If $0 < a \leq n$ and $0 < b \leq 1$, then from (2.13) we deduce that

$$\frac{z^\alpha f'(z)}{f^\alpha(z)} \prec \left(\frac{1}{1-bz} \right)^{a/nb} \equiv h_1(z). \tag{2.14}$$

The function $h_1(z)$ is analytic and convex univalent in U because

$$\operatorname{Re}\left\{ 1 + \frac{zh_1''(z)}{h_1(z)} \right\} = \operatorname{Re}\left\{ \frac{1+(a/n)z}{1-bz} \right\} \geq \frac{1-a/n}{1+b} \geq 0 \quad (z \in U). \tag{2.15}$$

Also, $h_1(U)$ is symmetric with respect to the real axis. Hence $\operatorname{Re}\{h_1(z)\} > h_1(-1)$ in U and it follows from (2.14) that

$$\operatorname{Re}\left\{ \frac{z^\alpha f'(z)}{f^\alpha(z)} \right\} > \left(\frac{1}{1+b} \right)^{a/nb} \quad (z \in U). \tag{2.16}$$

(ii) If $0 < a \leq n$ and $b = 0$, then from (2.13) we obtain

$$\frac{z^\alpha f'(z)}{f^\alpha(z)} \prec e^{(a/n)z} \equiv h_2(z). \tag{2.17}$$

Since $h_2(z)$ is analytic and convex univalent in U and $h_2(U)$ is symmetric with respect to the real axis, it follows from (2.17) that

$$\operatorname{Re}\left\{\frac{z^\alpha f'(z)}{f^\alpha(z)}\right\} > e^{-a/n} \quad (z \in U). \tag{2.18}$$

(iii) If $a \neq 0$ and $0 < b \leq 1$, then by (2.14) we have

$$\frac{z^\alpha f'(z)}{f^\alpha(z)} = \left(\frac{1}{1 - bw(z)}\right)^{a/nb} \quad (z \in U), \tag{2.19}$$

where $w(z)$ is analytic in U with $|w(z)| \leq |z|$ $(z \in U)$. Therefore we have

$$\left|\left(\frac{z^\alpha f'(z)}{f^\alpha(z)}\right)^{-nb/a} - 1\right| < |-bw(z)| < b \quad (z \in U). \tag{2.20}$$

(iv) If $a > 0$ and $b = 0$, then from (2.17) we get

$$\frac{z^\alpha f'(z)}{f^\alpha(z)} = e^{(a/n)w(z)} \quad (z \in U), \tag{2.21}$$

where $w(z)$ is analytic in U with $|w(z)| \leq |z|$ $(z \in U)$. Thus

$$\left|\frac{z^\alpha f'(z)}{f^\alpha(z)} - 1\right| = |e^{(a/n)w(z)} - 1| \leq e^{(a/n)|w(z)|} - 1 < e^{a/n} - 1 \quad (z \in U). \tag{2.22}$$

Therefore the proof of Theorem 2.4 is completed. □

By specifying the values of the parameters appearing in Theorem 2.4, we can obtain several useful corollaries.

Taking $0 < a = 2(\alpha - \beta) \leq n$ and $b = 1$, Theorem 2.4(i) reduces to the following.

Corollary 2.5. *Let $f \in A_n$ satisfy $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$\operatorname{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 2\alpha - \beta \quad (z \in U), \tag{2.23}$$

where α is a real number and $\alpha - n/2 \leq \beta < \alpha$, then

$$\operatorname{Re}\left\{\frac{z^\alpha f'(z)}{f^\alpha(z)}\right\} > \frac{1}{2^{(2(\alpha-\beta)/n)}} \quad (z \in U). \tag{2.24}$$

Remark 2.6. Owa et al. [4] proved that if $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and (2.23) for $\alpha \geq 0$ and $\alpha - n/2 \leq \beta < \alpha$, then

$$\operatorname{Re}\left\{\frac{z^\alpha f'(z)}{f^\alpha(z)}\right\} > \frac{n}{n + 2\alpha - 2\beta} \quad (z \in U). \tag{2.25}$$

In view of $2^x < 1 + x$ $(0 < x < 1)$, Corollary 2.5 is better than the main theorem of [4].

Corollary 2.7. *If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{a}{2} \quad (z \in U) \tag{2.26}$$

for some a $(0 < a \leq n)$, then $f \in S_n^*(2^{-a/n})$ and the order $2^{-a/n}$ is sharp.

Proof. Letting $\alpha = b = 1$ in Theorem 2.4(i) and using (2.26), we see that $f \in S_n^*(2^{-a/n})$. To show that the order $2^{-a/n}$ cannot be increased, we consider

$$f(z) = \exp \int_0^z \frac{(1+t^n)^{-a/n}}{t} dt \in A_n. \tag{2.27}$$

It is easy to verify that the function $f(z)$ defined by (2.27) satisfies (2.26) and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\left(\frac{1}{1+z^n}\right)^{a/n}\right\} \longrightarrow \left(\frac{1}{2}\right)^{a/n} \tag{2.28}$$

as $z \to 1$. Therefore the proof is completed. □

Putting $\alpha = 0$ and $b = 1$ in Theorem 2.4(i), we have the following.

Corollary 2.8. *If $f \in A_n$ satisfies $f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$-\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < \frac{a}{2} \quad (z \in U) \tag{2.29}$$

for some a $(0 < a \leq n)$, then $f \in C_n(2^{-a/n})$ and the order $2^{-a/n}$ is sharp.

Remark 2.9. Corollary 2.7 (with $0 < a = 2(1-\beta) \leq n$) and Corollary 2.8 (with $0 < a = 2\beta < n$) are better than the corresponding results in [4].

Setting $\alpha = 0$ and 1 in Theorem 2.4(ii), we have the following two corollaries.

Corollary 2.10. *If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$\left|\frac{zf''(z)}{f'(z)}\right| < a \quad (z \in U) \tag{2.30}$$

for some a $(0 < a \leq n)$, then $f \in C_n(e^{-a/n})$.

Corollary 2.11. *If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < a \quad (z \in U) \tag{2.31}$$

for some a $(0 < a \leq n)$, then $f \in S_n^*(e^{-a/n})$ and the order $e^{-a/n}$ is sharp with the extremal function

$$f(z) = \exp \int_0^z \frac{e^{-(a/n)t^n}}{t} dt. \tag{2.32}$$

For $\alpha = 1$ and $a = -nb$ $(0 < b \leq 1)$ in Theorem 2.4(iii), we have the following.

Corollary 2.12. *If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec -\frac{-nbz}{1-bz} \tag{2.33}$$

for some b $(0 < b \leq n)$, then $f \in S_n^*(1-b)$ and the order $1-b$ is sharp with the extremal function $f(z) = ze^{(b/n)z^n}$.

Next, applying Lemma 2.2, we obtain the following two results.

Theorem 2.13. *Let $f \in A$ satisfy $f(z) \neq 0$ for $z \in U\backslash\{0\}$ and*

$$\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} \prec h(z) \quad (z \in U), \tag{2.34}$$

where

$$h(z) = \frac{(1-2\alpha)^2 z^2 + 2(2-3\alpha)+1}{(1-z)^2} \quad (0 \le \alpha < 1;\ z \in U), \tag{2.35}$$

then $f \in S^*(\alpha)$ and the order α is sharp.

Proof. We put

$$\frac{zf'(z)}{f(z)} = (1-\alpha)p(z) + \alpha \tag{2.36}$$

for $0 \le \alpha < 1$. Then $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (2.36) logarithmically, we find that

$$\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} = (1-\alpha)zp'(z) + \big((1-\alpha)p(z)+\alpha\big)^2. \tag{2.37}$$

From (2.34) and (2.37), we have

$$(1-\alpha)zp'(z) + (1-\alpha)^2 p^2(z) + 2\alpha(1-\alpha)p(z) + \alpha^2 \prec h(z). \tag{2.38}$$

Now we choose

$$g(z) = \frac{1+z}{1-z}, \qquad \theta(w) = (1-\alpha)^2 w^2 + 2(1-\alpha)w + \alpha^2, \qquad \varphi(w) = 1-\alpha. \tag{2.39}$$

Then $g(z)$ is analytic and univalent in U, $\operatorname{Re}\{g(z)\} > 0$ $(z \in U)$, and $\theta(w)$ and $\varphi(w)$ are analytic with $\varphi(w) \neq 0$ in the w-plane.

The function

$$Q(z) = zg'(z)\varphi(z) = 2(1-\alpha)\frac{z}{(1-z)^2} \tag{2.40}$$

is univalent and starlike in U. Further,

$$\begin{aligned}\theta(g(z)) + Q(z) &= (1-\alpha)^2\left(\frac{1+z}{1-z}\right)^2 + 2\alpha(1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha^2 + 2(1-\alpha)\frac{z}{1-z} \\ &= \frac{(1-2\alpha)^2 z^2 + 2(2-3\alpha)z + 1}{(1-z)^2} = h(z),\end{aligned} \tag{2.41}$$

$$\begin{aligned}\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} &= \operatorname{Re}\left\{2(1-\alpha)g(z) + 2\alpha + \frac{zQ'(z)}{Q(z)}\right\} \\ &= (3-2\alpha)\operatorname{Re}\left\{\frac{1+z}{1-z}\right\} + 2\alpha > 0\end{aligned} \tag{2.42}$$

for $z \in U$. In view of (2.38)–(2.42), we see that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z). \tag{2.43}$$

Therefore, Lemma 2.2 leads to $p(z) \prec g(z)$, which implies that $f \in S^*(\alpha)$. Next, we consider

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \in A. \tag{2.44}$$

It is easy to see that

$$\begin{gathered} \frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} = h(z), \\ \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{1+(1-2\alpha)z}{1-z}\right\} \longrightarrow \alpha \end{gathered} \tag{2.45}$$

as $z \to -1$. The proof of the theorem is completed. □

Theorem 2.14. *If $f \in A$ satisfies $f(z) \neq 0$ for $z \in U \setminus \{0\}$ and*

$$\frac{zf'(z)}{f(z)} + 2\alpha \frac{z^2 f''(z)}{f(z)} \prec h(z), \tag{2.46}$$

where

$$h(z) = \frac{(2\alpha-1)^3 z^2 + 2\alpha(3-4\alpha)z + 1}{(1-z)^2} \quad (0 \le \alpha < 1;\ z \in U), \tag{2.47}$$

then $f \in S^*(\alpha)$ and the order α is sharp.

Proof. It suffices to prove the theorem for $0 < \alpha < 1$. We define the function $p(z)$ by (2.36). Then $p(z)$ is analytic in U and $p(0) = 1$. By a simple calculation, we find that

$$\begin{aligned} &\frac{zf'(z)}{f(z)} + 2\alpha \frac{z^2 f''(z)}{f(z)} \\ &\quad = 2\alpha(1-\alpha)zp'(z) + 2\alpha(1-\alpha)^2 p^2(z) + (1-\alpha)(1-2\alpha+4\alpha^2)p(z) \\ &\qquad + \alpha(1-2\alpha+2\alpha^2). \end{aligned} \tag{2.48}$$

Thus the subordination (2.46) becomes

$$\begin{aligned} &2\alpha(1-\alpha)zp'(z) + 2\alpha(1-\alpha)^2 p^2(z) + (1-\alpha)(1-2\alpha+4\alpha^2)p(z) \\ &\qquad + \alpha(1-2\alpha+2\alpha^2) \prec h(z). \end{aligned} \tag{2.49}$$

Set $g(z) = (1+z)/(1-z)$, $\theta(w) = 2\alpha(1-\alpha)^2 w^2 + (1-\alpha)(1-2\alpha+4\alpha^2)w + \alpha(1-2\alpha+2\alpha^2)$, and $\varphi(w) = 2\alpha(1-\alpha)$. Then $g(z)$, $\theta(w)$, and $\varphi(w)$ satisfy the conditions of Lemma 2.2. The function

$$Q(z) = zg'(z)\varphi(g(z)) = 4\alpha(1-\alpha)\frac{z}{(1-z)^2} \tag{2.50}$$

is univalent and starlike in U. Further,

$$\begin{aligned}\theta(g(z))+Q(z) &= 2\alpha(1-\alpha)^2\left(\frac{1+z}{1-z}\right)^2+(1-\alpha)(1-2\alpha+4\alpha^2)\left(\frac{1+z}{1-z}\right)\\&\quad+\alpha(1-2\alpha+2\alpha^2)+4\alpha(1-\alpha)\frac{z}{(1-z)^2}\\&=\frac{(2\alpha-1)^3z^2+2\alpha(3-4\alpha)z+1}{(1-z)^2}=h(z),\\ \operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}&=\operatorname{Re}\left\{2(1-\alpha)g(z)+\frac{1-2\alpha+4\alpha^2}{2\alpha}+\frac{zQ'(z)}{Q(z)}\right\}\\&=(3-2\alpha)\operatorname{Re}\left\{\frac{1+z}{1-z}\right\}+\frac{1-2\alpha+4\alpha^2}{2\alpha}>0,\end{aligned} \tag{2.51}$$

for $z\in U$. Note that

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(g(z))+zg'(z)\varphi(g(z))=h(z). \tag{2.52}$$

Hence, an application of Lemma 2.2 yields that $p(z)\prec g(z)$, that is, $f\in S^*(\alpha)$. For the function $f(z)$ defined by (2.44), we have

$$\begin{gathered}\frac{zf'(z)}{f(z)}+2\alpha\frac{z^2f''(z)}{f(z)}=h(z),\\ \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\longrightarrow\alpha\quad\text{as}\quad z\longrightarrow-1.\end{gathered} \tag{2.53}$$

Therefore we complete the proof of Theorem 2.14. □

Finally, by using Lemma 2.3, we prove the following.

Theorem 2.15. *Let $f\in A_n$ satisfy $f(z)\neq 0$ for $z\in U\setminus\{0\}$ and*

$$\left|\arg\left\{(1-\lambda)\frac{z^2(f'(z))^2}{f^2(z)}+\lambda\left(\frac{zf'(z)}{f(z)}+\frac{z^2f''(z)}{f(z)}\right)+\frac{n\lambda}{2}\right\}\right|<\pi\quad(z\in U) \tag{2.54}$$

for some λ $(\lambda>0)$. Then $f\in S_n^*(0)$ and the order 0 is sharp.

Proof. The function $g(z)$ defined by

$$g(z)=\frac{zf'(z)}{f(z)}=1+b_nz^n+b_{n+1}z^{n+1}+\cdots \tag{2.55}$$

is analytic in U and it is easily verified that

$$(1-\lambda)\frac{z^2(f'(z))^2}{f^2(z)}+\lambda\left(\frac{zf'(z)}{f(z)}+\frac{z^2f''(z)}{f(z)}\right)=g^2(z)+\lambda zg'(z)\quad(\lambda>0;\ z\in U). \tag{2.56}$$

Suppose that there exists a point $z_0\in U\setminus\{0\}$ such that

$$\operatorname{Re}\{g(z)\}>0\quad(|z|<|z_0|),\quad g(z_0)=i\beta, \tag{2.57}$$

where β is a real number. Then, applying Lemma 2.3, we have

$$z_0 g'(z_0) \leq -\frac{n(1+\beta^2)}{2}. \tag{2.58}$$

Thus it follows from (2.56), (2.57), and (2.58) that

$$\begin{aligned}
&(1-\lambda)\frac{z_0^2 (f'(z_0))^2}{f^2(z_0)} + \lambda\left(\frac{zf'(z_0)}{f(z_0)} + \frac{z_0^2 f''(z_0)}{f(z_0)}\right) + \frac{n\lambda}{2} \\
&\quad = (g(z_0))^2 + \lambda z_0 g'(z_0) + \frac{n\lambda}{2} \\
&\quad \leq -\beta^2 - \frac{n\lambda(1+\beta^2)}{2} + \frac{n\lambda}{2} \leq 0
\end{aligned} \tag{2.59}$$

for $\lambda > 0$, which contradicts (2.54). Hence $\operatorname{Re}\{g(z)\} > 0$ $(z \in U)$, that is $f \in S_n^*(0)$. If we let

$$f_n(z) = \frac{z}{(1-z^n)^{2/n}} \in A_n, \tag{2.60}$$

then

$$\begin{aligned}
&(1-\lambda)\frac{z^2 (f_n'(z))^2}{f_n^2(z)} + \lambda\left(\frac{zf_n'(z)}{f_n(z)} + \frac{z^2 f_n''(z)}{f_n(z)}\right) + \frac{n\lambda}{2} \\
&\quad = \left(1 + \frac{n\lambda}{2}\right)\left(\frac{1+z^n}{1-z^n}\right)^2 \quad (z \in U),
\end{aligned} \tag{2.61}$$

and so the function $f_n(z)$ satisfies (2.54). Noting that

$$\operatorname{Re}\frac{zf_n'(z)}{f_n(z)} = \operatorname{Re}\frac{1+z^n}{1-z^n} \longrightarrow 0 \tag{2.62}$$

as $z \to e^{i\pi/n}$, we conclude that the order 0 is the best possible. □

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