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# THE BESSEL-STRUVE INTERTWINING OPERATOR ON $\mathbb{C}$ AND MEAN-PERIODIC FUNCTIONS

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We give a description of all transmutation operators from the Bessel-Struve operator to the second-derivative operator. Next we define and characterize the mean-periodic functions on the space $\mathscr{H}$ of entire functions and we characterize the continuous linear mappings from $\mathscr{H}$ into itself which commute with Bessel-Struve operator.

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**1. Introduction.** Let $A$ and $B$ be two differential operators on a linear space $X$. We say that $\chi$ is a transmutation operator of $A$ into $B$ if $\chi$ is an isomorphism from $X$ into itself such that $A\chi = \chi B$. This notion was introduced by Delsarte in [2] and some generalization and applications were given in [1, 3, 7, 10].

In the case where $A$ and $B$ are two differential operators having the same order and without any singularity on the complex plan, acting on the space of entire functions on $\mathbb{C}$ denoted here by $\mathscr{H}$, Delsarte showed in [3] the existence of a transmutation operator between $A$ and $B$ and gave some applications on the theory of mean-periodic functions on $\mathbb{C}$.

In this paper, we consider the operator $\ell_\alpha$, $\alpha > -1/2$, on $\mathbb{C}$, given by

$$\ell_\alpha f(z) = \frac{d^2 f}{dz^2}(z) + \frac{2\alpha+1}{z}\left[\frac{df}{dz}(z) - \frac{df}{dz}(0)\right], \tag{1.1}$$

where $f$ is an entire function on $\mathbb{C}$. We call this operator Bessel-Struve operator on $\mathbb{C}$.

The Bessel-Struve kernel $S_\alpha(\lambda\cdot)$, $\lambda \in \mathbb{C}$, which is the unique solution of the initial value problem $\ell_\alpha u(z) = \lambda^2 u(z)$ with the initial conditions $u(0) = 1$ and $u'(0) = \lambda\Gamma(\alpha+1)/\sqrt{\pi}\,\Gamma(\alpha+3/2)$, is given by

$$S_\alpha(\lambda z) = j_\alpha(i\lambda z) - i h_\alpha(i\lambda z) \quad \forall z \in \mathbb{C}, \tag{1.2}$$

where $j_\alpha$ and $h_\alpha$ are the normalized Bessel and Struve functions (see [4]).

Moreover, the Bessel-Struve kernel is a holomorphic function on $\mathbb{C}\times\mathbb{C}$ and it can be expanded in a power series in the form

$$S_\alpha(\lambda z) = \sum_{n=0}^{+\infty} \frac{(\lambda z)^n}{c_n(\alpha)}, \quad c_n(\alpha) = \frac{\sqrt{\pi}\, n!\,\Gamma(n/2+\alpha+1)}{\Gamma(\alpha+1)\Gamma((n+1)/2)}. \tag{1.3}$$

The Bessel-Struve intertwining operator $\chi_\alpha$ is defined from the space $\mathcal{H}$ into itself by

$$\chi_\alpha f(z) = \sum_{n=0}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^n}{c_n(\alpha)} \quad \forall f \in \mathcal{H},\ z \in \mathbb{C}. \tag{1.4}$$

The dual intertwining operator ${}^t\chi_\alpha$ of $\chi_\alpha$ is defined on $\mathcal{H}'$ (the dual space of $\mathcal{H}$) by

$$\langle {}^t\chi_\alpha T, g\rangle = \langle T, \chi_\alpha g\rangle \quad \forall g \in \mathcal{H},\ T \in \mathcal{H}'. \tag{1.5}$$

The Bessel-Struve transform $\mathcal{F}_\alpha$ is defined on $\mathcal{H}'$ by

$$\mathcal{F}_\alpha(T)(\lambda) = \langle T, S_\alpha(-i\lambda\cdot)\rangle \quad \forall \lambda \in \mathbb{C}. \tag{1.6}$$

We use the transmutation operator $\chi_\alpha$ to define the Bessel-Struve translation operators $\tau_z, z \in \mathbb{C}$, associated with $\ell_\alpha$, and the Bessel-Struve convolution on $\mathcal{H}$ and $\mathcal{H}'$. A function $f$ in $\mathcal{H}$ is said to be mean periodic if the closed subspace $\Omega(f)$ generated by $\tau_z f, z \in \mathbb{C}$, satisfies $\Omega(f) \neq \mathcal{H}$.

The objective of this paper is to characterize every transmutation operator of $\ell_\alpha$ into the second derivative operator from $\mathcal{H}$ into itself. Next, we study the mean-periodic functions associated with the Bessel-Struve operator and we characterize the continuous linear mappings from $\mathcal{H}$ into itself which commute with $\ell_\alpha$.

We point out that the harmonic analysis associated with differential and differential-difference operators allows many applications as the study of integral representations (see [9]), Plancherel, and reconstruction formulas and other applications as the use of wavelets packets in the inversion of transmutation operators for the J. L. Lions operator and the Dunkl operator (see [5, 6]).

The content of this paper is as follows.

In Section 2, we prove that the Bessel-Struve intertwining operator $\chi_\alpha$ is a topological isomorphism from $\mathcal{H}$ into itself satisfying

$$\begin{gathered} \forall f \in \mathcal{H}, \quad \ell_\alpha \chi_\alpha f = \chi_\alpha \frac{d^2}{dz^2} f, \\ \chi_\alpha f(0) = f(0), \qquad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \end{gathered} \tag{1.7}$$

Using this operator and its dual, we study the harmonic analysis associated with the operator $\ell_\alpha$ (Bessel-Struve transform, Bessel-Struve translation operators, and Bessel-Struve convolution). Next, we determine all transmutation operators $W$ from the Bessel-Struve operator $\ell_\alpha$ to the second derivative operator $d^2/dz^2$.

In Section 3, we study the mean-periodic functions associated with $\ell_\alpha$. Next, we give the central result of the paper, which characterizes the continuous linear mappings from $\mathcal{H}$ into itself which commute with $\ell_\alpha$.

**2. Bessel-Struve transmutation operators.** In this section, we consider the normalized Bessel and Struve functions which allow to define the Bessel-Struve kernel. Next, we define the Bessel-Struve intertwining operator $\chi_\alpha$ and its dual ${}^t\chi_\alpha$; after that, we study the harmonic analysis associated with the operator $\ell_\alpha$. The aim of this section is to characterize every transmutation operator of $\ell_\alpha$ into $d^2/dz^2$ from $\mathscr{H}$ into itself.

Let $\alpha > -1/2$. The normalized Bessel function $j_\alpha$ is the kernel defined on $\mathbb{C}$ by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \tag{2.1}$$

where $J_\alpha$ is the Bessel function of order $\alpha$ (see [4, 12]).

The normalized Struve function $h_\alpha$ is the kernel defined on $\mathbb{C}$ by

$$h_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{\mathbf{H}_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n (z/2)^{2n+1}}{\Gamma(n+3/2)\Gamma(n+\alpha+3/2)}, \tag{2.2}$$

where $\mathbf{H}_\alpha$ is the Struve function of order $\alpha$ (see [4, 12]).

This function has the following Poisson integral representation:

$$h_\alpha(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_0^1 (1-t^2)^{\alpha-1/2}\sin(zt)\,dt. \tag{2.3}$$

The function $z \to h_\alpha(i\lambda z)$, $\lambda, z \in \mathbb{C}$, is the unique solution of the differential equation

$$\begin{gathered} \ell_\alpha u(z) = \lambda^2 u(z), \\ u(0) = 0, \qquad u'(0) = \frac{\lambda\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+3/2)}. \end{gathered} \tag{2.4}$$

The functions $h_\alpha$ and $j_\alpha$ are related by the formula

$$h_\alpha(z) = \frac{\Gamma(\alpha+1)z}{\sqrt{\pi}\,\Gamma(\alpha+3/2)}\int_0^{\pi/2} j_{\alpha+1/2}(z\sin\varphi)\sin\varphi\,d\varphi. \tag{2.5}$$

The Bessel-Struve kernel is the function $S_\alpha$ defined on $\mathbb{C}$ by

$$S_\alpha(z) = j_\alpha(iz) - ih_\alpha(iz). \tag{2.6}$$

This kernel can be expanded in a power series in the form

$$S_\alpha(z) = \sum_{n=0}^{+\infty}\frac{z^n}{c_n(\alpha)}, \quad c_n(\alpha) = \frac{\sqrt{\pi}\,n!\Gamma(n/2+\alpha+1)}{\Gamma(\alpha+1)\Gamma((n+1)/2)}, \tag{2.7}$$

and has the following integral representation:

$$S_\alpha(z) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_0^1 (1-t^2)^{\alpha-1/2}\exp(zt)\,dt. \tag{2.8}$$

The function $z \to S_\alpha(\lambda z)$, $\lambda \in \mathbb{C}$, is the unique solution of the differential equation

$$\begin{aligned} &\ell_\alpha u(z) = \lambda^2 u(z), \\ &u(0) = 1, \qquad u'(0) = \frac{\lambda \Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+3/2)}. \end{aligned} \tag{2.9}$$

**Notations.**

(i) We denote by $\mathcal{H}$, the space of entire functions on $\mathbb{C}$, with the topology of the uniform convergence on compact subsets of $\mathbb{C}$. Thus $\mathcal{H}$ is a Fréchet space.

(ii) We denote by $\mathcal{H}'$, the dual space of $\mathcal{H}$.

**Proposition 2.1.** *The operator $\chi_\alpha$ defined by*

$$\chi_\alpha f(z) = \sum_{n=0}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^n}{c_n(\alpha)}, \quad \forall f \in \mathcal{H},\ z \in \mathbb{C}, \tag{2.10}$$

*is an isomorphism from $\mathcal{H}$ into itself satisfying the transmutation relation*

$$\begin{aligned} &\forall f \in \mathcal{H}, \quad \ell_\alpha \chi_\alpha f = \chi_\alpha \frac{d^2}{dz^2} f, \\ &\chi_\alpha f(0) = f(0), \qquad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \end{aligned} \tag{2.11}$$

*The inverse of $\chi_\alpha$ is given by*

$$\chi_\alpha^{-1}(f)(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n(f)(0) \frac{z^{2n}}{(2n)!} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(0) \frac{z^{2n+1}}{(2n+1)!} \quad \forall f \in \mathcal{H},\ z \in \mathbb{C}. \tag{2.12}$$

**Proof.** First we prove that the image of the function $f$ in $\mathcal{H}$ by $\chi_\alpha$ is an entire function, and that $\chi_\alpha$ is a continuous linear operator.

Since $f$ is an entire function, from the Cauchy integral formula, we have

$$\forall n \in \mathbb{N}, \quad \frac{d^n f}{dz^n}(0) = \frac{n!}{2i\pi} \int_{C_R} \frac{f(w)}{w^{n+1}} dw, \tag{2.13}$$

where $C_R$ is a circle with center 0 and radius $R > 0$. Hence there exists a positive constant $M$ such that

$$\forall n \in \mathbb{N}, \quad \left| \frac{d^n f}{dz^n}(0) \frac{1}{c_n(\alpha)} \right| \leq M R^{-n} \|f\|_R, \tag{2.14}$$

where

$$\|f\|_R = \max_{|z| \leq R} |f(z)|. \tag{2.15}$$

As $R$ is arbitrary, the radius of convergence of the power series in (2.10) is infinite. Thus $\chi_\alpha(f)$ is an entire function.

Using (2.14), we obtain

$$\forall f \in \mathcal{H}, \quad \|\chi_\alpha(f)\|_R \leq 2M\|f\|_{2R}. \tag{2.16}$$

Thus $\chi_\alpha$ defines a continuous linear mapping from $\mathcal{H}$ into itself. Furthermore, using the fact that

$$\forall n \geq 2, \quad \ell_\alpha(z^n) = \frac{c_n(\alpha)}{c_{n-2}(\alpha)} z^{n-2}, \tag{2.17}$$

we get

$$\forall z \in \mathbb{C}, \quad \ell_\alpha \chi_\alpha f(z) = \sum_{n=2}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^{n-2}}{c_{n-2}(\alpha)} = \sum_{n=0}^{+\infty} \frac{d^{n+2} f}{dz^{n+2}}(0) \frac{z^n}{c_n(\alpha)} = \chi_\alpha \frac{d^2}{dz^2} f(z). \tag{2.18}$$

It is clear that

$$\chi_\alpha f(0) = f(0), \qquad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \tag{2.19}$$

Suppose now that $\chi_\alpha f = 0$ for a certain $f \in \mathcal{H}$. Then, according to (2.10), $(d^n f/dz^n)(0) = 0$, $n \in \mathbb{N}$. Hence $f = 0$, thus we prove that $\chi_\alpha$ is a one-to-one mapping from $\mathcal{H}$ into itself.

Now we consider the operator $\psi$ on $\mathcal{H}$ defined by

$$\psi f(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n f(0) \frac{z^{2n}}{(2n)!} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(0) \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbb{C}. \tag{2.20}$$

In the same way as for $\chi_\alpha$ and by a simple calculation, we prove that $\psi$ is a continuous linear mapping from $\mathcal{H}$ into itself and

$$\forall f \in \mathcal{H}, \quad \chi_\alpha \psi f = \psi \chi_\alpha f = f. \tag{2.21}$$

Then $\chi_\alpha$ is a topological isomorphism from $\mathcal{H}$ into itself. □

**Remarks 2.2.** (i) The operator $\chi_\alpha$ which is a transmutation operator from $\ell_\alpha$ into $d^2/dz^2$ on $\mathcal{H}$ will be called the Bessel-Struve intertwining operator on $\mathbb{C}$.

(ii) Formula (2.10) means that the Taylor coefficients of the image of an entire function by $\chi_\alpha$ are multiplied by the Taylor coefficients of the Bessel-Struve kernel.

**Corollary 2.3.** (i) *For* $\lambda, z \in \mathbb{C}$,

$$S_\alpha(\lambda z) = \chi_\alpha(e^{\lambda \cdot})(z). \tag{2.22}$$

(ii) *Every function* $f$ *in* $\mathcal{H}$ *can be expanded in a power series:*

$$\forall z \in \mathbb{C}, \quad f(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n f(0) \frac{z^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(0) \frac{z^{2n+1}}{c_{2n+1}(\alpha)}. \tag{2.23}$$

**Definition 2.4.** The dual intertwining operator ${}^t\chi_\alpha$ of $\chi_\alpha$ is defined on $\mathcal{H}'$ by

$$\langle {}^t\chi_\alpha(T), g\rangle = \langle T, \chi_\alpha(g)\rangle \quad \forall g \in \mathcal{H}. \tag{2.24}$$

**Remark 2.5.** From the properties of the operator $\chi_\alpha$, we deduce that the operator ${}^t\chi_\alpha$ is an isomorphism from $\mathcal{H}'$ into itself; the inverse operator $({}^t\chi_\alpha)^{-1}$ is given by

$$\langle ({}^t\chi_\alpha)^{-1}(T), g\rangle = \langle T, \chi_\alpha^{-1}(g)\rangle \quad \forall g \in \mathcal{H}. \tag{2.25}$$

**Notations.**

(i) We denote by $\mathrm{Exp}_a(\mathbb{C})$, $a > 0$, the space of functions of exponential type $a$. It is the space of functions $f \in \mathcal{H}$ such that

$$N_a(f) = \sup_{z\in\mathbb{C}} |f(z)|e^{-a|z|} < +\infty. \tag{2.26}$$

(ii) We denote by $\mathrm{Exp}(\mathbb{C})$, the space of functions with exponential type. It is given by

$$\mathrm{Exp}(\mathbb{C}) = \cup_{a>0} \underset{a}{\mathrm{Exp}}(\mathbb{C}). \tag{2.27}$$

The space $\mathrm{Exp}(\mathbb{C})$ is endowed with the inductive limit topology.

(iii) We denote by $\mathcal{F}$, the classical Fourier transform defined on $\mathcal{H}'$ by

$$\mathcal{F}(T)(\lambda) = \langle T, e^{-i\lambda\cdot}\rangle \quad \forall \lambda \in \mathbb{C}. \tag{2.28}$$

(iv) We denote by $*_o$, the classical convolution product given by

$$T *_o f(z) = \langle T_w, f(w+z)\rangle \quad \forall T \in \mathcal{H}',\ f \in \mathcal{H},\ z \in \mathbb{C}. \tag{2.29}$$

**Definition 2.6.** The Bessel-Struve transform $\mathcal{F}_\alpha$ of $T \in \mathcal{H}'$ is given by

$$\mathcal{F}_\alpha(T)(\lambda) = \langle T, S_\alpha(-i\lambda\cdot)\rangle \quad \forall \lambda \in \mathbb{C}. \tag{2.30}$$

**Remark 2.7.** From Corollary 2.3(i) and Definition 2.4, we obtain

$$\forall T \in \mathcal{H}', \quad \mathcal{F}_\alpha(T)(\lambda) = \mathcal{F}_\alpha({}^t\chi_\alpha(T))(\lambda). \tag{2.31}$$

**Proposition 2.8.** *The Bessel-Struve transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{H}'$ into* $\mathrm{Exp}(\mathbb{C})$.

**Proof.** According to [8], the classical Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{H}'$ into $\mathrm{Exp}(\mathbb{C})$. Then the result follows from (2.25) and (2.31). □

**Lemma 2.9.** *Let $f\in\mathcal{H}$. The Cauchy problem*

$$\begin{aligned} \ell_{\alpha,z}u(z,w) &= \ell_{\alpha,w}u(z,w), \\ u(0,w) = f(w), &\qquad \frac{\partial}{\partial z}u(0,w) = f'(w) \end{aligned} \tag{2.32}$$

*has a unique solution that is an entire function on $\mathbb{C}\times\mathbb{C}$ given by*

$$u(z,w) = \chi_{\alpha,z}\chi_{\alpha,w}\left[\chi_{\alpha}^{-1}(f)(z+w)\right] \quad \forall z,w\in\mathbb{C}. \tag{2.33}$$

**Proof.** From Proposition 2.1, (2.32) is equivalent to the Cauchy problem

$$\begin{aligned} \frac{\partial^2}{\partial z^2}v(z,w) &= \frac{\partial^2}{\partial w^2}v(z,w), \\ v(0,w) = \chi_{\alpha}^{-1}(f)(w), &\qquad \frac{\partial}{\partial z}v(0,w) = \frac{d(\chi_{\alpha}^{-1}f)}{dz}(w), \end{aligned} \tag{2.34}$$

where

$$v(z,w) = \chi_{\alpha,z}^{-1}\chi_{\alpha,w}^{-1}u(z,w). \tag{2.35}$$

But the solution of (2.34) is given by

$$v(z,w) = \chi_{\alpha}^{-1}(f)(z+w) \quad \forall z,w\in\mathbb{C}. \tag{2.36}$$

□

**Definition 2.10.** The Bessel-Struve translation operators $\tau_z$, $z\in\mathbb{C}$, associated with the operator $\ell_\alpha$, is defined on $\mathcal{H}$ by

$$\tau_z f(w) = \chi_{\alpha,z}\chi_{\alpha,w}\left[\chi_{\alpha}^{-1}(f)(z+w)\right] \quad \forall w\in\mathbb{C}. \tag{2.37}$$

The operator $\tau_z$, $z\in\mathbb{C}$, satisfies the following properties.

(i) For all $z\in\mathbb{C}$, the operator $\tau_z$ is linear continuous from $\mathcal{H}$ into itself.

(ii) For all $f\in\mathcal{H}$ and $z,w\in\mathbb{C}$,

$$\begin{aligned} \tau_z f(w) = \tau_w f(z), &\qquad \tau_0 f(w) = f(w), \\ \tau_z(\tau_w f) = \tau_w(\tau_z f), &\qquad \ell_\alpha\tau_z f = \tau_z\ell_\alpha f. \end{aligned} \tag{2.38}$$

(iii) The following product formula holds:

$$\forall z,w\in\mathbb{C}, \quad \tau_z(S_\alpha(\lambda\cdot))(w) = S_\alpha(\lambda w)S_\alpha(\lambda z). \tag{2.39}$$

**Corollary 2.11.** *Let $f\in\mathcal{H}$ and $z\in\mathbb{C}$. Then the function $w\to\tau_z f(w)$ can be expanded in the Taylor series:*

$$\forall w\in\mathbb{C}, \quad \tau_z f(w) = \sum_{n=0}^{+\infty}\ell_\alpha^n f(z)\frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha)\sum_{n=0}^{+\infty}\frac{d(\ell_\alpha^n f)}{dz}(z)\frac{w^{2n+1}}{c_{2n+1}(\alpha)}. \tag{2.40}$$

**Proof.** For $z, w \in \mathbb{C}$, we have

$$\tau_z f(w) = \chi_{\alpha,z}\chi_{\alpha,w}[\chi_\alpha^{-1}(f)(z+w)]. \tag{2.41}$$

Applying Corollary 2.3(ii) to the function $w \to \tau_z f(w)$, we obtain

$$\begin{aligned}\tau_z f(w) &= \sum_{n=0}^{+\infty} \ell_\alpha^n[\tau_z f](0)\frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha)\sum_{n=0}^{+\infty}\frac{d(\ell_\alpha^n[\tau_z f])}{dz}(0)\frac{w^{2n+1}}{c_{2n+1}(\alpha)} \\ &= \sum_{n=0}^{+\infty}\tau_z[\ell_\alpha^n f](0)\frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha)\sum_{n=0}^{+\infty}\tau_z\left[\frac{d(\ell_\alpha^n f)}{dz}\right](0)\frac{w^{2n+1}}{c_{2n+1}(\alpha)},\end{aligned} \tag{2.42}$$

which proves the result. □

**Definition 2.12.** (i) The convolution product of two elements $T$ and $K$ in $\mathcal{H}'$ is defined by

$$\langle T * K, f\rangle = \langle T_z, \langle K_w, \tau_z f(w)\rangle\rangle \quad \forall f \in \mathcal{H}. \tag{2.43}$$

(ii) Let $T \in \mathcal{H}'$ and $f \in \mathcal{H}$. The convolution product of $T$ and $f$ is the function in $\mathcal{H}$ defined by

$$T * f(z) = \langle T_w, \tau_z f(w)\rangle \quad \forall z \in \mathbb{C}. \tag{2.44}$$

The convolution $*$ satisfies the following properties.

(i) Let $T$, $K \in \mathcal{H}'$ and let $f \in \mathcal{H}$. Then

$$T * (K * f) = (T * K) * f. \tag{2.45}$$

(ii) Let $T$, $K \in \mathcal{H}'$. Then

$$\mathcal{F}_\alpha(T * K) = \mathcal{F}_\alpha(T)\mathcal{F}_\alpha(K). \tag{2.46}$$

**Proposition 2.13.** *Let $T \in \mathcal{H}'$ and let $f \in \mathcal{H}$. Then*

$$\begin{aligned}({}^t\chi_\alpha)^{-1}(T) * \chi_\alpha(f) &= \chi_\alpha(T *_o f), \\ {}^t\chi_\alpha(T) *_o \chi_\alpha^{-1}(f) &= \chi_\alpha^{-1}(T * f),\end{aligned} \tag{2.47}$$

*where $*_o$ is the classical convolution product given by (2.29).*

**Proof.** From Definition 2.12, we have

$$\begin{aligned}\forall z \in \mathbb{C}, \quad &({}^t\chi_\alpha)^{-1}(T) * \chi_\alpha(f)(z) \\ &= \left\langle ({}^t\chi_\alpha)^{-1}(T)_\xi, \tau_z(\chi_\alpha(f))(\xi)\right\rangle = \langle T_\xi, \chi_{\alpha,\xi}^{-1}\tau_z(\chi_\alpha(f))(\xi)\rangle.\end{aligned} \tag{2.48}$$

But from Definition 2.10, we obtain

$$\forall \xi \in \mathbb{C}, \quad \chi_{\alpha,\xi}^{-1}\tau_z(\chi_\alpha(f))(\xi) = \chi_{\alpha,z}(f)(\xi - z). \tag{2.49}$$

Thus

$$\begin{aligned}&({}^t\chi_\alpha)^{-1}(T)*\chi_\alpha(f)(z)\\&\quad=\langle T_\xi,\chi_{\alpha,z}(f)(\xi-z)\rangle=\chi_{\alpha,z}(\langle T_\xi,f(\xi-z)\rangle)=\chi_\alpha(T*_o f)(z),\end{aligned}\tag{2.50}$$

which proves the first relation.

For the second relation, we have

$$\begin{aligned}\forall z\in\mathbb{C},\quad &{}^t\chi_\alpha(T)*_o({}^t\chi_\alpha)^{-1}(f)(z)\\&=\langle {}^t\chi_\alpha(T)_\xi,\chi_\alpha^{-1}(f)(\xi-z)\rangle=\langle T_\xi,\chi_{\alpha,\xi}\chi_\alpha^{-1}(f)(\xi-z)\rangle.\end{aligned}\tag{2.51}$$

But

$$\forall z,\xi\in\mathbb{C},\quad \chi_{\alpha,\xi}\chi_\alpha^{-1}(f)(\xi-z)=\chi_{\alpha,z}^{-1}(\tau_z f)(\xi).\tag{2.52}$$

So

$$\forall z\in\mathbb{C},\quad {}^t\chi_\alpha(T)*(\chi_\alpha)^{-1}(f)(z)=\chi_{\alpha,z}^{-1}\langle T_\xi,\tau_z f(\xi)\rangle=\chi_\alpha^{-1}(T*f)(z),\tag{2.53}$$

which finishes the proof. □

Now we are in position to derive the main result of this section.

**Notations.**
(i) We denote $D=d/dz$.
(ii) We denote by $\mathcal{G}_{D^2}$, the group of isomorphisms $Y$ from $\mathcal{H}$ into itself such that

$$YD^2=D^2Y.\tag{2.54}$$

**Theorem 2.14.** *Every transmutation operator $W$ of $\ell_\alpha$ into $D^2$ from $\mathcal{H}$ into itself is of the form*

$$Wf(z)=({}^t\chi_\alpha)^{-1}T_0*\chi_\alpha(f)(z)+({}^t\chi_\alpha)^{-1}T_1*\chi_\alpha(f)(-z)\quad\forall z\in\mathbb{C},\tag{2.55}$$

*where $T_0,T_1\in\mathcal{H}'$.*

**Proof.** It is clear that every transmutation operator $W$ of $\ell_\alpha$ into $D^2$ from $\mathcal{H}$ into itself is of the form $W=\chi_\alpha Y$, where $Y\in\mathcal{G}_{D^2}$. Then according to [3], every element $Y$ of $\mathcal{G}_{D^2}$ has the form

$$Yf(z)=T_0*_o f(z)+T_1*_o f(-z),\tag{2.56}$$

where $T_0,T_1\in\mathcal{H}'$. Thus, we can write

$$\forall z\in\mathbb{C},\quad Wf(z)=\chi_\alpha(T_0*_o f)(z)+\chi_\alpha(T_1*_o f)(-z).\tag{2.57}$$

Hence the result follows from Proposition 2.13. □

## 3. Mean-periodic functions and commutators of $\ell_\alpha$

### 3.1. Mean-periodic functions

**Definition 3.1.** A function $f$ in $\mathscr{H}$ is said to be mean periodic if the closed subspace $\Omega(f)$ generated by $\tau_z f$, $z \in \mathbb{C}$, satisfies

$$\Omega(f) \neq \mathscr{H}. \tag{3.1}$$

From Hahn-Banach theorem, this definition is equivalent to the following.

**Definition 3.2.** A function $f$ in $\mathscr{H}$ is said to be mean periodic if there exists $T \in \mathscr{H}' \setminus \{0\}$ such that

$$\forall z \in \mathbb{C}, \quad T * f(z) = 0. \tag{3.2}$$

**Definition 3.3.** Let $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. The function $S_{\alpha,\ell}(\lambda, \cdot)$ is defined by

$$S_{\alpha,\ell}(\lambda, z) = \frac{d^\ell}{d\mu^\ell} S_\alpha(\mu z)\Big|_{\mu=-i\lambda} \quad \forall z \in \mathbb{C}. \tag{3.3}$$

**Lemma 3.4.** *Let $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. Then the function $S_{\alpha,\ell}(\lambda, \cdot)$ is mean periodic and*

$$\forall z \in \mathbb{C}, \quad S_{\alpha,\ell}(\lambda, z) = \chi_\alpha\left(\xi^\ell \exp(-i\lambda\xi)\right)(z). \tag{3.4}$$

**Proof.** Let $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. According to Proposition 2.8, there exists $T \in \mathscr{H}' \setminus \{0\}$ such that

$$\forall j = 0, \dots, \ell, \quad \frac{d^j}{d\mu^j}\left(\mathscr{F}_\alpha(T)\right)(\mu)\Big|_{\mu=\lambda} = 0. \tag{3.5}$$

Then from the properties of the Bessel-Struve translation for every $z \in \mathbb{C}$, we can write

$$\begin{aligned}
(T * S_{\alpha,\ell}(\lambda \cdot))(z) &= \left\langle T(w), \frac{d^\ell}{d\mu^\ell}\left(\tau_w\left(S_\alpha(\mu \cdot)\right)(z)\right)\Big|_{\mu=-i\lambda}\right\rangle \\
&= \left\langle T(w), \frac{d^\ell}{d\mu^\ell}\left(S_\alpha(\mu z) S_\alpha(\mu w)\right)\Big|_{\mu=-i\lambda}\right\rangle \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{d^{\ell-j}}{d\mu^{\ell-j}}\left(S_\alpha(\mu z)\right)\Big|_{\mu=-i\lambda} \frac{d^j}{d\mu^j}\mathscr{F}_\alpha(T)(\mu)\Big|_{\mu=\lambda} \\
&= 0.
\end{aligned} \tag{3.6}$$

Thus we prove that $S_{\alpha,\ell}(\lambda, \cdot)$ is a mean-periodic function. The result follows from (1.3) and (2.10). □

Let $f \in \mathscr{H}$. The following proposition characterizes the functions which belong to $\Omega(f)$.

**Proposition 3.5.** *Let $f\in\mathscr{H}$, $\ell\in\mathbb{N}$, and $\lambda\in\mathbb{C}$. The function $S_{\alpha,j}(\lambda,\cdot)$, $0\le j\le\ell$, belongs to $\Omega(f)$ if and only if for all $T$ in $\mathscr{H}'$ satisfying*

$$\forall z\in\mathbb{C},\quad T*f(z)=0, \tag{3.7}$$

*then*

$$\frac{d^j}{d\mu^j}(\mathscr{F}_\alpha(T))(\mu)\Big|_{\mu=\lambda}=0,\quad 0\le j\le\ell. \tag{3.8}$$

**Proof.** If $S_{\alpha,j}(\lambda,\cdot)$, $0\le j\le\ell$, belongs to $\Omega(f)$, then for all $T\in\mathscr{H}'$ satisfying (3.7) we have

$$\langle T,S_{\alpha,j}(\lambda,\cdot)\rangle=0. \tag{3.9}$$

Then

$$\begin{aligned}\langle T,S_{\alpha,j}(\lambda,\cdot)\rangle&=\frac{d^j}{d\mu^j}\Big\langle T,S_\alpha(\mu\cdot)\Big|_{\mu=-i\lambda}\Big\rangle\\&=\frac{d^j}{d\mu^j}\mathscr{F}_\alpha(T)(\mu)\Big|_{\mu=\lambda}=0.\end{aligned} \tag{3.10}$$

The converse follows from the Hahn-Banach theorem. □

**Definition 3.6.** Let $f\in\mathscr{H}$ be a mean-periodic function. The spectrum $\mathrm{Sp}(f)$ of $f$ is the set

$$\mathrm{Sp}(f)=\{(\lambda,\ell),\lambda\in\mathbb{C},\ \ell\in\mathbb{N},\ S_{\alpha,j}(\lambda\cdot)\in\Omega(f),\ 0\le j\le\ell\}. \tag{3.11}$$

**Remarks 3.7.** (i) From Proposition 3.5, we have

$$\mathrm{Sp}(f)=\left\{(\lambda,\ell),\ \lambda\in\mathbb{C},\ \ell\in\mathbb{N},\ \frac{d^j}{d\mu^j}\mathscr{F}_\alpha(T)(\mu)\Big|_{\mu=\lambda}=0,\ j=0,1,\dots,\ell,\ T\in(\Omega(f))^{\perp}\right\}. \tag{3.12}$$

(ii) If $\mathrm{Sp}(f)\ne\emptyset$, we say that $\Omega(f)$ admits a spectral analysis associated with $\ell_\alpha$.

**Proposition 3.8.** *Let $f\in\mathscr{H}$. Denote by $\mathcal{S}(f)$ the closed subspace of $\mathscr{H}$ generated by $\{D^k\ell_\alpha^n f\}_{n\in\mathbb{N};k=0,1}$. Then $\Omega(f)=\mathcal{S}(f)$.*

**Proof.** According to Corollary 2.11, we have, for every $g\in\mathscr{H}$,

$$Dg=\lim_{w\to0}\frac{1}{w}[\tau_w g-g], \tag{3.13}$$

$$\ell_\alpha g=\lim_{w\to0}\frac{c_2(\alpha)}{w^2}[\tau_w g-g-wDg], \tag{3.14}$$

$$D\ell_\alpha g=\lim_{w\to0}\frac{c_3(\alpha)}{c_1(\alpha)w^2}\left[\tau_w g-g-wg-\frac{w^2}{c_2(\alpha)}\ell_\alpha g\right] \tag{3.15}$$

in the sense of the convergence in $\mathscr{H}$.

Suppose that $g \in \Omega(f)$. Then, for every $w \in \mathbb{C}$, $\tau_w g \in \Omega(f)$. Hence we conclude that for $k = 0,1$, $D^k \ell_\alpha g \in \Omega(f)$. By induction, we can prove that, for every $n \in \mathbb{N}$ and $k = 0,1$, $D^k \ell_\alpha^n g \in \Omega(f)$. In particular, for every $n \in \mathbb{N}$ and $k = 0,1$, $D^k \ell_\alpha^n f \in \Omega(f)$. Thus we conclude that $\mathcal{S}(f) \subset \Omega(f)$.

Let now $g \in \mathcal{S}(f)$. Using once more Corollary 2.11, we prove that, for every $w \in \mathbb{C}$, $\tau_w g \in \mathcal{S}(f)$. In particular, for every $w \in \mathbb{C}$, $\tau_w f \in \mathcal{S}(f)$. Hence, $\Omega(f) = \mathcal{S}(f)$. □

**Corollary 3.9.** *Let $f \in \mathcal{H}$. Then $f$ is a mean periodic if and only if $\mathcal{S}(f) \neq \mathcal{H}$.*

**Corollary 3.10.** *Let $f \in \mathcal{H}$. Then $f$ is a mean-periodic function if and only if $\chi_\alpha^{-1}(f)$ is a classical mean-periodic function.*

**Theorem 3.11.** *Let $f \in \mathcal{H}$. Then $f$ is a mean-periodic function if and only if $f$ is a limit of finite linear combination of the functions $S_{\alpha,j}(\lambda,\cdot)$, $0 \le j \le \ell$, such that $(\lambda,\ell) \in \mathrm{Sp}(f)$.*

**Proof.** To see this property, we can use Lemma 3.4 and a celebrated result about classical mean-periodic functions established in [11, page 926]. □

**Corollary 3.12.** *Every mean-periodic function such that $\mathrm{Sp}(f) = \varnothing$ is zero.*

### 3.2. The commutator of $\ell_\alpha$

**Notations.**

(i) We denote by $\mathcal{G}_\alpha$, the group of isomorphisms $Y$ of $\mathcal{H}$ into itself such that

$$Y\ell_\alpha = \ell_\alpha Y; \tag{3.16}$$

(ii) We denote by $\vartheta_\alpha(f)$ (resp., $\vartheta_{D^2}(f)$), the closed subspaces of $\mathcal{H}$ generated by $Yf$, $Y \in \mathcal{G}_\alpha$, (resp., $\mathcal{G}_{D^2}$).

**Proposition 3.13.** (i) *The group $\mathcal{G}_\alpha$ is isomorphic to $\mathcal{G}_{D^2}$.*
(ii)

$$\forall f \in \mathcal{H}, \quad \vartheta_\alpha(f) = \chi_\alpha \vartheta_{D^2}(\chi_\alpha^{-1}(f)). \tag{3.17}$$

**Proposition 3.14.** *The set of functions $f$ in $\mathcal{H}$ satisfying*

$$\vartheta_\alpha(f) \neq \mathcal{H} \tag{3.18}$$

*with the set of mean-periodic functions is identified.*

**Proof.** From Proposition 3.13, $f \in \mathcal{H}$ satisfies (3.18) if and only if $\chi_\alpha^{-1}(f)$ satisfies

$$\vartheta_{D^2}\chi_\alpha^{-1}(f) \neq \mathcal{H}. \tag{3.19}$$

But these functions are classical mean-periodic functions. The result follows from Proposition 3.13. □

Now we are able to state the main result of this paper.

**Theorem 3.15.** *Let $L$ be a continuous linear mapping from $\mathcal{H}$ into itself. The following statements are equivalent.*

(i) *$L$ commutes with Bessel-Struve translation operators $\tau_z$, $z \in \mathbb{C}$, on $\mathcal{H}$, that is, $\tau_z L = L\tau_z$, $z \in \mathbb{C}$, on $\mathcal{H}$.*

(ii) *$L$ commutes with the Bessel-Struve operator $\ell_\alpha$ on $\mathcal{H}$, that is, $\ell_\alpha L = L\ell_\alpha$ on $\mathcal{H}$.*

(iii) *There exists a unique element $T$ in $\mathcal{H}'$ such that $Lf = T * f$, $f \in \mathcal{H}$.*

(iv) *There exists a complex Borel regular measure $\gamma$ having compact support on $\mathbb{C}$, for which for all $f \in \mathcal{H}$,*

$$L(f)(z) = \int_{\mathbb{C}} (\tau_z f)(w)\, d\gamma(w) \quad \forall z \in \mathbb{C}. \tag{3.20}$$

(v) *There exists $\Psi, \Phi \in \operatorname{Exp}(\mathbb{C})$ such that for all $f \in \mathcal{H}$, $Lf = \Psi(\ell_\alpha)f + D\Phi(\ell_\alpha)f$, where $\Psi(\ell_\alpha)f$ and $D\Phi(\ell_\alpha)f$ are given by*

$$\begin{aligned}
&[\Psi(\ell_\alpha)f](z) = \sum_{n=0}^{+\infty} a_{2n}\ell_\alpha^n f(z), \quad \forall z \in \mathbb{C}, \\
&[D\Phi(\ell_\alpha)f](z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_\alpha^n f)}{dz}(z), \quad \forall z \in \mathbb{C},
\end{aligned} \tag{3.21}$$

*where $\Psi(z) = \sum_{n=0}^{+\infty} a_{2n} z^n$ and $\Phi(z) = c_1(\alpha)\sum_{n=0}^{+\infty} a_{2n+1} z^n$.*

**Proof.** (i)⇒(ii). From (3.13) and (3.14), we have

$$\begin{aligned}
D(Lg) &= \lim_{w\to 0} \frac{1}{w}[\tau_w Lg - Lg - wDLg] = L\left(\lim_{w\to 0} \frac{1}{w}[\tau_w g - g]\right) = L(Dg), \\
\ell_\alpha(Lg) &= \lim_{w\to 0} \frac{c_2(\alpha)}{w^2}[\tau_w Lg - g - wDLg] = L\left(\lim_{w\to 0} \frac{c_2(\alpha)}{w^2}[\tau_w g - g - wDg]\right) = L(\ell_\alpha g).
\end{aligned} \tag{3.22}$$

Hence (i) implies (ii).

(ii)⇒(i). We decide the results from Corollary 2.11.

(i)⇒(iii). Assume that (i) holds. We define the functional $T$ on $\mathcal{H}$ as follows:

$$\langle T, f\rangle = L(f)(0), \quad f \in \mathcal{H}. \tag{3.23}$$

It is clear that $T$ is in $\mathcal{H}'$ and $Lf = T * f$, $f \in \mathcal{H}$.

(iii)⇒(iv). It follows immediately from Hahn-Banach and Riesz representation theorems.

(iv)⇒(v). Suppose that for all $f \in \mathcal{H}$, we have

$$\forall z \in \mathbb{C}, \quad L(f)(z) = \int_{\mathbb{C}} (\tau_z f)(w)\,d\gamma(w), \tag{3.24}$$

where $\gamma$ is a complex Borel regular measure with compact support.

According to Corollary 2.11, we obtain for all $z \in \mathbb{C}$,

$$L(f)(z) = \sum_{n=0}^{+\infty} \ell_\alpha^n f(z) \int_{\mathbb{C}} \frac{w^{2n}}{c_{2n}(\alpha)}\,d\gamma(w) + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_\alpha^n f)}{dz}(z) \int_{\mathbb{C}} \frac{w^{2n+1}}{c_{2n+1}(\alpha)}\,d\gamma(w). \tag{3.25}$$

Hence

$$Lf = \Psi(\ell_\alpha) f + D\Phi(\ell_\alpha) f, \tag{3.26}$$

where

$$\Psi(z) = \sum_{n=0}^{+\infty} a_{2n} z^n, \qquad \Phi(z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} z^n, \tag{3.27}$$

with, for every $n \in \mathbb{N}$,

$$a_n = \int_{\mathbb{C}} \frac{w^n}{c_n(\alpha)}\,d\gamma(w). \tag{3.28}$$

Since $\gamma$ has compact support on $\mathbb{C}$, for certain $a$ and $C$, we have

$$\forall n \in \mathbb{N}, \quad |a_n| \le C \frac{a^n}{c_n(\alpha)}. \tag{3.29}$$

Then we have

$$\forall z \in \mathbb{C}, \quad |\Psi(z)| \le C \sum_{n=0}^{+\infty} \frac{(|z|a)^n}{c_n(\alpha)} = C S_\alpha(|z|a) \le C e^{|z|a}. \tag{3.30}$$

Similarly we have

$$\forall z \in \mathbb{C}, \quad |\Phi(z)| \le c_1(\alpha) C e^{|z|a}. \tag{3.31}$$

Thus we have proved that (v) is true.

(v)⇒(i). Suppose now that, for every $f \in \mathcal{H}$ and $z \in \mathbb{C}$,

$$(Lf)(z) = \sum_{n=0}^{+\infty} a_{2n} (\ell_\alpha^n f)(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_\alpha^n f)}{dz}(z), \tag{3.32}$$

for a certain $a_k \in \mathbb{C}$, $k \in \mathbb{N}$, where the series converges in $\mathcal{H}$.

Hence, if $f \in \mathscr{H}$, since $\tau_z \ell_\alpha f = \ell_\alpha \tau_z f$, $z \in \mathbb{C}$, using (2.38) and the fact that $\tau_z$ is a continuous linear mapping from $\mathscr{H}$ into itself, we obtain for every $z, w \in \mathbb{C}$,

$$\begin{aligned}\tau_w(Lf)(z) &= \sum_{n=0}^{+\infty} a_{2n}\tau_w(\ell_\alpha^n f)(z) + c_1(\alpha)\sum_{n=0}^{+\infty} a_{2n+1}\tau_w\left(\frac{d(\ell_\alpha^n f)}{dz}\right)(z) \\ &= \sum_{n=0}^{+\infty} a_{2n}\ell_\alpha^n(\tau_w f)(z) + c_1(\alpha)\sum_{n=0}^{+\infty} a_{2n+1}\frac{d(\ell_\alpha^n(\tau_w f))}{dz}(z) \\ &= L(\tau_w f)(z).\end{aligned} \tag{3.33}$$

Hence (v) implies (i). □

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