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ON MIXED-TYPE REVERSE-ORDER LAWS FOR THE MOORE-PENROSE INVERSE OF A MATRIX PRODUCT

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Some mixed-type reverse-order laws for the Moore-Penrose inverse of a matrix product are established. Necessary and sufficient conditions for these laws to hold are found by the matrix rank method. Some applications and extensions of these reverse-order laws to the weighted Moore-Penrose inverse are also given.

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If A and B are a pair of invertible matrices of the same size, then the product AB is nonsingular, too, and the inverse of the product AB satisfies the reverse-order law $(AB)^{-1} = B^{-1}A^{-1}$. This law can be used to find the properties of $(AB)^{-1}$, as well as to simplify various matrix expressions that involve the inverse of a matrix product. However, this formula cannot trivially be extended to the Moore-Penrose inverse of matrix products. For a general $m \times n$ complex matrix A, the Moore-Penrose inverse $A^\dagger$ of A is the unique $n \times m$ matrix X that satisfies the following four Penrose equations:

(i) $AXA = A$,
(ii) $XAX = X$,
(iii) $(AX)^* = AX$,
(iv) $(XA)^* = XA$,

where $(\cdot)^*$ denotes the conjugate transpose of a complex matrix. A matrix X is called a $\{1\}$-inverse (inner inverse) of A if it satisfies (i) and is denoted by A^-. General properties of the Moore-Penrose inverse can be found in [2, 4, 16].

Let A and B be a pair of matrices such that AB exists. In many situations, one needs to find the Moore-Penrose inverse of the product AB and its properties. Because $A^\dagger A$, $BB^\dagger$, and $BB^\dagger A^\dagger A$ are not necessarily identity matrices, the relationship between $(AB)^\dagger$ and $B^\dagger A^\dagger$ is quite complicated and the reverse-order law $(AB)^\dagger = B^\dagger A^\dagger$ does not necessarily hold. Therefore, it is not easy to simplify matrix expressions that involve the Moore-Penrose inverse of matrix products. Theoretically speaking, for any matrix product AB, the Moore-Penrose inverse $(AB)^\dagger$ can be written as

$$(AB)^\dagger = B^\dagger A^\dagger \quad \text{or} \quad (AB)^\dagger = B^\dagger A^\dagger + X, \tag{1}$$

where X is a residue matrix. For these two situations, one can consider the following two problems:

(I) necessary and sufficient conditions for $(AB)^\dagger = B^\dagger A^\dagger$ to hold,

(II) if $(AB)^\dagger \neq B^\dagger A^\dagger$, find possible expressions of X in $(AB)^\dagger = B^\dagger A^\dagger + X$, and then determine necessary and sufficient conditions for $(AB)^\dagger = B^\dagger A^\dagger + X$ to hold.

The investigation of the Moore-Penrose inverse of the product AB was started in 1960s. For the standard situation $(AB)^\dagger = B^\dagger A^\dagger$, a well-known result due to Greville [9] asserts that

$$(AB)^\dagger = B^\dagger A^\dagger \Longleftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B), \qquad \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*), \tag{2}$$

where $\mathcal{R}(\cdot)$ denotes the range (column space) of a matrix. Many other equivalent conditions for $(AB)^\dagger = B^\dagger A^\dagger$ to hold can be found in [2, 4, 16, 26]. Generally speaking, the two range inclusions in (2) are strict conditions for any pair of matrices A and B to satisfy. Therefore, it is necessary to seek various weaker reverse-order laws for $(AB)^\dagger$ to satisfy. In addition to (2), $(AB)^\dagger$ may satisfy some other mixed-type reverse-order laws. For example,

$$\begin{gathered}(AB)^\dagger = (A^\dagger AB)^\dagger (ABB^\dagger)^\dagger, \\ (AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger, \qquad (AB)^\dagger = B^* (A^*ABB^*)^\dagger A^*.\end{gathered} \tag{3}$$

These reverse-order laws were studied in [6, 8, 11, 26]. Although these matrix equalities are more complicated than the law in (2), the conditions for these equalities to hold are weaker than that for (2) to hold. In fact, mixed-type reverse-order laws also stem from various reasonable operations for the Moore-Penrose inverse of matrix products (see Remark 10). Although $(AB)^\dagger$ can be written as $(AB)^\dagger = B^\dagger A^\dagger + X$ in general, it is not easy to give an explicit expression for the residue matrix X for the given matrices A and B. Some discussion for the expression of X and its properties were given in [8].

In the investigation of $(AB)^\dagger$, we observe that a possible expression for $(AB)^\dagger$ is

$$(AB)^\dagger = B^\dagger A^\dagger - B^\dagger [(I_n - BB^\dagger)(I_n - A^\dagger A)]^\dagger A^\dagger. \tag{4}$$

A direct motivation for us to find out the residue matrix in (4) arises from two different decompositions of the following block matrix:

$$M = \begin{bmatrix} I_n & B \\ A & 0 \end{bmatrix} \tag{5}$$

and its generalized inverses. In fact, it is easy to verify that M can be decomposed as the following two forms:

$$\begin{gathered} M = \begin{bmatrix} I_n & 0 \\ A & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -AB \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & I_k \end{bmatrix} := P_1 N_1 Q_1, \\ M = \begin{bmatrix} I_n & (I_n - BB^\dagger)A^\dagger \\ 0 & I_m \end{bmatrix} \begin{bmatrix} T & B \\ A & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ B^\dagger & I_k \end{bmatrix} := P_2 N_2 Q_2, \end{gathered} \tag{6}$$

where $T = (I_n - BB^\dagger)(I_n - A^\dagger A)$. From these two decompositions, one can find two $\{1\}$-inverses of M as follows:

$$\begin{aligned} M^- &= Q_1^{-1}N_1^\dagger P_1^{-1} = \begin{bmatrix} I_n & -B \\ 0 & I_k \end{bmatrix}\begin{bmatrix} I_n & 0 \\ 0 & -(AB)^\dagger \end{bmatrix}\begin{bmatrix} I_n & 0 \\ -A & I_m \end{bmatrix} \\ &= \begin{bmatrix} I_n & B(AB)^\dagger \\ (AB)^\dagger A & -(AB)^\dagger \end{bmatrix}, \\ M^- &= Q_2^{-1}N_2^\dagger P_2^{-1} = \begin{bmatrix} I_n & 0 \\ -B^\dagger & I_k \end{bmatrix}\begin{bmatrix} T^\dagger & A^\dagger \\ B^\dagger & 0 \end{bmatrix}\begin{bmatrix} I_n & -(I_n - BB^\dagger)A^\dagger \\ 0 & I_m \end{bmatrix} \\ &= \begin{bmatrix} T^\dagger & A^\dagger - T^\dagger A^\dagger \\ B^\dagger - B^\dagger T^\dagger & B^\dagger T^\dagger A^\dagger - B^\dagger A^\dagger \end{bmatrix}. \end{aligned} \tag{7}$$

These two $\{1\}$-inverses of M are not necessarily equal. Therefore, it is natural to consider under what conditions the two $\{1\}$-inverses of M in (7) are equal; or some blocks of them are equal. The mixed-type reverse-order law (4) is noticed by comparing the lower right blocks of (7).

Because the right-hand side of (4) involves complicated matrix operations, it is not easy to establish necessary and sufficient conditions for (4) to hold by definitions, as well as various matrix decompositions associated with A, B, and AB. In the investigation of various problems on generalized inverses of matrices, the present author notices that the rank of matrix is a simple and powerful method for dealing with the relationship between any two matrix expressions involving generalized inverses. In fact, any two matrices A and B of the same size are equal if and only if $r(A-B) = 0$, where $r(\cdot)$ denotes the rank of a matrix. If one can find some nontrivial formulas for the rank of $A - B$, then necessary and sufficient conditions for $A = B$ to hold can be derived from these rank formulas. This method can be used for investigating the relations between any two matrix expressions that involve generalized inverses. Several simple rank formulas for the differences of matrices found by the present author are given below:

$$\begin{gathered} r(A^kA^\dagger - A^\dagger A^k) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k, A^*] - 2r(A), \\ r(A^*A^\dagger - A^\dagger A^*) = r(AA^*A^2 - A^2A^*A), \\ r(AB - ABB^\dagger A^\dagger AB) = r[A^*, B] + r(AB) - r(A) - r(B), \\ r\left([A,B]^\dagger - \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}\right) = r[AA^*B, BB^*A], \\ r\left([A,B]^\dagger[A,B] - \begin{bmatrix} A^\dagger A & 0 \\ 0 & B^\dagger B \end{bmatrix}\right) = r(A) + r(B) - r[A,B], \\ \min_{A^-,B^-} r(A^- - B^-) = r(A-B) - r\begin{bmatrix} A \\ B \end{bmatrix} - r[A,B] + r(A) + r(B), \end{gathered} \tag{8}$$

where $[A,B]$ denotes a row block matrix; see Tian [20, 23, 25, 26]. The significance of these simple rank formulas is: they connect different matrix expressions through the rank of these matrices. From these rank equalities, one can derive some basic properties for the matrices on the left-hand sides. For instance, let the right-hand sides of the above rank equalities be zero and simplify by some elementary methods, one can immediately obtain necessary and sufficient conditions for the matrices on the left-hand sides to be zero.

In this paper, we establish a rank formula associated with (4) and then derive from the rank formula a necessary and sufficient condition for (4) to hold.

The following rank formulas are well known:

$$r[A,B] = r(A) + r(B - AA^{\dagger}B) = r(B) + r(A - BB^{\dagger}A), \tag{9}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[(I - BB^{\dagger})A(I - C^{\dagger}C)]; \tag{10}$$

if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^{\dagger}B), \tag{11}$$

see Marsaglia and Styan [12].

Recall the equality $A^*(A^*AA^*)^{\dagger}A^* = A^{\dagger}$ (see Zlobec [28]) and notice that $\mathcal{R}(A) = \mathcal{R}(AA^*A)$ and $\mathcal{R}(A^*) = \mathcal{R}(A^*AA^*)$. By appealing to (11), the rank of the Schur complement $D - CA^{\dagger}B$ is

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{12}$$

This rank equality is quite useful in dealing with various matrix expressions involving the Moore-Penrose inverse. Rank formulas for the Schur complement $D - CA^{-}B$, where A^{-} is a $\{1\}$-inverse of A, can be found in [23].

The main result of this paper is given below.

Theorem 1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and let $T = (I_n - BB^{\dagger})(I_n - A^{\dagger}A)$. Then,*

$$r[(AB)^{\dagger} - B^{\dagger}A^{\dagger} + B^{\dagger}T^{\dagger}A^{\dagger}] = r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[AB, AA^*AB] - 2r(AB). \tag{13}$$

Hence, the reverse-order law (4) holds if and only if A and B satisfy the following two range equalities:

$$\mathcal{R}(AA^*AB) = \mathcal{R}(AB), \qquad \mathcal{R}\left[(ABB^*B)^*\right] = \mathcal{R}[(AB)^*]. \tag{14}$$

Proof. From (12), we first obtain

$$r[(AB)^{\dagger} - B^{\dagger}A^{\dagger} + B^{\dagger}T^{\dagger}A^{\dagger}] = r\begin{bmatrix} T^*TT^* & T^*A^{\dagger} \\ B^{\dagger}T^* & B^{\dagger}A^{\dagger} - (AB)^{\dagger} \end{bmatrix} - r(T). \tag{15}$$

Applying (10) to T yields

$$r(T)=r\begin{bmatrix} I_n & B \\ A & 0 \end{bmatrix}-r(A)-r(B)=n+r(AB)-r(A)-r(B). \tag{16}$$

It is easy to verify that $T^*TT^*=(I_n-A^\dagger A)T(I_n-BB^\dagger)$. Recall that elementary matrix operations and block elementary matrix operations do not change the rank of matrix. Thus, we can find by (10) and block elementary matrix operations that

$$\begin{aligned}
& r\begin{bmatrix} T^*TT^* & T^*A^\dagger \\ B^\dagger T^* & B^\dagger A^\dagger-(AB)^\dagger \end{bmatrix} \\
&\quad = r\begin{bmatrix} (I_n-A^\dagger A)T(I_n-BB^\dagger) & (I_n-A^\dagger A)(I_n-BB^\dagger)A^\dagger \\ B^\dagger(I_n-A^\dagger A)(I_n-BB^\dagger) & B^\dagger A^\dagger-(AB)^\dagger \end{bmatrix} \\
&\quad = r\begin{bmatrix} T & (I_n-BB^\dagger)A^\dagger & A^\dagger \\ B^\dagger(I_n-A^\dagger A) & B^\dagger A^\dagger-(AB)^\dagger & 0 \\ B^\dagger & 0 & 0 \end{bmatrix}-r(A)-r(B) \\
&\quad = r\begin{bmatrix} I_n+BB^\dagger A^\dagger A & -BB^\dagger A^\dagger & A^\dagger \\ -B^\dagger A^\dagger A & B^\dagger A^\dagger-(AB)^\dagger & 0 \\ B^\dagger & 0 & 0 \end{bmatrix}-r(A)-r(B) \\
&\quad = r\begin{bmatrix} I_n & -BB^\dagger A^\dagger & A^\dagger \\ -(AB)^\dagger A & B^\dagger A^\dagger-(AB)^\dagger & 0 \\ B^\dagger & 0 & 0 \end{bmatrix}-r(A)-r(B) \\
&\quad = r\begin{bmatrix} I_n & 0 & 0 \\ 0 & B^\dagger A^\dagger-(AB)^\dagger-(AB)^\dagger ABB^\dagger A^\dagger & (AB)^\dagger \\ 0 & B^\dagger A^\dagger & -B^\dagger A^\dagger \end{bmatrix}-r(A)-r(B) \\
&\quad = n+r\begin{bmatrix} B^\dagger A^\dagger-(AB)^\dagger ABB^\dagger A^\dagger & (AB)^\dagger \\ 0 & -B^\dagger A^\dagger \end{bmatrix}-r(A)-r(B).
\end{aligned} \tag{17}$$

Also note that $\mathcal{R}\{[I_n-(AB)^\dagger AB]B^\dagger A^\dagger\}\cap\mathcal{R}[(AB)^\dagger]=\{0\}$. It follows that

$$\mathcal{R}\begin{bmatrix} B^\dagger A^\dagger-(AB)^\dagger ABB^\dagger A^\dagger \\ 0 \end{bmatrix}\cap\mathcal{R}\begin{bmatrix} (AB)^\dagger \\ -B^\dagger A^\dagger \end{bmatrix}=\{0\}. \tag{18}$$

Thus,

$$\begin{aligned}
& r\begin{bmatrix} B^\dagger A^\dagger-(AB)^\dagger ABB^\dagger A^\dagger & (AB)^\dagger \\ 0 & -B^\dagger A^\dagger \end{bmatrix} \\
&\quad = r\begin{bmatrix} B^\dagger A^\dagger-(AB)^\dagger ABB^\dagger A^\dagger \\ 0 \end{bmatrix}+r\begin{bmatrix} (AB)^\dagger \\ -B^\dagger A^\dagger \end{bmatrix} \\
&\quad = r[B^\dagger A^\dagger-(AB)^\dagger ABB^\dagger A^\dagger]+r\begin{bmatrix} (AB)^\dagger \\ B^\dagger A^\dagger \end{bmatrix}.
\end{aligned} \tag{19}$$

Applying (9) gives

$$\begin{aligned} r[B^\dagger A^\dagger-(AB)^\dagger(AB)B^\dagger A^\dagger] &= r[(AB)^\dagger, B^\dagger A^\dagger]-r[(AB)^\dagger] \\ &= r[(AB)^*, B^\dagger A^*]-r(AB) \\ &= r[BB^*(AB)^*, B^*A^*]-r(AB) \\ &= r\begin{bmatrix} ABB^*B \\ AB \end{bmatrix}-r(AB), \\ r\begin{bmatrix} (AB)^\dagger \\ B^\dagger A^\dagger \end{bmatrix} = r\begin{bmatrix} (AB)^* \\ B^*A^\dagger \end{bmatrix} &= r\begin{bmatrix} (AB)^*AA^* \\ B^*A^* \end{bmatrix} = r[AA^*AB, AB]. \end{aligned} \tag{20}$$

Substituting (20) into (19), and (19) into (17), and then (16) and (17) into (15) gives us (13). Let the right-hand side of (13) be zero and note that

$$r(AB) = r(AA^*AB) = r(ABB^*B). \tag{21}$$

Then the equivalence of (4) and (14) follows. □

The establishment of (13) is not easy, because the matrix expression on the left-hand side of (13) involves three terms consisting of the Moore-Penrose inverse and the right-hand side of (13) involve ranks of block matrices with the products AB, AA^*AB, and ABB^*B. However, the two block matrices on the right-hand side of (13) and the two range equalities in (14) are easy to simplify when A and B satisfy some conditions. For example, if both A and B are partial isometries, that is, $A^\dagger = A^*$ and $B^\dagger = B^*$, then (14) is satisfied. In this case, (4) becomes

$$(AB)^\dagger = (AB)^* - B^*[(I_n - BB^*)(I_n - A^*A)]^\dagger A^*. \tag{22}$$

The most valuable consequence of (4) is concerned with the Moore-Penrose inverse of the product of two orthogonal projectors.

Corollary 2. *Let P and Q be a pair of orthogonal projectors of order n. Then, the product PQ satisfies the following two identities:*

$$(PQ)^\dagger = QP - Q(Q^\perp P^\perp)^\dagger P, \tag{23}$$

$$(PQ)^2 = PQ + PQ(Q^\perp P^\perp)^\dagger PQ, \tag{24}$$

where $P^\perp = I_n - P$ and $Q^\perp = I_n - Q$. In particular,

$$\begin{aligned} (PQ)^\dagger = QP &\Longleftrightarrow Q(Q^\perp P^\perp)^\dagger P = 0, \\ (PQ)^2 = PQ &\Longleftrightarrow PQ(Q^\perp P^\perp)^\dagger PQ = 0. \end{aligned} \tag{25}$$

Proof. Note that $P^2 = P = P^* = P^\dagger$, $Q^2 = Q = Q^* = Q^\dagger$ for any pair of orthogonal projectors P and Q. Thus, P and Q satisfy (14), and (4) is reduced to (23). Premultiplying and postmultiplying both sides of (23) by PQ yield (24). □

Recall that for any pair of orthogonal projectors P and Q,

$$(PQ)^{\dagger} = QP \Longleftrightarrow (PQ)^2 = PQ \Longleftrightarrow PQ = QP, \tag{26}$$

see, for example, [1]. Thus, (25) could be regarded as two new equivalent statements for the commutativity of two orthogonal projectors. There are many results in the literature on products of orthogonal projectors and related topics; see, for example, [1, 13]. The two identities (23) and (24) are two fundamental results for the product of two orthogonal projectors. They can be used for dealing with various matrix expressions that involve products of two orthogonal projectors. For example, the product PQP satisfies the following identity:

$$\begin{aligned}(PQP)^{\dagger} &= [(PQ)(PQ)^*]^{\dagger} = (QP)^{\dagger}(PQ)^{\dagger} \\ &= P\left[I_n - (P^{\perp}Q^{\perp})^{\dagger}\right]Q\left[I_n - (Q^{\perp}P^{\perp})^{\dagger}\right]P.\end{aligned} \tag{27}$$

Moreover, it is easy to verify that

$$[(PQ)^2]^{\dagger} = (PQ)^{\dagger}(QP)^{\dagger}(PQ)^{\dagger}, \qquad (PQ)^{\#} = (QP)^{\dagger}(PQ)^{\dagger}(QP)^{\dagger}, \tag{28}$$

where $(PQ)^{\#}$ is the group inverse of PQ. Hence, one can also derive from (23) two identities for $[(PQ)^2]^{\dagger}$ and $(PQ)^{\#}$. From (23), one can also derive some valuable expressions for $(P \pm Q)^{\dagger}$ and $(PQ \pm QP)^{\dagger}$; see [5].

Let $\|A\|$ denote the spectral norm of a matrix A, that is, the maximal singular value of A. For a nonnull orthogonal projector P, $\|P\| = 1$. For any pair of orthogonal projectors P and Q with $PQ \neq 0$, it can be derived from (23) the following norm equality:

$$\|(PQ)^{\dagger}\| \leq \left\|I_n - (Q^{\perp}P^{\perp})^{\dagger}\right\|. \tag{29}$$

It was shown in [3, 15] that if P is idempotent with $P \neq 0$ and $P \neq I$, then $\|I - P\| = \|P\|$. If P and Q are two orthogonal projectors, then $(PQ)^{\dagger}$ is idempotent; see [14]. Note that $I_n - P$ and $I_n - Q$ are orthogonal projectors. Hence, $[(I_n - Q)(I_n - P)]^{\dagger}$ is idempotent. Thus, if $(I_n - Q)(I_n - P) \neq 0$, then

$$\left\|I_n - (Q^{\perp}P^{\perp})^{\dagger}\right\| = \left\|(Q^{\perp}P^{\perp})^{\dagger}\right\|. \tag{30}$$

Applying this equality to (29), we see that if $PQ \neq 0$ and $Q^{\perp}P^{\perp} \neq 0$, then

$$\|(PQ)^{\dagger}\| \leq \left\|(Q^{\perp}P^{\perp})^{\dagger}\right\|. \tag{31}$$

Replacing P with $I_n - Q$ and Q with $I_n - P$ in (31) also gives

$$\left\|(Q^{\perp}P^{\perp})^{\dagger}\right\| \leq \|(PQ)^{\dagger}\|. \tag{32}$$

Hence, we have the following result.

Theorem 3. *Let P and Q be a pair of orthogonal projectors with both $PQ \neq 0$ and $Q^{\perp}P^{\perp} \neq 0$. Then,*

$$\|(PQ)^{\dagger}\| = \left\|(Q^{\perp}P^{\perp})^{\dagger}\right\|. \tag{33}$$

Identity (23) can be used to establish some identities for the Moore-Penrose inverse of ABC, where A, B, and C are three orthogonal projectors.

Theorem 4. *Let A, B, and C be a triple of orthogonal projectors of order n. Then,*

$$(ABC)^{\dagger} = \left(P_{(AB)^*}C\right)^{\dagger}\left(AP_{BC}\right)^{\dagger}, \tag{34}$$

$$(ABC)^{\dagger} = C\left[I_n - \left(C^{\perp}P^{\perp}_{(AB)^*}\right)^{\dagger}\right](AB)^{\dagger}B(BC)^{\dagger}\left[I_n - \left(P^{\perp}_{BC}A^{\perp}\right)^{\dagger}\right]A, \tag{35}$$

where $P_{(AB)^*} = (AB)^{\dagger}AB$ and $P_{BC} = BC(BC)^{\dagger}$.

Proof. Recall a simple result in [7] that any matrix product UV satisfies

$$(UV)^{\dagger} = \left(U^{\dagger}UV\right)^{\dagger}\left(UVV^{\dagger}\right)^{\dagger}. \tag{36}$$

Applying this formula to $ABC = (AB)(BC)$ gives (34). Applying (23) to $(P_{(AB)^*}C)^{\dagger}$ and $(AP_{BC})^{\dagger}$ also gives

$$\begin{aligned}\left(P_{(AB)^*}C\right)^{\dagger} &= CP_{(AB)^*} - C\left(C^{\perp}P^{\perp}_{(AB)^*}\right)^{\dagger}P_{(AB)^*},\\ \left(AP_{BC}\right)^{\dagger} &= P_{BC}A - P_{BC}\left(P^{\perp}_{BC}A^{\perp}\right)^{\dagger}A.\end{aligned} \tag{37}$$

Substituting these two results into (34) yields (35). □

Although reverse-order laws for the Moore-Penrose inverse of matrix products have many different expressions, some of these reverse-order laws may be equivalent. A simple example is

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Longleftrightarrow \left[(A^{\dagger})^*B\right]^{\dagger} = B^{\dagger}A^* \Longleftrightarrow \left[A(B^{\dagger})^*\right]^{\dagger} = B^*A^{\dagger}; \tag{38}$$

see [21]. When investigating (4), we also find that (4) is equivalent to the following two mixed-type reverse-order laws:

$$(AB)^{\dagger} = B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}, \tag{39}$$

$$(AB)^{\dagger} = B^*\left(A^*ABB^*\right)^{\dagger}A^*. \tag{40}$$

The reverse-order law (39) was first studied by Galperin and Waksman [8], and then by Izumino [11] for a product of two linear operators. They showed that (39) holds if and only if

$$\mathcal{R}\left[(A^*)^{\dagger}B\right] = \mathcal{R}(AB), \qquad \mathcal{R}(B^{\dagger}A^*) = \mathcal{R}[(AB)^*]. \tag{41}$$

This result is also true for a complex matrix product. Because the Moore-Penrose inverses of A and B are contained in the condition (41), it cannot be regarded as a satisfactory necessary and sufficient condition for (39) to hold. In fact, (41) is equivalent to (14) by noticing

$$\begin{aligned} r\left[AB, (A^*)^{\dagger}B\right] &= r\left[AA^*AB, AA^*(A^*)^{\dagger}B\right] = r[AA^*AB, AB],\\ r\begin{bmatrix} AB \\ A(B^*)^{\dagger}\end{bmatrix} &= r\begin{bmatrix} ABB^*B \\ A(B^*)^{\dagger}B^*B\end{bmatrix} = r\begin{bmatrix} ABB^*B \\ AB\end{bmatrix}.\end{aligned} \tag{42}$$

Without much effort, one can show that

$$r\left[(AB)^{\dagger}-B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]=r\begin{bmatrix}AB\\ABB^{*}B\end{bmatrix}+r[AB,AA^{*}AB]-2r(AB);\tag{43}$$

see [26]. Equality (43) is derived from the following result.

Lemma 5 [22]. *Let X_1 and X_2 be a pair of outer inverses of a matrix A, that is, $X_1AX_1 = X_1$ and $X_2AX_2 = X_2$. Then,*

$$r(X_1-X_2)=r\begin{bmatrix}X_1\\X_2\end{bmatrix}+r[X_1,X_2]-r(X_1)-r(X_2).\tag{44}$$

Hence,

$$X_1=X_2\Longleftrightarrow\mathcal{R}(X_1)=\mathcal{R}(X_2),\qquad\mathcal{R}(X_1^{*})=\mathcal{R}(X_2^{*}).\tag{45}$$

Obviously, the matrix $(AB)^{\dagger}$ is an outer inverse of AB by the definition of the Moore-Penrose inverse. It is also easy to verify that $B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$ is an outer inverse of AB. Thus, it follows by (44) that

$$\begin{aligned}r&\left[(AB)^{\dagger}-B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]\\&=r\begin{bmatrix}(AB)^{\dagger}\\B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\end{bmatrix}+r\left[(AB)^{\dagger},B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]\\&\quad-r[(AB)^{\dagger}]-r\left[B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right].\end{aligned}\tag{46}$$

Hence, one can derive from (44) that (39) holds if and only if

$$\mathcal{R}[(AB)^{\dagger}]=\mathcal{R}\left[B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right],\qquad\mathcal{R}\left\{[(AB)^{\dagger}]^{*}\right\}=\mathcal{R}\left\{\left[B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]^{*}\right\}.\tag{47}$$

It is also easy to verify that

$$\begin{gathered}r\begin{bmatrix}(AB)^{\dagger}\\B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\end{bmatrix}=r[AB,AA^{*}AB],\\r\left[(AB)^{\dagger},B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]=r\begin{bmatrix}AB\\ABB^{*}B\end{bmatrix},\\r[(AB)^{\dagger}]=r\left[B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]=r(AB).\end{gathered}\tag{48}$$

Then, (43) follows. Another rank equality related to (39) is

$$r\left[(AB)^{\dagger}-B^{\dagger}\left(A^{\dagger}ABB^{\dagger}\right)^{\dagger}A^{\dagger}\right]=r\left[(AB)^{\dagger}-\left(A^{\dagger}AB\right)^{\dagger}A^{\dagger}\right]+r\left[(AB)^{\dagger}-B^{\dagger}\left(ABB^{\dagger}\right)^{\dagger}\right],\tag{49}$$

which is shown in [26]. Two rank formulas associated with (40) are given below.

Theorem 6. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$
\begin{aligned}
r\left[(AB)^{\dagger} - B^*(A^*ABB^*)^{\dagger}A^*\right] = r\begin{bmatrix} (AB)^{\dagger} \\ B^*(A^*ABB^*)^{\dagger}A^* \end{bmatrix} &+ r\left[(AB)^{\dagger}, B^*(A^*ABB^*)^{\dagger}A^*\right] \\
&- r\left[(AB)^{\dagger}\right] - r\left[B^*(A^*ABB^*)^{\dagger}A^*\right],
\end{aligned} \tag{50}
$$

$$
r\left[(AB)^{\dagger} - B^*(A^*ABB^*)^{\dagger}A^*\right] = r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} + r[AB, AA^*AB] - 2r(AB). \tag{51}
$$

Hence, the following statements are equivalent:

(a) *(40) holds,*

(b) $\mathcal{R}[(AB)^{\dagger}] = \mathcal{R}[B^*(A^*ABB^*)^{\dagger}A^*]$, $\mathcal{R}\{[(AB)^{\dagger}]^*\} = \mathcal{R}\{[B^*(A^*ABB^*)^{\dagger}A^*]^*\}$,

(c) *(14) holds.*

Proof. Note that $B^*(A^*ABB^*)^{\dagger}A^*(AB)B^*(A^*ABB^*)^{\dagger}A^* = B^*(A^*ABB^*)^{\dagger}A^*$, that is, $B^*(A^*ABB^*)^{\dagger}A^*$ is an outer inverse of AB. Thus, (50) is derived from (44), and (51) is a simplification of (50). □

Applying (13), (43), and (51) to the two products $A^{\dagger}AB$ and $ABB^{\dagger}$ gives us the following result.

Theorem 7. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and let $T = (I_n - BB^{\dagger})(I_n - A^{\dagger}A)$. Then,*

$$
\begin{aligned}
r\left[(A^{\dagger}AB)^{\dagger} - B^{\dagger}A^{\dagger}A + B^{\dagger}T^{\dagger}A^{\dagger}A\right] &= r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} - r(AB), \\
r\left[(ABB^{\dagger})^{\dagger} - BB^{\dagger}A^{\dagger} + BB^{\dagger}T^{\dagger}A^{\dagger}\right] &= r[AB, AA^*AB] - r(AB), \\
r\left[(A^{\dagger}AB)^{\dagger} - B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}A\right] &= r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} - r(AB), \\
r\left[(ABB^{\dagger})^{\dagger} - BB^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}\right] &= r[AB, AA^*AB] - r(AB), \\
r\left[(A^{\dagger}AB)^{\dagger} - B^*(A^{\dagger}ABB^*)^{\dagger}A^{\dagger}A\right] &= r\begin{bmatrix} AB \\ ABB^*B \end{bmatrix} - r(AB), \\
r\left[(ABB^{\dagger})^{\dagger} - BB^{\dagger}(A^*ABB^{\dagger})^{\dagger}A^*\right] &= r[AB, AA^*AB] - r(AB).
\end{aligned} \tag{52}
$$

Hence,

(a) *the following statements are equivalent:*

  (i) $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A - B^{\dagger}T^{\dagger}A^{\dagger}A$,

  (ii) $(A^{\dagger}AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}$,

  (iii) $(A^{\dagger}AB)^{\dagger} = B^*(A^{\dagger}ABB^*)^{\dagger}$,

  (iv) $\mathcal{R}[(ABB^*B)^*] = \mathcal{R}[(AB)^*]$,

(b) *the following statements are equivalent:*

  (i) $(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger} - BB^{\dagger}T^{\dagger}A^{\dagger}$,

  (ii) $(ABB^{\dagger})^{\dagger} = (A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$,

  (iii) $(ABB^{\dagger})^{\dagger} = (A^*ABB^{\dagger})^{\dagger}A^*$,

  (iv) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB)$.

The combination of (4), (14), (39), (40), (41), (47) and Theorems 6 and 7 gives us the following result.

Theorem 8. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and let $T = (I_n - BB^\dagger)(I_n - A^\dagger A)$. Then, the following statements are equivalent:*

(a) $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$,

(b) $(A^\dagger ABB^\dagger)^\dagger = B^\dagger(AB)^\dagger A^\dagger$,

(c) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger = B^\dagger(ABB^\dagger)^\dagger$,

(d) $(A^\dagger AB)^\dagger = (AB)^\dagger A$ *and* $(ABB^\dagger)^\dagger = B(AB)^\dagger$,

(e) $(A^\dagger AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^\dagger ABB^\dagger)^\dagger A^\dagger$,

(f) $(A^\dagger AB)^\dagger = B^*(A^\dagger ABB^*)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^* ABB^\dagger)^\dagger A^*$,

(g) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$,

(h) $(A^*ABB^*)^\dagger = (B^*)^\dagger(AB)^\dagger(A^*)^\dagger$,

(i) $(AB)^\dagger = B^\dagger A^\dagger - B^\dagger T^\dagger A^\dagger$,

(j) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A - B^\dagger T^\dagger A^\dagger A$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger - BB^\dagger T^\dagger A^\dagger$,

(k) $\mathcal{R}[(AB)^\dagger] = \mathcal{R}(B^\dagger T_1^\dagger A^\dagger)$ *and* $\mathcal{R}\{[(AB)^\dagger]^*\} = \mathcal{R}[(B^\dagger T_1^\dagger A^\dagger)^*]$, *where* $T_1 = A^\dagger ABB^\dagger$,

(l) $\mathcal{R}[(AB)^\dagger] = \mathcal{R}(B^* T_2^\dagger A^*)$ *and* $\mathcal{R}\{[(AB)^\dagger]^*\} = \mathcal{R}[(B^* T_2^\dagger A^*)^*]$, *where* $T_2 = A^*ABB^*$,

(m) $\mathcal{R}[(A^*)^\dagger B] = \mathcal{R}(AB)$ *and* $\mathcal{R}(B^\dagger A^*) = \mathcal{R}[(AB)^*]$,

(n) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB)$ *and* $\mathcal{R}[(ABB^*B)^*] = \mathcal{R}[(AB)^*]$.

The results given above can be extended to the weighted Moore-Penrose inverse of a matrix product. The weighted Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ with respect to a pair of Hermitian positive definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ is defined to be the unique $n\times m$ matrix that satisfies the following four matrix equations:

(i) $AXA = A$,

(ii) $XAX = X$,

(iii) $(MAX)^* = MAX$,

(iv) $(NXA)^* = NXA$,

and this X is denoted as $X = A^\dagger_{M,N}$. In particular, when $M = I_m$ and $N = I_n$, $A^\dagger_{I_m,I_n}$ is the standard Moore-Penrose inverse $A^\dagger$ of A. Reverse-order laws for the weighted Moore-Penrose inverse of matrix products have been studied; see [17, 24, 27]. As is well known (see, e.g., [2]), the weighted Moore-Penrose inverse $A^\dagger_{M,N}$ of A can be rewritten as

$$A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2}, \tag{53}$$

where $M^{1/2}$ and $N^{1/2}$ are the positive definite square roots of M and N, respectively. By appealing to (12), one can obtain the following basic rank formula:

$$r\left(D - CA^\dagger_{M,N}B\right) = r\begin{bmatrix} A^*MAN^{-1}A^* & A^*MB \\ CN^{-1}A^* & D \end{bmatrix} - r(A); \tag{54}$$

see also [26]. Applying Theorem 8 to $(M^{1/2}ABN^{-1/2})^\dagger$ in

$$(AB)^\dagger_{M,N} = N^{-1/2}(M^{1/2}ABN^{-1/2})^\dagger M^{1/2}, \tag{55}$$

and noting that

$$A^{\dagger}_{M,I} = (M^{1/2}A)^{\dagger}M^{1/2}, \qquad B^{\dagger}_{I,N} = N^{-1/2}(BN^{-1/2})^{\dagger} \tag{56}$$

yields the following result.

Theorem 9. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and let $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{p\times p}$ be a pair of Hermitian positive definite matrices. Then, the following statements are equivalent:*

(a) $(AB)^{\dagger}_{M,N} = B^{\dagger}_{I,N}(A^{\dagger}_{M,I}ABB^{\dagger}_{I,N})^{\dagger}A^{\dagger}_{M,I}$,
(b) $(AB)^{\dagger}_{M,N} = (A^{\dagger}_{M,P}AB)^{\dagger}_{P,N}A^{\dagger}_{M,P} = B^{\dagger}_{P,N}(ABB^{\dagger}_{P,N})^{\dagger}_{M,P}$,
(c) $(AB)^{\dagger}_{M,N} = N^{-1}B^*(A^*MABN^{-1}B^*)^{\dagger}A^*M$,
(d) $(AB)^{\dagger}_{M,N} = B^{\dagger}_{I,N}A^{\dagger}_{M,I} - B^{\dagger}_{I,N}[(I_n - BB^{\dagger}_{I,N})(I_n - A^{\dagger}_{M,I}A)]^{\dagger}A^{\dagger}_{M,I}$,
(e) $\mathcal{R}(AA^*MAB) = \mathcal{R}(AB)$ *and* $\mathcal{R}[(ABN^{-1}B^*B)^*] = \mathcal{R}[(AB)^*]$.

Remark 10. Reverse-order laws can be established from any reasonable operations for the Moore-Penrose inverse of matrix products. For example, write

$$AB = AA^{\dagger}ABB^{\dagger}B = AA^*(A^{\dagger})^*(B^{\dagger})^*B^*B = AA^*\left[(A^{\dagger})^*(B^{\dagger})^*\right]B^*B := PNQ. \tag{57}$$

The reverse-order law $(PNQ)^{\dagger} = Q^{\dagger}N^{\dagger}P^{\dagger}$ is equivalent to

$$(AB)^{\dagger} = (B^*B)^{\dagger}\left[(B^{\dagger}A^{\dagger})^{\dagger}\right]^*(AA^*)^{\dagger}. \tag{58}$$

The law $(AB)^{\dagger} = B^*(A^*ABB^*)^{\dagger}A^*$ is obtained from writing

$$AB = (A^*)^{\dagger}(A^*ABB^*)(B^*)^{\dagger} := PNQ \tag{59}$$

and $(PNQ)^{\dagger} = Q^{\dagger}N^{\dagger}P^{\dagger}$. On the other hand, mixed-type reverse-order laws can be introduced by comparing different decompositions of a block matrix and their generalized inverses. The introduction of (4) is such an example. In addition, one can consider some variations of (4). For instance, replacing the Moore-Penrose inverses in (4) with $\{1\}$-inverses yields the following reverse-order law for $(AB)^-$:

$$(AB)^- = B^-A^- - B^-F_A(E_BF_A)^-E_BA^-, \tag{60}$$

where $F_A = I_n - A^-A$ and $E_B = I_n - BB^-$. One can also establish some rank equalities associated with this reverse-order law and then derive necessary and sufficient conditions for this law to hold.

For a triple matrix product ABC, the Moore-Penrose inverse $(ABC)^{\dagger}$ can be written as either $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$ or $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger} + X$. The law $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$ was studied in [10, 19]. Necessary and sufficient conditions for this law to hold are quite strict and complicated. Two reasonable extensions of (4) to $(ABC)^{\dagger}$ are

$$\begin{aligned}(ABC)^{\dagger} &= (BC)^{\dagger}B(AB)^{\dagger} - (BC)^{\dagger}[P^{\perp}_{BC}BP^{\perp}_{(AB)^*}]^{\dagger}(AB)^{\dagger},\\ (ABC)^{\dagger} &= C^{\dagger}\left[I_p - (P^{\perp}_{C}P^{\perp}_{B^*})^{\dagger}\right]B^{\dagger}\left[I_n - (P^{\perp}_{B}P^{\perp}_{A^*})^{\dagger}\right]A^{\dagger}.\end{aligned} \tag{61}$$

Some reasonable extensions of (39) and (40) to $(ABC)^\dagger$ are

$$\begin{aligned}(ABC)^\dagger &= C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger,\\ (ABC)^\dagger &= C^*(A^*ABCC^*)^\dagger A^*,\\ (ABC)^\dagger &= (BC)^\dagger[(AB)^\dagger ABC(BC)^\dagger]^\dagger(AB)^\dagger.\end{aligned} \tag{62}$$

Various rank formulas associated with these reverse-order laws can be established. For more details, see [18].

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