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FINITE-PART SINGULAR INTEGRAL APPROXIMATIONS IN HILBERT SPACES

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Some new approximation methods are proposed for the numerical evaluation of the finite-part singular integral equations defined on Hilbert spaces when their singularity consists of a homeomorphism of the integration interval, which is a unit circle, on itself. Therefore, some existence theorems are proved for the solutions of the finite-part singular integral equations, approximated by several systems of linear algebraic equations. The method is further extended for the proof of the existence of solutions for systems of finite-part singular integral equations defined on Hilbert spaces, when their singularity consists of a system of diffeomorphisms of the integration interval, which is a unit circle, on itself.

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1. Introduction. In recent years, finite-part singular integral equations are of increasing technological importance in engineering mechanics. Fields like elasticity, plasticity, and fracture mechanics are covered with success with finite-part singular integral equations methods. This type of singular integral equations consists in the generalization of the Cauchy singular integral equations, which have been systematically studied during the last decades.

The concept of finite-part singular integrals was firstly introduced by Hadamard [2, 3]. Many years later, Schwartz [13] analyzed some basic properties of the above type of singular integrals.

Beyond the above, Kutt [4] proposed and investigated several numerical formulas for the evaluation of the finite-part singular integrals. He also explained the difference between a finite-part integral and a "generalized principle value integral."

Some years later, Golberg [1] introduced some algorithms for the numerical evaluation of the finite-part singular integrals and proposed several theorems for the convergence of the numerical solutions. The method, which he investigated, is an extension of the Galerkin's and the collocation method.

On the other hand, Ladopoulos [5, 6, 7, 8, 9] has generalized the Sokhotski-Plemelj formulas, in order to determine the limiting values of the finite-part singular integrals defined over a smooth open or closed contour. He further introduced several numerical methods for the evaluation of the finite-part singular integral equations of the first and the second kind and has applied them to some important problems of fracture mechanics.

Moreover, Ladopoulos et al. [10, 11] investigated basic concepts of functional analysis in order to prove some general properties of finite-part singular integral equations.

These integral equations were defined on Hilbert and L_p spaces, and by a proposed method, they have been reduced to the equivalent Fredholm equations. Furthermore, the above singular integral equations have been applied to the solution of several important crack problems.

Beyond the above, in the present study, the finite-part singular integral equations defined on Hilbert spaces are investigated, when their singularity consists of a homeomorphism of the integration interval, which is a unit circle, on itself. Hence, some existence theorems are proposed for the solution of the above type of singular integral equations, approximated by several systems of linear algebraic equations.

The method is further used in order to prove the existence of solutions for systems of finite-part singular integral equations, too. The singularity of the above systems consists of a system of diffeomorphisms of the integration interval, which is a unit circle, on itself.

2. Existence theorems of finite-part singular integral approximations in Hilbert spaces

Definition 2.1. Consider the finite-part singular integral equation

$$\begin{aligned}\Phi u(t) \equiv A(t)u(t) &+ \frac{B(t)}{\pi i}\,\oint_{\Gamma}\frac{u(x)}{(x-t)^{\mu}}dx + C(t)u[\varphi(t)]\\ &+ \frac{D(t)}{\pi i}\,\oint_{\Gamma}\frac{u(x)}{(x-\varphi(t))^{\mu}}dx\\ &+ \int_{\Gamma}k(t,x)u(x)dx = f(t), \quad t\in\Gamma,\ \mu\in\mathbb{N},\end{aligned}\tag{2.1}$$

where Γ denotes the unit circle $\Gamma=\{t:|t|=1\}$, $A(t),B(t),C(t),D(t),k(t,x)\in H_{\beta,m\times m}$ [12] (Banach space of $m\times m$ matrix-valued functions satisfying Hölder's condition with exponent β, $0<\beta\le 1$), $f(t)\in H_{\beta,m}$, $u(t)$ is the unknown function, and $\varphi(t)$ is a homeomorphism of Γ on itself.

Theorem 2.2. *Consider the finite-part singular integral equation (2.1), where $\varphi(t)$ satisfies the following condition:*

$$\varphi_i(t)=t,\quad \varphi_j(t)=\varphi[\varphi_{j-1}(t)],\quad 1\le j\le i,\ i\in\mathbb{N}.\tag{2.2}$$

An approximate solution of (2.1) is of the form

$$u_n(t)=\sum_{l=-n}^{n}\varepsilon_l t^l,\quad t=e^{is},\tag{2.3}$$

where the coefficients ε_l, $l=-n,\dots,n$, are obtained by solving the following system of linear algebraic equations:

$$\sum_{l=0}^{n}A_{j-l}\varepsilon_l+\sum_{l=-n}^{-1}B_{j-l}\varepsilon_l+\sum_{l=0}^{n}C_{jl}\varepsilon_l+\sum_{l=-n}^{-1}D_{jl}\varepsilon_l+\sum_{l=-n}^{n}K_{jl}\varepsilon_l=f_j\quad (j=-n,\dots,n),\tag{2.4}$$

with A_j, B_j, C_{jl}, D_{jl}, and K_{jl}, $j = \pm 1, \pm 2, \dots$, being the Fourier coefficients of the corresponding matrix-valued functions

$$A_1(t) = A(t) + B(t), \qquad B_1(t) = A(t) - B(t), \qquad C_{1l}(t) = [C(t) + D(t)][\varphi(t)]^l,$$
$$D_{1l}(t) = [C(t) - D(t)][\varphi(t)]^l, \qquad K_{1l} = \int_\Gamma k(t,x)x^l dx. \tag{2.5}$$

Moreover, consider the following conditions to be satisfied:

$$D(t) = C(t)A^{-1}[\varphi(t)]B[\varphi(t)], \tag{2.6}$$

$$\varphi'(t) \neq 0, \quad t \in \Gamma, \ \varphi'(t) \in H_\beta, \tag{2.7}$$

and the operator Φ is invertible in the Hilbert space $L_{2,m}$ [1], $\det A(t)$ *does not vanish if $t \in \Gamma$, and $\varphi(t)$ satisfies the inequality*

$$\left\| C[\varphi^{-1}(t)]A^{-1}(t)[\varphi^{-1}(t)]^{1/2} \right\| < 1, \tag{2.8}$$

where $\varphi^{-1}(t)$ is the inverse of $\varphi(t)$.

Then, for sufficiently large n, the system (2.4) has a unique solution and the approximate solution (2.3) converges to the exact solution of (2.1) at a rate described by the inequality

$$\|u(t) - u_n(t)\|_{L_{2,m}} \leq \xi n^{-\beta}, \tag{2.9}$$

where ξ denotes a constant independent of n.

Proof. Consider the following operators to be valid on the space L_2:

$$Fu(t) = \frac{1}{\pi i} ⨍_\Gamma \frac{u(x)}{(x-t)^\mu}, \qquad G = \frac{1}{2}(I + F), \qquad H = \frac{1}{2}(I - F),$$
$$Ku(t) = \int_\Gamma k(t,x)u(x)dx, \qquad Lu(t) = u[\varphi(t)], \qquad M_n u(t) = \sum_{i=-n}^{n} \varepsilon_i t^i \tag{2.10}$$

with ε_i being the Fourier coefficients of $u(t)$ and I the identity operator.

Furthermore, we introduce the operator

$$\Pi = AI + BF + Ku, \tag{2.11}$$

where A and B are the operators corresponding to multiplication by $A(t)$ and $B(t)$, respectively.

Moreover, the operator Π_n is on the subspace $\mathrm{Im} M_n$ of the space $L_{2,m}$ and is defined by the relation

$$\Pi_n = M_n \Pi M_n. \tag{2.12}$$

Hence, from (2.8), it follows that the operator $\Pi_2 = I + CA_*L$ has an inverse, where A_* is the operation of multiplying by $A_*(t) = A^{-1}[\varphi(t)]$. Therefore, from (2.6), we obtain

$$\Phi = (I + CA_*L)(A_1G + B_1H + K_*), \tag{2.13}$$

where

$$K_* = \Pi_2^{-1}K. \tag{2.14}$$

As the operator Φ is invertible, then the following operator is invertible too:

$$\Phi_* = \Pi_2^{-1}\Phi = A_1G + B_1H + K_*. \tag{2.15}$$

Furthermore, we consider the following operators:

$$\begin{gathered} \Pi_{2n} = M_n\Pi_2M_n = M_n(I + CA_*L)M_n = M_n + M_nCA_*LM_n, \\ \Phi_{2n} = (A_{1n}G + B_{1n}H + M_nK_*)M_n, \end{gathered} \tag{2.16}$$

where A_{1n} and B_{1n} denote multiplication by the matrices of polynomials of degree not higher than n, uniformly approximating most accurately the matrices $A_1(t)$ and $B_1(t)$, respectively.

Moreover, the operators Π_{2n} are invertible for all n and the operators Φ_{2n} are invertible for sufficiently large n.

By putting

$$\varepsilon_n = ||M_n\Phi M_n - \Pi_{2n}\Phi_{2n}||, \tag{2.17}$$

we obtain $\lim_{n\to\infty}\varepsilon_n = 0$.

Therefore, the operators $M_n\Phi M_n$ are invertible, beginning with some $n = n_1$, and hence the system (2.4) has a unique solution and the proof of the requested inequality (2.9) is obvious. □

Theorem 2.3. *Let the finite-part singular integral equation (2.1), where $\varphi(t)$ satisfies condition (2.2). An approximate solution of (2.1) is of the form (2.3), where the coefficients ε_l, $l = -n,\dots,n$, are obtained by solving the following system of linear algebraic equations:*

$$\begin{aligned} &\sum_{k=0}^{n}\left[A_{j-k} + \sum_{l=-n}^{n} D_{jl}A_{l-k}\right]\varepsilon_l + \sum_{k=-n}^{-1}\left[B_{j-k} + \sum_{l=-n}^{n} D_{jl}B_{l-k}\right]\varepsilon_k + \sum_{k=-n}^{n} K_{jl}\varepsilon_k \\ &\quad = f_j \quad (j = \overline{-n,n}), \end{aligned} \tag{2.18}$$

where A_j, B_j, D_{jl}, and K_{jl}, $j = \pm 1, \pm 2, \dots$, are the Fourier coefficients of the matrix-valued functions given by (2.5).

Moreover, if conditions (2.7) and (2.8) are satisfied, and the operator K is continuously invertible in the Hilbert space $L_{2,m}$, then the system (2.18) has a unique solution for sufficiently large n and the approximate solutions $u_n(t)$ converge to the exact solution $u(t)$ of the finite-part singular integral equation (2.1) with a rate given by (2.9).

Proof. By using inequality (2.8), the linear system (2.18) is equivalent to the operational-equation system

$$\Phi_n u_n \equiv M_n(I+CA_*L)M_n(AI+BF)M_n u_n + M_n K M_n u_n = M_{nf}, \tag{2.19}$$

where the operators M_n, L, F, and K are given by (2.10), I denotes the identity operator, and $A_*(t) = A^{-1}[\varphi(t)]$.

Furthermore, the operator $AI+BF$ is invertible and because of the conditions of the theorem, beginning with some $n = n_*$, the operators $Z_n = M_n(AI+BF)M_n$ are also invertible. Hence, the operators $Z_n^{-1}M_n$ converge strongly to $(AI+BF)^{-1}$.

Moreover, the operators Π_{2n} are invertible for all n and therefore the operators $X_n = M_n(I+CA_*L)M_n(AI+BF)M_n$ are invertible for $n \geq n_*$ and the operators $X_n^{-1}M_n$ converge strongly to $(AI+BF)^{-1}(I+CA_*L)^{-1}$. Hence, the system (2.18) has a unique solution for sufficiently large n and the proof of Theorem 2.3 is completed. □

3. Existence theorems of other kinds of finite-part singular integral approximations in Hilbert spaces

Definition 3.1. Consider the finite-part singular integral equation

$$\begin{aligned} Nu(t) \equiv A(t)u(t) + \frac{B(t)}{\pi i}\,⨎_\Gamma \frac{u(x)}{(x-t)^\mu}dx + C(t)u[\varphi(t)] \\ + \frac{D(t)}{\pi i}\,⨎_\Gamma \frac{u[\varphi(t)]}{(x-t)^\mu}dx + \int_\Gamma k(t,x)u(x)dx = f(t), \quad t\in\Gamma,\ \mu\in\mathbb{N}, \end{aligned} \tag{3.1}$$

where Γ denotes the unit circle $\Gamma = \{t : |t| = 1\}$, $A(t), B(t), C(t), D(t), k(x,t) \in H_{\beta,m\times m}$, $f(t) \in H_{\beta,m}$, $u(t)$ is the unknown function, and $\varphi(t)$ is a homeomorphism of Γ on itself.

Theorem 3.2. *Let the finite-part singular integral equation (3.1), where an approximate solution is of the form (2.3) with the coefficients ε_l, $l = -n,\dots,n$, to be obtained by solving the system of linear algebraic equations*

$$\sum_{k=0}^{n} A_{j-k}\varepsilon_k + \sum_{k=-n}^{-1} B_{j-k}\varepsilon_k + \sum_{l=-n}^{n}\left[C_{jl}^* + D_{jl}^* + K_{jl}\right]\varepsilon_l = f_j \quad (j=-n,\dots,n), \tag{3.2}$$

where A_j, B_j, and K_{jl}, $j = \pm1, \pm2, \dots$, are the Fourier coefficients of the matrix-valued functions given by (2.5) and C_{jl}^* and D_{jl}^* are the Fourier coefficients of the matrix-valued functions $C(t)[\varphi(t)]^l$ and $(D(t)/\pi i)⨎_\Gamma(|\varphi(t)|^l/(x-t))$, respectively.

Moreover, if condition (2.7) is satisfied and the operator N is invertible in $L_{2,m}$, then the system (3.2) has unique solutions for sufficiently large n and the approximate solutions of (3.2) converge to the exact solution with a rate given by (2.9).

Proof. We use the following representation for the finite-part singular integral equation (3.1):

$$N = \Phi + FL - LF, \tag{3.3}$$

where Φ denotes the finite-part singular integral equation (2.1) and F, L are the operators given by (2.10).

Therefore, since the operator $FL - LF$, under the assumption concerning $\varphi(t)$, is completely continuous, it is easily proved that the system (3.2) has unique solutions for sufficiently large n. □

Definition 3.3. Consider the system of finite-part singular integral equations

$$\begin{aligned} Tu(t) \equiv A(t)u(t) &+ \frac{B(t)}{\pi i} \int_{\Gamma} \frac{u(x)}{(x-t)^{\mu}} dx \\ &+ \sum_{k=1}^{\xi} \left[C_k(t) u[\varphi_k(t)] + \frac{D_k(t)}{\pi i} \int_{\Gamma} \frac{u(x)}{(x-\varphi_k(t))^{\mu}} dx \right] \\ &+ \int_{\Gamma} k(t,x) u(x) dx = f(t), \end{aligned} \tag{3.4}$$

where Γ denotes the unit circle $\Gamma = \{t : |t| = 1\}$, $A(t), B(t), C_k(t), D_k(t), k(t,x) \in H_{\beta,m\times m}$ $(0 < \beta \leq 1)$, $f(t) \in H_{\beta,m}$, $u(t)$ is the unknown function, and φ_k, $k = 1,2,\dots,\xi$, is a system of diffeomorphisms of Γ on itself.

The proof of the following theorem is analogous to the proof of Theorem 2.2.

Theorem 3.4. *Let the system of finite-part singular integral equations (3.4), while an approximate solution is of the form (2.3) with the coefficients ε_l, $l = -n,\dots,n$, to be obtained by solving the system of linear algebraic equations*

$$\begin{aligned} &\sum_{l=0}^{n} A_{j-l}\varepsilon_l + \sum_{l=-n}^{-1} B_{j-l}\varepsilon_l + \sum_{k=1}^{\xi} \left[\sum_{l=0}^{n} C_{jl}^{(k)} \varepsilon_l + \sum_{l=-n}^{-1} D_{jl}^{(k)} \varepsilon_l \right] + \sum_{l=-n}^{n} K_{jl}\varepsilon_l \\ &\quad = f_j \quad (j = -n,\dots,n), \end{aligned} \tag{3.5}$$

where A_j, B_j, and K_{jl} are the Fourier coefficients of the corresponding matrix-valued functions given by (2.5) and $C_{jl}^{(k)}$ and $D_{jl}^{(k)}$ are the Fourier coefficients of $[C_k(t) + D_k(t)](\varphi_k(t))^l$ and $[C_k(t) - D_k(t)](\varphi_k(t))^l$, respectively.

Moreover, if condition (2.7) is satisfied and the operator T is invertible in $L_{2,m}$, then the system (3.5) has unique solutions for sufficiently large n.

4. Conclusions. A finite-part singular integral equations analysis has been presented by proposing several approximation methods. Some existence theorems were proved for the solutions of the systems of linear algebraic equations on which the finite-part singular integral equations are approximated. The singularity of the above type of singular integral equations consists of a homeomorphism of the integration interval (unit circle) on itself.

The method was further extended in order to prove the existence of solutions for systems of finite-part singular integral equations, when their singularity consists of a system of diffeomorphisms of the integration interval (unit circle) on itself.

Hence, the present study was devoted to a basic description of numerical schemes, the vigorous foundation and comparison of a series of approximate methods and algorithms, and their application to the numerical solution of finite-part singular integral equations defined on Hilbert spaces.

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