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ON VOLTERRA INEQUALITIES AND THEIR APPLICATIONS

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We present certain variants of two-dimensional and n-dimensional Volterra integral inequalities. In particular, generalizations of the Gronwall inequality are obtained. These results are applied in various problems for differential and integral equations.

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1. Introduction. Many authors have considered integral inequalities in two variables of the form

$$u(x,y) \leq f(x,y) + \int_0^x \int_0^y b(s,t)u(s,t)\,ds\,dt \tag{1.1}$$

in domain $D = \{(x,y) : x \geq 0,\ y \geq 0\}$ (see [1, 2, 3]). They have got the following result.

Theorem 1.1. *If b and f are nonnegative continuous functions in D and f is nondecreasing with respect to each variable, then inequality (1.1) implies the following Gronwall inequality:*

$$u(x,y) \leq f(x,y)\exp\left[\int_0^x \int_0^y b(s,t)\,ds\,dt\right]. \tag{1.2}$$

Our aim is to consider special cases of two-dimensional inequalities of the Volterra type:

$$u(x,y) \leq f(x,y) + \int_0^x \int_0^y k(x,y,s,t)u(s,t)\,ds\,dt, \tag{1.3}$$

where f is a given function in D and k is defined in domain $\Omega = \{(x,y,s,t) : 0 \leq s \leq x < \infty,\ 0 \leq t \leq y < \infty\}$.

The obtained results for integral inequalities in two variables are applied in various differential and integral problems.

Some similar problems were considered in [4] for integral inequalities of the Volterra-Fredholm type:

$$u(x,y) \leq f(x,t) + \int_0^t \int_a^b k(x,t;y,s)u(y,s)\,dy\,ds. \tag{1.4}$$

In this paper, we obtain better estimations than in [4], because in (1.3) double Volterra operator which plays there a dominant role arises. Moreover, integral inequalities in n independent variables are considered and applied to study boundedness and stability of solutions to n-dimensional nonlinear integral equation of the Volterra type.

2. Two-dimensional Volterra inequalities. From the theory of Volterra linear integral equations, the following result follows.

Lemma 2.1. *Let f and k be continuous functions in D and Ω, respectively. If k is nonnegative, and continuous function u satisfies inequality (1.3) in D, then*

$$u(x,y) \le f(x,y) + \int_0^x \int_0^y r(x,y,s,t) f(s,t)\,ds\,dt, \tag{2.1}$$

where

$$r(x,y,s,t) = \sum_{n=1}^{\infty} k_n(x,y,s,t) \tag{2.2}$$

is the resolvent kernel defined by formulas

$$\begin{aligned} &k_n(x,y,s,t) = \int_s^x \int_t^y k(x,y,\xi,\eta) k_{n-1}(\xi,\eta,s,t)\,d\xi\,d\eta \quad \text{for } n = 2,3,\dots, \\ &k_1(x,y,s,t) = k(x,y,s,t). \end{aligned} \tag{2.3}$$

Moreover, if f is nondecreasing with respect to every variable, then

$$u(x,y) \le f(x,y)\left[1 + \int_0^x \int_0^y r(x,y,s,t)\,ds\,dt\right]. \tag{2.4}$$

Using this lemma we can prove the following theorem.

Theorem 2.2. *Let a, b, and f be nonnegative continuous functions in D. If the continuous function u satisfies the inequality*

$$u(x,y) \le f(x,y) + a(x,y) \int_0^x \int_0^y b(s,t) u(s,t)\,ds\,dt, \tag{2.5}$$

then

$$\begin{aligned} u(x,y) \le{}& f(x,y) \\ &+ a(x,y) \int_0^x \int_0^y b(s,t) \exp\left[\int_s^x \int_t^y a(\xi,\eta) b(\xi,\eta)\,d\xi\,d\eta\right] f(s,t)\,ds\,dt. \end{aligned} \tag{2.6}$$

If f is nondecreasing, then

$$u(x,y) \le f(x,y)\left[1 + a(x,y) \int_0^x \int_0^y b(s,t) \exp\left[\int_s^x \int_t^y a(\xi,\eta) b(\xi,\eta)\,d\xi\,d\eta\,ds\,dt\right]\right]. \tag{2.7}$$

Proof. We notice that inequality (2.5) is a special case of (1.3) with $k(x,y,s,t) = a(x,y)b(s,t)$. In virtue of formula (2.4), we have

$$\begin{aligned} k_1(x,y,s,t) &= a(x,y)b(s,t), \\ k_2(x,y,s,t) &= \int_s^x \int_t^y k(x,y,p,q)k_1(p,q,s,t)\,dp\,dq \\ &= \int_s^x \int_t^y a(x,y)b(p,q)a(p,q)b(s,t)\,dp\,dq \\ &= a(x,y)b(s,t)\int_s^x \int_t^y a(p,q)b(p,q)\,dp\,dq \\ &= a(x,y)b(s,t)M(x,y,s,t), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M(x,y,s,t) &= \int_s^x \int_t^y a(p,q)b(p,q)\,dp\,dq, \\ \frac{\partial^2 M}{\partial x \partial y} &= a(x,y)b(x,y), \\ k_3(x,y,s,t) &= \int_s^x \int_t^y k(x,y,p,q)k_2(p,q,s,t)\,dp\,dq \\ &= \int_s^x \int_t^y a(x,y)b(p,q)a(p,q)b(s,t)M(p,q)\,dp\,dq \\ &= a(x,y)b(s,t)\int_s^x \int_t^y b(p,q)a(p,q)M(p,q)\,dp\,dq \\ &= a(x,y)b(s,t)\int_s^x \int_t^y \frac{\partial^2 M}{\partial p \partial q}M(p,q)\,dp\,dq \\ &= a(x,y)b(s,t)\int_s^x \int_t^y \frac{1}{2}\frac{\partial^2 (M^2)}{\partial p \partial q}\,dp\,dq \\ &\le a(x,y)b(s,t)\frac{[M(x,y,s,t)]^2}{2!}, \end{aligned} \tag{2.9}$$

since

$$M\frac{\partial^2 (M^2)}{\partial x \partial y} \le \frac{1}{2!}\frac{\partial^2 (M^2)}{\partial x \partial y}. \tag{2.10}$$

By induction, we obtain

$$k_i(x,y,s,t) \le a(x,y)b(s,t)\frac{[M(x,y,s,t)]^{i-1}}{(i-1)!}. \tag{2.11}$$

Then from (2.2), we get

$$\begin{aligned} r(x,y,s,t) &\le a(x,y)b(s,t)\sum_{i=1}^{\infty}\frac{[M(x,y,s,t)]^{i-1}}{(i-1)!} \\ &= a(x,y)b(s,t)\exp[M(x,y,s,t)] \\ &= a(x,y)b(s,t)\exp\left[\int_s^x \int_t^y a(p,q)b(p,q)\,dp\,dq\right]. \end{aligned} \tag{2.12}$$

Hence, using Lemma 2.1, the proof is finished. □

The results were obtained by estimating the iterated kernels k_i. We can get stronger inequality in the following case.

Remark 2.3. If $a(x,y) = a_1(x)a_2(y)$ and $b(s,t) = b_1(s)b_2(t)$, then

$$\begin{aligned} r(x,y,s,t) &= a_1(x)a_2(y)b_1(s)b_2(t)\sum_{i=0}^{\infty}\frac{\left[\int_s^x\int_t^y a_1(p)a_2(q)b_1(p)b_2(q)\,dp\,dq\right]^n}{(n!)^2} \\ &\le a_1(x)a_2(y)b_1(s)b_2(t)\exp\left[\int_s^x\int_t^y a_1(p)a_2(q)b_1(p)b_2(q)\,dp\,dq\right] \end{aligned} \tag{2.13}$$

(if a_1, a_2, b_1, b_2 are positive, then this inequality is strict).

Corollary 2.4. *If the assumptions of Theorem 2.2 are satisfied, then the following inequality holds:*

$$\begin{aligned} u(x,y) \le F(x,y)\Big[1+a(x,y)\int_0^x\int_0^y b(s,t) \\ \times\exp\left[\int_s^x\int_t^y a(p,q)b(p,q)\,dp\,dq\right]ds\,dt\Big], \end{aligned} \tag{2.14}$$

where

$$F(x,y) = \sup\{f(s,t) : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\}. \tag{2.15}$$

Lemma 2.5. *If b is nonnegative continuous function in D, then*

$$1+\int_0^x\int_0^y b(s,t)\exp\left[\int_s^x\int_t^y b(p,q)\,dp\,dq\right]ds\,dt \le \exp\left[\int_0^x\int_0^y b(s,t)\,ds\,dt\right] \tag{2.16}$$

(if b is positive, then this inequality is strict).

Using Lemma 2.5 in Corollary 2.4, we get the following result.

Corollary 2.6. *If the assumptions of Corollary 2.4 are fulfilled and a is nonincreasing with respect to each variable, then inequality (2.14) leads to*

$$u(x,y) \le F(x,y)\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right]. \tag{2.17}$$

Remark 2.7. If f is nondecreasing with respect to each variable, then $F(x,y) = f(x,y)$ and the Gronwall-type inequality

$$u(x,y) \le f(x,y)\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right] \tag{2.18}$$

follows, which is a generalization of the results obtained in [1, 2, 3] (if $a = 1$, we get (1.2)).

Remark 2.8. If $f = c > 0$ and a is nonincreasing with respect to each variable, then the inequality

$$u(x,y) \le c + a(x,y)\int_0^x \int_0^y b(s,t)u(s,t)\,ds\,dt \tag{2.19}$$

implies that

$$u(x,y) \le c\exp\left[\int_0^x \int_0^y a(s,t)b(s,t)\,ds\,dt\right]. \tag{2.20}$$

Theorem 2.9. *Let a, b, and f be nonnegative and continuous in D. If the continuous function u satisfies inequality (2.5) and $a(x,y) \neq 0$, then*

$$u(x,y) \le a(x,y)T(x,y)\exp\left[\int_0^x \int_0^y a(s,t)b(s,t)\,ds\,dt\right], \tag{2.21}$$

where

$$T(x,y) = \sup\left\{\frac{f(s,t)}{a(s,t)} : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\right\}. \tag{2.22}$$

Proof. From (2.5), we get

$$\frac{u(x,y)}{a(x,y)} \le T(x,y) + \int_0^x \int_0^y a(s,t)b(s,t)\frac{u(s,t)}{a(s,t)}\,ds\,dt. \tag{2.23}$$

Using Theorem 2.2, we obtain the inequality

$$\frac{u(x,y)}{a(x,y)} \le T(x,y)\exp\left[\int_0^x \int_0^y a(s,t)b(s,t)\,ds\,dt\right], \tag{2.24}$$

which finishes the proof. □

Corollary 2.10. *If the assumptions of Theorem 2.9 are satisfied and f/a is nondecreasing in D, then inequality (2.5) implies (2.18).*

Remark 2.11. If $a = 1$, we get the Gronwall inequality in two variables (1.2).

Theorem 2.12. *Let u, a, and b be nonnegative continuous functions in D and let f be positive and continuous in D. If a/f is nonincreasing with respect to each variable, then inequality (2.5) implies (2.18).*

Proof. From (2.5), we obtain

$$\frac{u(x,y)}{f(x,y)} \le 1 + \frac{a(x,y)}{f(x,y)}\int_0^x \int_0^y b(s,t)f(s,t)\frac{u(s,t)}{f(s,t)}\,ds\,dt. \tag{2.25}$$

Using Remark 2.8, we get

$$\frac{u(x,y)}{f(x,y)} \le \exp\left[\int_0^x \int_0^y \frac{a(s,t)}{f(s,t)}b(s,t)f(s,t)\,ds\,dt\right]. \tag{2.26}$$

Hence (2.21) follows. □

Theorem 2.13. *If f, a, and b are nonnegative continuous functions in D, then inequality (2.5) implies*

$$u(x,y) \le W(x,y)\exp\int_0^x\int_0^y W(s,t)b(s,t)\,ds\,dt, \tag{2.27}$$

where

$$W(x,y) = \max_D\left[a(x,y), f(x,y)\right] \neq 0. \tag{2.28}$$

Proof. From (2.5), we get

$$u(x,y) \le W(x,y)\left[1+\int_0^x\int_0^y b(s,t)u(s,t)\,ds\,dt\right] \tag{2.29}$$

or

$$\frac{u(x,y)}{W(x,y)} \le 1+\int_0^x\int_0^y W(s,t)b(s,t)\frac{u(s,t)}{W(s,t)}\,ds\,dt. \tag{2.30}$$

In virtue of Theorem 1.1, we have

$$\frac{u(x,y)}{W(x,y)} \le \exp\int_0^x\int_0^y W(s,t)b(s,t)\,ds\,dt. \tag{2.31}$$

That finishes the proof. □

Corollary 2.14. *If the assumptions of Theorem 2.13 are fulfilled, then*

$$\begin{aligned} u(x,y) &\le a(x,y)\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right] \quad \text{for } a(x,y)\le f(x,y),\\ u(x,y) &\le f(x,y)\exp\left[\int_0^x\int_0^y f(s,t)b(s,t)\,ds\,dt\right] \quad \text{for } a(x,y)\ge f(x,y). \end{aligned} \tag{2.32}$$

Theorem 2.15. *Let f be nonnegative continuous function in D and nondecreasing with respect to each variable, and let k be nonnegative function in Ω such that*

$$k(x,y,s,t) \le k(s,t,s,t) \tag{2.33}$$

for $0\le s\le x$, $0\le t\le y$. If the nonnegative and continuous function u satisfies inequality (1.3), then

$$u(x,y) \le f(x,y)\exp\left[\int_0^x\int_0^y k(s,t,s,t)\,ds\,dt\right]. \tag{2.34}$$

Proof. From (1.3), we obtain

$$u(x,y) \le f(x,y)+\int_0^x\int_0^y k(s,t,s,t)u(s,t)\,ds\,dt. \tag{2.35}$$

The proof follows using Theorem 1.1. □

Theorem 2.16. *Let f be nonnegative continuous function in D and let k be positive function in Ω such that*

$$k(x,y,s,t) \le k(x,y,x,y) \tag{2.36}$$

for $0 \le s \le x$, $0 \le t \le y$. If $f(x,y)/k(x,y,x,y)$ is nondecreasing with respect to variables x and y, then inequality (1.3) implies (2.34) for nonnegative and continuous function u in D.

Proof. In virtue of the assumptions of Theorem 2.16, inequality (1.3) leads to

$$u(x,y) \le f(x,y) + k(x,y,x,y)\int_0^x\int_0^y u(s,t)\,ds\,dt. \tag{2.37}$$

Using Corollary 2.10, we get (2.34). □

Theorem 2.17. *Let f be positive continuous function in D and let k be nonnegative continuous function in Ω such that $k(x,y,s,t) \le f(x,y)$. If the continuous and nonnegative function u satisfies inequality (1.3), then*

$$u(x,y) \le f(x,y)\exp\int_0^x\int_0^y f(s,t)\,ds\,dt. \tag{2.38}$$

Proof. We notice that inequality (1.3) leads to

$$u(x,y) \le f(x,y)\left[1 + \int_0^x\int_0^y u(s,t)\,ds\,dt\right] \tag{2.39}$$

or

$$\frac{u(x,y)}{f(x,y)} \le 1 + \int_0^x\int_0^y f(s,t)\frac{u(s,t)}{f(s,t)}\,ds\,dt. \tag{2.40}$$

From Theorem 1.1, we obtain the inequality

$$\frac{u(x,y)}{f(x,y)} \le \exp\int_0^x\int_0^y f(s,t)\,ds\,dt, \tag{2.41}$$

which concludes the proof. □

Theorem 2.18. *Suppose that f is a positive continuous function in D and K/f is nonincreasing with respect to each variable, where*

$$K(x,y) = \sup\{k(x,y,s,t) : 0 \le s \le x,\ 0 \le t \le y\}, \tag{2.42}$$

and let k be nonnegative continuous function in Ω. Then inequality (1.3) implies the following inequality:

$$u(x,y) \le f(x,y)\exp\int_0^x\int_0^y K(s,t)\,ds\,dt, \tag{2.43}$$

for nonnegative and continuous function u in D.

Proof. From inequality (1.3), we get

$$u(x,y) \le f(x,y) + K(x,y)\int_0^x\int_0^y u(s,t)\,ds\,dt. \tag{2.44}$$

Using Theorem 2.12, we obtain (2.43). □

Remark 2.19. Estimate (2.43) follows from (1.3) if f/K is nondecreasing with respect to each variable and K is positive and f is nonnegative.

Remark 2.20. Results of this paper can be extended on the class L^2.

3. Some applications of two-dimensional inequalities. In this section, we present some applications of the given inequalities to study the boundedness, stability and uniqueness solutions of certain nonlinear integral equations, and value boundary problems for nonlinear hyperbolic partial differential equations. Moreover, the boundedness of solutions to a system of two-dimensional Volterra integral equations is studied.

3.1. We consider two-dimensional Volterra nonlinear integral equation

$$u(x,y) = f(x,y) + \int_0^x\int_0^y H[x,y,s,t,u(s,t)]\,ds\,dt \tag{3.1}$$

with the following assumptions:

(1) f is continuous in D,

(2) H is continuous in domain

$$W = \{(x,y,s,t) : 0 \le s \le x,\ 0 \le t \le y,\ |u| < \infty\} \tag{3.2}$$

and satisfying the Lipschitz condition

$$|H(x,y,s,t,\overline{u}) - H(x,y,s,t,\overline{\overline{u}})| \le a(x,y)b(s,t)|\overline{u} - \overline{\overline{u}}|, \tag{3.3}$$

or

(3) H is continuous in W and satisfies the condition

$$|H(x,y,s,t,u)| \le a(x,y)b(s,t)|u|, \tag{3.4}$$

where a and b are positive continuous functions in D, such that $ab \in L(\mathbb{R}_+^2)$, that is,

$$\int_0^\infty\int_0^\infty a(s,t)b(s,t)\,ds\,dt < \infty. \tag{3.5}$$

Remark 3.1. Existence, uniqueness, and stability of solutions to (3.1) follow by assumptions (1) and (2).

Proposition 3.2 (bounds of solutions). *Let assumptions (1) and (3) be satisfied. If aT is bounded in D, where*

$$T(x,y)=\sup\left\{\frac{|f(s,t)|}{a(s,t)}:0\le s\le x,\ 0\le t\le y\right\},\tag{3.6}$$

then solutions of (3.1) are bounded in D.

Proof. Applying (1) and (3) to (3.1), we get the following inequality:

$$|u(x,y)|\le|f(x,y)|+a(x,y)\int_0^x\int_0^y b(s,t)|u(s,t)|\,ds\,dt.\tag{3.7}$$

Using Theorem 2.9, we obtain the inequality

$$|u(x,y)|\le a(x,y)T(x,y)\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right],\tag{3.8}$$

which gives the boundedness of solutions of (3.1). □

Remark 3.3. If $|f|/a$ is nondecreasing in D, then (3.8) leads to the following estimate:

$$|u(x,y)|\le|f(x,y)|\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right]\tag{3.9}$$

of the solution to (3.1) that is bounded as f is bounded.

3.2. Consider the following partial differential equation of the hyperbolic type:

$$u_{xy}=(pu)_y+f(x,y)+F[x,y,u(x,y)]\tag{3.10}$$

with value boundary conditions

$$u(x,0)=\alpha(x),\qquad u(0,y)=\beta(y).\tag{3.11}$$

This problem is equivalent to the integral equation

$$u(x,y)=g(x,y)+\int_0^x p(s,y)u(s,y)\,ds+\int_0^x\int_0^y F[s,t,u(s,t)]\,ds\,dt,\tag{3.12}$$

where

$$g(x,y)=\alpha(x)+\beta(y)-u(0,0)-\int_0^x p(s,0)\alpha(s)\,ds+\int_0^x\int_0^y f(s,t)\,dt\,ds.\tag{3.13}$$

Let the following assumptions be fulfilled:

($1^\diamond$) g and p are continuous functions in D,

($2^\diamond$) F is a continuous function satisfying one of the following conditions:

$$|F(x,y,\overline{u})-F(x,y,\overline{\overline{u}})|\le\varphi(x,y)|\overline{u}-\overline{\overline{u}}|\quad\text{in }\Theta\tag{3.14}$$

or

$$|F(x,y,u)|\le\varphi(x,y)|u|\quad\text{in }\Theta,\tag{3.15}$$

where

$$\Theta = \{(x,y,u) : x,y \geq 0,\ -\infty < u < \infty\} \tag{3.16}$$

and continuous nonnegative function $\varphi \in L(\mathbb{R}_+^2)$.

From (3.12), we get

$$\begin{aligned} |u(x,y)| \leq |g(x,y)| &+ \int_0^x |p(s,y)|\,|u(s,y)|\,ds \\ &+ \int_0^x \int_0^y |F[s,t,u(s,t)]|\,ds\,dt. \end{aligned} \tag{3.17}$$

We notice that

$$r(x,y) = \int_0^x \int_0^y |F[s,t,u(s,t)]|\,ds\,dt \tag{3.18}$$

is nonnegative and nondecreasing function with respect to variables x and y.

Then we have

$$|u(x,y)| \leq |g(x,y)| + r(x,y) + \int_0^x |p(s,y)|\,|u(s,y)|\,ds. \tag{3.19}$$

Treating (3.19) as one-dimensional inequality for every $y \in \mathbb{R}_+$ and using Gronwall inequality (see [5]), we get

$$|u(x,y)| \leq (G(x,y) + r(x,y)) \exp\left[\int_0^x |p(s,y)|\,ds\right], \tag{3.20}$$

where

$$G(x,y) = \sup\{|g(s,t)| : 0 \leq s \leq x,\ 0 \leq t \leq y\} \tag{3.21}$$

is nondecreasing function in D.

Using (3.15), we have

$$r(x,y) = \int_0^x \int_0^y |F[s,t,u(s,t)]|\,ds\,dt \leq \int_0^x \int_0^y \varphi(s,t)\,|u(s,t)|\,ds\,dt. \tag{3.22}$$

Then we obtain the following integral inequality:

$$|u(x,y)| \leq P(x,y)\left[G(x,y) + \int_0^x \int_0^y \varphi(s,t)\,|u(s,t)|\,ds\,dt\right], \tag{3.23}$$

where

$$P(x,y) = \exp\left(\int_0^x |p(s,y)|\,ds\right). \tag{3.24}$$

From (3.23), we get

$$\frac{|u(x,y)|}{P(x,y)} \leq G(x,y) + \int_0^x \int_0^y \frac{|u(s,t)|}{P(s,t)} \varphi(s,t) P(s,t)\,ds\,dt, \tag{3.25}$$

and using Gronwall inequality we have

$$|u(x,y)| \le P(x,y)G(x,y)\exp\left[\int_0^x\int_0^y \varphi(s,t)P(s,t)\,ds\,dt\right]. \tag{3.26}$$

3.3. Consider the following system of two-dimensional integral equations of the Volterra type:

$$u_i(x,y) = w_i(x,y) + \sum_{j=1}^{n}\int_0^x\int_0^y k_{ij}(x,y,s,t)u_i(s,t)\,ds\,dt, \quad i = 1,2,\dots,n, \tag{3.27}$$

where functions w_i $(i = 1,2,\dots,n)$ and k_{ij} $(i,j = 1,2,\dots,n)$ are continuous in D and Ω, respectively.

From (3.27), we get

$$u(x,y) \le w(x,y) + \int_0^x\int_0^y k(x,y,s,t)u(s,t)\,ds\,dt, \tag{3.28}$$

where

$$\begin{aligned} u(x,y) &= \sum_{i=1}^{n}|u_i(x,y)|, \\ w(x,y) &= \sum_{i=1}^{n}|w_i(x,y)|, \\ k(x,y,s,t) &= \sum_{i=1}^{n}\max\{|k_{ij}(x,y,s,t)| : 1 \le j \le n\}. \end{aligned} \tag{3.29}$$

Using Corollary 2.6, we obtain the following theorem.

Theorem 3.4. *If*

$$\sum_{i=1}^{n}\max\{|k_{ij}(x,y,s,t)| : 1 \le j \le n\} \le a(s,t) \tag{3.30}$$

for nonnegative and continuous function a, then the estimate

$$\begin{aligned} \sum_{i=1}^{n}|u_i(x,y)| \le \sup&\left\{\sum_{i=1}^{n}|w_i(s,t)| : 0 \le s \le x,\ 0 \le t \le y\right\} \\ &\times\exp\left[\int_0^x\int_0^y a(s,t)\,ds\,dt\right] \end{aligned} \tag{3.31}$$

holds. Moreover, if $a \in L(\mathbb{R}_+^2)$ and

$$\sup\left\{\sum_{i=1}^{n}|w_i(s,t)| : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\right\} < \infty, \tag{3.32}$$

then the solution $\{u_i(x,y)\}$, $i = 1,2,\dots,n$, of (3.27) is bounded in D.

Theorem 3.5. *If*

$$\sum_{i=1}^{n} \max\{|k_{ij}(x,y,s,t)| : 1 \le j \le n\} = b(x,y)a(s,t) \tag{3.33}$$

for nonnegative and continuous functions a and b in D $(b \neq 0)$, then

$$\sum_{i=1}^{n} |u_i(x,y)| \le b(x,y) \sup_{\substack{0\le s\le x<\infty\\0\le t\le y<\infty}} \sum_{i=1}^{n} \frac{|w_i(s,t)|}{b(s,t)} \exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right]. \tag{3.34}$$

Additionally, if $ab \in L(\mathbb{R}_+^2)$ and

$$\sup_{\substack{0\le s\le x<\infty\\0\le t\le y<\infty}} \sum_{i=1}^{n} \frac{|w_i(s,t)|}{b(s,t)} < \infty, \tag{3.35}$$

then the solution of (3.27) is bounded.

Proof. The proof follows from Theorem 2.9. □

By Theorem 2.16, we get the following theorem.

Theorem 3.6. *If*

$$\sup_{\substack{0\le s\le x\\0\le t\le y}} \sum_{i=1}^{n} \max|k_{ij}(x,y,s,t)| \le p(x,y) \tag{3.36}$$

for $p \in L(\mathbb{R}_+^2)$, then

$$\sum_{i=1}^{n} |u_i(x,y)| \le p(x,y) \sup_{\substack{0\le s\le x\\0\le t\le y}} \frac{w(s,t)}{p(s,t)} \exp\left[\int_0^x\int_0^y p(s,t)\,ds\,dt\right], \tag{3.37}$$

and solution $\{u_i(x,y)\}$, $i = 1,2,\dots,n$, of (3.27) is bounded, when

$$p(x,y) \sup_{\substack{0\le s\le x\\0\le t\le y}} \frac{w(s,t)}{p(s,t)} < \infty \quad \text{in } D, \tag{3.38}$$

where

$$w(x,y) = \sum_{i=1}^{n} |w_i(x,y)|. \tag{3.39}$$

From Remark 2.19, the next theorem follows.

Theorem 3.7. *Let*

$$\sum_{i=1}^{n} \max_{1\le j\le n} |k_{ij}(x,y,s,t)| \le \sum_{i=1}^{n} \max_{1\le j\le n} |k_{ij}(s,t,s,t)| \tag{3.40}$$

for $0 \le s \le x$, $0 \le t \le y$. *Then the estimate*

$$\sum_{i=1}^{n} |u_i(x,y)| \le g(x,y) \exp\left[\int_0^x \int_0^y \sum_{i=1}^{n} \max_{1\le j\le n} |k_{ij}(s,t,s,t)|\,ds\,dt\right] \tag{3.41}$$

holds, where

$$g(x,y) = \sup\left\{\sum_{i=1}^{n} |w_i(s,t)| : 0 \le s \le x,\ 0 \le t \le y\right\}. \tag{3.42}$$

If

$$\int_0^\infty \int_0^\infty \sum_{i=1}^{n} \max_{1\le j\le n} |k_{ij}(s,t,s,t)|\,ds\,dt < \infty, \tag{3.43}$$

then solution $\{u_i(x,y)\}$, $i = 1,2,\dots,n$, *of (3.27) is bounded in* D *for bounded function* g *in* D.

4. Integral inequalities of n variables and their applications. In this section, we establish n-independent variable generalizations of the integral inequalities established in Section 2. For this purpose, we introduce the following notations.

A point $(x_1,\dots,x_n)$ in the n-dimensional Euclidean space $\mathbb{R}^n$ is denoted by x and the origin of $\mathbb{R}^n$ is $0 = (0,\dots,0)$. For $x, y, s \in \mathbb{R}^n$, we denote that $x \le y \Leftrightarrow x_i \le y_i$ for every $i = 1,2,\dots,n$.

A function f is said to be nondecreasing if $x \le y \Rightarrow f(x) \le f(y)$. An interval $[c,d]$, where $c, d \in \mathbb{R}^n$ $(c \le d)$, is defined by the inequality $c \le x \le d$. Let $\int_c^d f(s)ds$ be the n-fold integral taken over $[c,d]$.

Let $d > 0$, $I = [0,d]$, and $J = \{(x,y) : 0 \le s \le x \le d\}$. Consider the following Volterra nonlinear integral equation in $\mathbb{R}^n$:

$$u(x) = f(x) + \int_0^x h(x,s,u(s))\,ds. \tag{4.1}$$

Lemma 4.1. *If* $f \in C(I)$ *and* $h \in C(J \times R)$ *satisfies a Lipschitz condition*

$$|h(x,y,z) - h(x,y,\overline{z})| \le L|z - \overline{z}| \quad \text{in } J \times \mathbb{R}, \tag{4.2}$$

then (4.1) has a unique solution in (3.14) which depends continuously on f *and* h.

Proof. Let $C(I)$ be normed by

$$\|u\|_\alpha = \max\{|u(x)|e^{-\alpha\sigma(x)} : x \in I\}, \quad \sigma(x) = \sum_{i=1}^{n} x_i,\ \alpha > 0. \tag{4.3}$$

Abbreviating (4.1) as

$$u = Su, \tag{4.4}$$

one has $S : X \to X$ continuous and, for $v, w \in C(I)$,

$$|Sv - Sw|(x) \le \int_0^x L\,|v(x) - w(s)|\,ds \le L\|v - w\|_\alpha \int_0^x e^{-\alpha\sigma(x)}ds. \tag{4.5}$$

The integral is less than or equal to $e^{\alpha\sigma(x)}\alpha^{-n}$, and one obtains, after division by $e^{\alpha\sigma(x)}$,

$$\|Sv - Sw\|_\alpha \le L\alpha^{-n}\|v - w\|_\alpha. \tag{4.6}$$

We choose α such that $L\alpha^{-n} = 1/2$. When u is the solution,

$$\|u - v\|_\alpha \le \|Su - Sv\|_\alpha + \|Su - Sv\|_\alpha + \|Sv - v\|_\alpha \Longrightarrow \|u - v\|_\alpha \le 2\|Sv - v\|_\alpha, \tag{4.7}$$

in particular, $\|u\|_\alpha \le 2\|S(0)\|_\alpha$ which leads to

$$|u(x)| \le 2||S(0)||_\alpha e^{\alpha\sigma(x)}, \quad ||S(0)||_\alpha \le \left\|\,|f(x)| + \int_0^x |h(x,s,0)|\,ds\right\|_\alpha. \tag{4.8}$$

□

The theorem now follows. This also answers the existence and uniqueness in the infinite case, where the "quadrant" $Q = \mathbb{R}_+^n$ is considered (stability in Q is another story).

Introduce the notation $\partial_i = \partial/\partial x_i$, $D = \partial_1, \partial_2, \ldots, \partial_n$.

Lemma 4.2 (see [6]). *The solution of*

$$u(x) = f(x) + \int_0^x k(x,s)u(s)ds \tag{4.9}$$

is given by

$$u(x) = f(x) + \int_0^x r(x,s)f(s)ds, \tag{4.10}$$

where the resolvent kernel r is defined by

$$r(x,s) = \sum_{n=1}^\infty k_n(x,s), \tag{4.11}$$

with iterated kernels k_n constructed by formulas

$$\begin{aligned} k_n(x,s) &= \int_s^x k(x,\xi)k_{n-1}(\xi,s)d\xi \\ &= \int_s^x k_{n-1}(x,\xi)k(\xi,s)d\xi \quad \textit{for } n = 1,2,3,\ldots, \\ k_1(x,s) &= k(x,s). \end{aligned} \tag{4.12}$$

Lemma 4.3. *If $b \in C(I)$ is nonnegative in (3.14), then*

$$1 + \int_0^x b(s)e^{B(s)}ds \le e^{B(x)}, \tag{4.13}$$

where

$$B(x) = \int_0^x b(s)\,ds, \quad x > 0. \tag{4.14}$$

Proof. We notice that $D_iB = B_i$ and $D_{ij}B = B_{ij}$ are nonnegative and

$$\begin{aligned} DB(x) &= b(x), \\ De^{B(x)} &\ge b(x)e^{B(x)}, \\ \int_0^x De^{B(s)}\,ds &= e^{B(x)} - 1, \\ \int_0^x b(s)e^{B(s)}\,ds &\le e^{B(s)} - 1. \end{aligned} \tag{4.15}$$

In the case $n = 3$, we have

$$De^B = e^B\left(B_1B_2B_3 + B_1B_{23} + B_2B_{13} + B_3B_{12} + b\right). \tag{4.16}$$

□

Theorem 4.4. *If $f \in C(I)$ and a nonnegative function $k \in C(J)$, then $u \in C(I)$ satisfying the inequality*

$$u(x) \le f(x) + \int_0^x k(x,s)u(s)\,ds \tag{4.17}$$

has the estimation

$$u(x) \le f(x) + \int_0^x r(x,s)f(s)\,ds, \tag{4.18}$$

where r is a resolvent kernel (4.11).

Proof. Abbreviate the right-hand side of (4.9) as $f + Ku = Su$. Then S is a monotone increasing operator ($v \le w \Rightarrow Sv \le Sw$). According to (4.9), u is a subsolution, $u \le Su$, hence the sequence $u_n = S^n u$ obtained by successive approximation is increasing and converges to the solution $\varphi = S\varphi$. Since φ is the right-hand side of inequality (4.17), the theorem is proved. □

Theorem 4.5. *Let f be continuous, nonnegative, and nonincreasing function in I and let $b \in C(I)$ be nonnegative. Then for $u \in C(I)$, the inequality*

$$u(x) \le f(x) + \int_0^x b(s)u(s)\,ds \tag{4.19}$$

implies

$$u(x) \le f(x)e^{B(x)}, \quad B(x) = \int_0^x b(s)\,ds. \tag{4.20}$$

Proof. We assume first that $f(x) = 1$. With the same notation (but with $k(x,s) = b(s)$), we have $u \le Su$ and we show that $w = e^{B(x)}$ satisfies $s \ge Sw$, that is,

$$e^{B(x)} \ge 1 + \int_0^x b(s)e^{B(s)}\,ds, \tag{4.21}$$

according to Lemma 4.3. The sequence $u_n = S^n u$ is increasing and $w_n = S^n w$ is decreasing and both have the same limit φ = solution of $\varphi = S\varphi$. It follows that $u \le w$.

The theorem holds for $f(x) = 1$ and hence for $f(x) = \text{const} = c \ge 0$. Now we fix $x_0 > 0$, put $c = f(x_0)$, and consider the inequality in $I = [0, x_0]$. We get $u(x) \le ce^{B(x)}$ in I, in particular at x_0. Since x_0 is arbitrary, this theorem follows. □

Theorem 4.6. *If $u, a, b, f \in C(I)$ and a, b, and f are nonnegative, then the inequality*

$$u(x) \le f(x) + \int_0^x a(x)b(s)u(s)ds \tag{4.22}$$

implies

$$u(x) \le f(x) + \int_0^x a(x)b(s)e^{M(x,s)}f(s)ds, \quad M(x,s) = \int_s^x a(t)b(t)dt. \tag{4.23}$$

Proof. We notice that $DM(x,s)^m \ge mb(s)M(x,s)^{m-1}$. Using Lemmas 4.2, 4.3, and Theorem 4.5 and proceeding similarly as in the case of a two-dimensional inequality, we get (4.23). □

We get the optimal estimation in the case of a special kernel. That is important since it allows to give the kernel explicitly and hence gives a much better bound.

Remark 4.7. In the case $k(x,s) = a(x)b(s)$ with $a(x) = \prod_{i=1}^n a_i(x_i)$ and $b(s) = \prod_{i=1}^n b_i(s_i)$, the resolvent kernel is

$$r(x,s) = a(x)b(s)E_n\left(\prod_{i=1}^n \int_0^x a_i(t)b_i(t)dt\right), \tag{4.24}$$

where

$$E_n(z) = \sum_{p=0}^{\infty} \frac{z^p}{(p!)^n}. \tag{4.25}$$

For the case $a(x) = 1$, we refer to [6, pages 142-143].

Remark 4.8. Inequality (4.19), with $k(x,s) \le k(s,s)$ for $s \le x$, implies (4.20) with $b(s) = k(s,s)$.

Remark 4.9. Inequality (4.22), with $k(x,s) \le k(x,x)$ for $s \le x$, implies (4.23) with $a(x) = k(x,x)$, $b(s) = 1$.

Introducing the function

$$F(x) = \max\{f(s) : 0 \le s \le x\}, \tag{4.26}$$

which is nondecreasing in (3.14), we get the following proposition.

Proposition 4.10. *Inequality (4.22) implies*

$$u(x) \le F(x)\left(1 + a(x)\int_0^x b(s)e^{M(x,s)}ds\right), \tag{4.27}$$

where

$$u(x) \le F(x)\exp\left(\int_0^x a(s)b(s)\,ds\right) \tag{4.28}$$

if a is nonincreasing,

$$u(x) \le f(x)\exp\left(\int_0^x a(s)b(s)\,ds\right) \tag{4.29}$$

if f/a is nondecreasing,

$$u(x) \le a(x)\varphi(x)\exp\left(\int_0^x a(s)b(s)\,ds\right) \tag{4.30}$$

if

$$\varphi(x) = \max\left\{\frac{f(s)}{a(s)} : 0 \le s \le x\right\} \tag{4.31}$$

for a positive function a, and

$$u(x) \le W(x)\exp\left(\int_0^x W(s)b(s)\,ds\right), \quad W(x) = \max\{a(x), f(x)\} > 0. \tag{4.32}$$

We can use the presented theory of n-dimensional inequalities to study boundedness and stability of solutions for n-dimensional integral equation (4.1).

Example 4.11. In the integral (4.1), the assumption

$$|h(x,s,z)| \le a(x)b(s)|z| \quad (z \text{ is real}) \tag{4.33}$$

leads immediately to the integral inequality

$$|u(x)| \le |f(x)| + \int_0^x a(x)b(s)|u(s)|\,ds. \tag{4.34}$$

By Proposition 4.10, we obtain

$$|u(x)| \le a(x)\max\left\{\frac{|f(s)|}{a(s)} : 0 \le s \le x\right\}\exp\int_0^x a(s)b(s)\,ds. \tag{4.35}$$

This applies in particular to the linear equation (4.9) with

$$|k(x,s)| \le a(x)b(s). \tag{4.36}$$

If $|f(x)|/a(x)$ is nondecreasing in (3.14), then we get the estimation

$$|u(x)| \le f(x)\exp\int_0^x a(s)b(s)\,ds. \tag{4.37}$$

Stability in a finite interval has been discussed earlier. In $Q = \mathbb{R}_+^n$, one has to define what it means. One possibility is to require that $\overline{u}$ is an approximate solution that satisfies

$$\left|\overline{u}(x) - f(x) - \int_0^x h(x,s,\overline{u}(s))\,ds\right| < \delta. \tag{4.38}$$

If the assumption

$$|h(x,s,z)| - |h(x,s,\overline{z})| \le \overline{a}(x)\overline{b}(s)|z - \overline{z}| \tag{4.39}$$

is fulfilled, then we have

$$|\overline{u} - u|(x) \le \delta + \int_0^x \overline{a}(x)\overline{b}(s)|u - \overline{u}|(s)\,ds \tag{4.40}$$

and it gives a bound. In the linear case, one can consider two equations with coefficients f, k and $\overline{f}$, $\overline{k}$ and require $|f - \overline{f}|$, $|k - \overline{k}|$ to derive a bound for $|\overline{u} - u|$.

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