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SKEW-SYMMETRIC VECTOR FIELDS ON A CR-SUBMANIFOLD OF A PARA-KÄHLERIAN MANIFOLD

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We deal with a CR-submanifold M of a para-Kählerian manifold $\widetilde{M}$, which carries a J-skew-symmetric vector field X. It is shown that X defines a global Hamiltonian of the symplectic form Ω on $M^{\top}$ and JX is a relative infinitesimal automorphism of Ω. Other geometric properties are given.

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1. Introduction. CR-submanifolds M of some pseudo-Riemannian manifolds $\widetilde{M}$ have been first investigated by Rosca [10], and also studied in [2, 3, 11].

If $\widetilde{M}$ is a para-Kählerian manifold, it has been proved that any coisotropic submanifold M of $\widetilde{M}$ is a CR-submanifold (such CR-submanifolds have been denominated CICR-submanifolds [6]).

In this note, one considers a foliate CICR-submanifold M of a para-Kählerian manifold $\widetilde{M}(J,\widetilde{\Omega},\widetilde{g})$. It is proved that the necessary and sufficient condition in order that the leaf $M^{\top}$ of the horizontal distribution $D^{\top}$ on M carries a J-skew-symmetric vector field X, that is, $\nabla X = X \wedge JX$, is that the vertical distribution $D^{\perp}$ on M is autoparallel.

In this case, M may be viewed as the local Riemannian product $M = M^{\top} \times M^{\perp}$, where $M^{\top}$ is an invariant totally geodesic submanifold of M and $M^{\perp}$ is an isotropic totally geodesic submanifold.

Furthermore, if Ω is the symplectic form of $M^{\top}$, it is shown that X is a global Hamiltonian of Ω and JX is a relative infinitesimal automorphism of Ω (a similar discussion can be made for proper CR-submanifolds of a Kählerian manifold).

2. Preliminaries. Let $\widetilde{M}(J,\widetilde{\Omega},\widetilde{g})$ be a $2m$-dimensional para-Kählerian manifold, where, as is well known [7], the triple $(J,\widetilde{\Omega},\widetilde{g})$ of tensor fields is the *paracomplex operator*, the *symplectic form*, and the *para-Hermitian metric tensor field*, respectively.

If $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$, these manifolds satisfy

$$J^2 = Id, \quad d\widetilde{\Omega} = 0, \qquad (\widetilde{\nabla} J)\widetilde{Z} = 0, \quad \widetilde{Z} \in \Gamma T\widetilde{M}. \tag{2.1}$$

Let $x : M \to \widetilde{M}$ be the immersion of an l-codimensional submanifold M, $l < m$, in $\widetilde{M}$ and let $T_p^{\perp}M$ and T_pM be the normal space and the tangent space at each point $p \in M$.

If $J(T_p^{\perp}M) \subset T_pM$, then M is said to be a *coisotropic* submanifold of $\widetilde{M}$ (see [2]). If $\widetilde{W} = \operatorname{vect}\{h_a, h_{a^*};\ a = 1,\dots,m,\ a^* = a+m\}$ is a real Witt vector basis on $\widetilde{M}$, one has

$$\tilde{g}(h_a,h_b) = \tilde{g}(h_{a^*},h_{b^*}) = \delta_{ab}. \tag{2.2}$$

Next, if $\widetilde{W}^* = \{\omega^a, \omega^{a^*}\}$ denotes the associated cobasis of $\widetilde{W}$, then $\tilde{g}$ and $\tilde{\Omega}$ are expressed by

$$\tilde{g} = 2\sum \omega^a \otimes \omega^{a^*}, \tag{2.3}$$

$$\tilde{\Omega} = \sum \omega^a \wedge \omega^{a^*}. \tag{2.4}$$

We recall also that $\widetilde{W}$ may split as

$$\widetilde{W} = \tilde{S} + \tilde{S}^*, \tag{2.5}$$

where the pairing $(\tilde{S}, \tilde{S}^*)$ defines an involutive automorphism of square 1, that is,

$$Jh_a = h_{a^*}, \qquad Jh_{a^*} = h_a, \tag{2.6}$$

and the local connection forms $\tilde{\theta}_B^A \in \Lambda^1\widetilde{M}$, $A,B \in \{1,2,\dots,2m\}$ satisfy

$$\tilde{\theta}_b^{a^*} = 0, \qquad \tilde{\theta}_{b^*}^{a} = 0, \qquad \tilde{\theta}_b^a + \tilde{\theta}_{a^*}^{b^*} = 0. \tag{2.7}$$

It has been proved in [10] that any coisotropic submanifold M of a para-Kählerian manifold $\widetilde{M}$ is a CR-submanifold of $\widetilde{M}$ and such a submanifold has been called a CICR-submanifold [6].

Let $D^{\top} : p \to D_p^{\top} = T_pM \setminus J(T_p^{\perp}M)$ and $D^{\perp} : p \to D_p^{\perp} = J(T_p^{\perp}M) \subset T_pM$ be the two complementary differentiable distributions on M. One has

$$JD_p^{\top} = D_p^{\top}, \qquad JD_p^{\perp} = T_p^{\perp}M, \tag{2.8}$$

and $D^{\top}$ (resp., $D^{\perp}$) is called the *horizontal* (resp., *vertical*) *distribution* on M.

As in the Kählerian case, the vertical distribution $D^{\perp}$ is always involutive.

If M is defined by the Pfaffian system

$$\omega^r = 0, \quad r = 2m+1-l,\dots,2m, \tag{2.9}$$

then one has

$$\begin{aligned} D_p^{\top} &= \{h_i, h_{i^*},\ i = 1,\dots,m-l,\ i^* = i+m\}, \\ D_p^{\perp} &= \{h_r,\ r = m-l+1,\dots,m\}. \end{aligned} \tag{2.10}$$

Further denote by

$$\varphi^{\perp} = \omega^{m-l+1} \wedge \cdots \wedge \omega^m \tag{2.11}$$

the simple unit form which corresponds to $D^{\perp}$.

Then, in order that the distribution $D^\top$ be also involutive, it is necessary and sufficient that $\varphi^\perp$ be a conformal integral invariant of $D^\top$, that is,

$$\mathcal{L}_{D^\top}\varphi^\perp = f\varphi^\perp \tag{2.12}$$

for a certain scalar function f.

By a standard calculation, one derives that the above equation implies

$$\theta_i^r = 0, \tag{2.13}$$

and in this case, one may write

$$d\varphi^\perp = -\left(\sum \theta_r^r\right) \wedge \varphi^\perp, \tag{2.14}$$

that is, $\varphi^\perp$ is exterior recurrent.

In this case, as is known [2, 10], M is a foliated CR-submanifold of $\widetilde{M}$.

We will investigate now the case when the leaf $M^\top$ of $D^\top$ carries a J-skew-symmetric vector field X, that is,

$$\nabla X = X \wedge JX. \tag{2.15}$$

One may express ∇X as

$$\nabla X = (JX)^\flat \otimes X - X^\flat \otimes JX, \tag{2.16}$$

where

$$X = X^i h_i + X^{i^*} h_{i^*} = X^i \omega^{i^*} + X^{i^*} \omega^i. \tag{2.17}$$

Recalling Cartan structure equations [4],

$$\begin{aligned} \nabla h &= \theta \otimes e \in A^1(M, TM), \\ d\omega &= -\theta \wedge \omega, \\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned} \tag{2.18}$$

In the above equations, θ, respectively Θ, are the local connection forms in the bundle W, respectively the curvature forms on M.

Then making use of Cartan structure equations, one finds by a standard calculation that (2.16) implies that the vertical distribution $D^\perp$ is autoparallel, that is, $\nabla_{Z'} Z'' \in D^\perp$, for all $Z', Z'' \in D^\perp$, which, in terms of connection forms, is expressed by

$$\theta_r^i = 0. \tag{2.19}$$

We agree to call θ_r^i and θ_i^r the *mixed connection forms*.

Taking account of (2.13) and (2.19), one derives from (2.16)

$$dX^\flat = 2(JX)^\flat \wedge X^\flat, \tag{2.20}$$

which agrees with the general equation of skew-symmetric killing vector fields [5, 8].

Next, by (2.1), one has

$$\nabla JX = (JX)^\flat \otimes JX - X^\flat \otimes X, \tag{2.21}$$

which shows that JX is a gradient vector field.

Hence, we may state the following theorem.

Theorem 2.1. *Let $x : M \to \widetilde{M}$ be an improper immersion of a* CR*-submanifold in a para-Kählerian manifold $\widetilde{M}(J,\widetilde{\Omega},\widetilde{g})$ and let $D^\top$ (resp., $D^\perp$) be the horizontal distribution (resp., the vertical distribution) on M. If M is a foliate* CR*-submanifold, then the necessary and sufficient condition in order that the leaf $M^\top$ of $D^\top$ carries a J-skew-symmetric vector field X is that $D^\perp$ is an autoparallel foliation. In this case, the* CR*-submanifold M under consideration may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is an invariant totally geodesic submanifold of M and $M^\perp$ is an isotropic totally geodesic submanifold. In addition, in this case, JX is a gradient vector field.*

3. Properties. In this section, we will pointout some additional properties of X involving the symplectic form Ω of $M^\top$ and the exterior covariant differential d^∇ of ∇X. Operating on (2.16) and (2.21), one derives by a short calculation

$$\begin{aligned} d^\nabla(\nabla X) &= \nabla^2 X = 2\left(X^\flat \wedge (JX)^\flat\right) \otimes JX, \\ d^\nabla(\nabla JX) &= \nabla^2 JX = 2\left(X^\flat \wedge (JX)^\flat\right) \otimes X, \end{aligned} \tag{3.1}$$

which gives

$$\begin{aligned} \nabla^2(X+JX) &= 2\left(X^\flat \wedge (JX)^\flat\right) \otimes (X+JX), \\ \nabla^2(X-JX) &= -2\left(X^\flat \wedge (JX)^\flat\right) \otimes (X-JX). \end{aligned} \tag{3.2}$$

Therefore, we agree to define $X+JX$ and $X-JX$ as 2-*covariant recurrent vector fields*.

It should also be noticed that by reference to the general formula

$$\nabla_V(X_1 \wedge \cdots \wedge X_p) = \sum (X_1 \wedge \cdots \wedge \nabla_V X_j \wedge \cdots \wedge X_p), \quad V \in \Gamma TM, \tag{3.3}$$

one finds by (2.15) and (2.21)

$$\nabla_V(X \wedge JX) = 2g(V,JX)(X \wedge JX). \tag{3.4}$$

This shows that the covariant derivative of $X \wedge JX$ with respect to any vector field V is proportional to $X \wedge JX$.

On the other hand, by the general formula

$$\nabla^2 V(Z,Z') = R(Z,Z')V, \tag{3.5}$$

where R denotes the curvature tensor field and V, Z, Z' are vector fields, one has (see also [9])

$$\mathcal{R}(Z,V) = TrR(\cdot,Z)V, \tag{3.6}$$

where $\mathcal{R}$ is the Ricci tensor field of ∇.

Since in the case under consideration one must take in (3.6) the para-Hermitian trace, then setting in (3.6) $Z = V = X$, one finds

$$\mathcal{R}(X,X) = 0, \tag{3.7}$$

that is, the Ricci curvature of X vanishes.

Denote by $\tilde{\Omega}$ the symplectic form of $\widetilde{M}$, then $\Omega = \tilde{\Omega}|_{M^\top}$ is a symplectic form of rank equal to the dimension of $M^\top$, that is, in our case, $2(m-l)$.

Then, if ${}^\flat Z : Z \to -i_Z\Omega$ is the symplectic isomorphism, by a short calculation and on behalf of (2.4), one gets

$${}^\flat X = -(JX)^\flat, \tag{3.8}$$

and since JX is a gradient vector field, we conclude according to a known definition (see also [1]) that X is a *global Hamiltonian* of Ω.

In a similar manner, one finds

$${}^\flat(JX) = X^\flat, \tag{3.9}$$

and by (2.20), it follows that

$$d(\mathcal{L}_{JX}\Omega) = 0, \tag{3.10}$$

which shows that JX is a relative infinitesimal automorphism of Ω [1].

We state the following theorem.

Theorem 3.1. *Let M be a* CR*-submanifold of a para-Kählerian manifold $\widetilde{M}$ and let Ω be the symplectic form on $M^\top$. If M carries a J-skew-symmetric vector field X, then the following properties hold:*

(i) *X is a global Hamiltonian of Ω and JX is a relative infinitesimal automorphism of Ω;*

(ii) *the Ricci tensor field $\mathcal{R}(X,X)$ vanishes;*

(iii) *the vector fields $X+JX$ and $X-JX$ are 2-covariant recurrent.*

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