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A LOWER BOUND FOR RATIO OF POWER MEANS

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Let n and m be natural numbers. Suppose that $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by $P_n(r) = ((1/n)\sum_{i=1}^{n} a_i^r)^{1/r}$. Then $P_n(r)/P_{n+m}(r) \ge a_n/a_{n+m}$. The lower bound is the best possible.

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1. Introduction. It is well known that the inequality

$$\frac{n}{n+1} < \left(\frac{(1/n)\sum_{i=1}^{n} i^r}{(1/(n+1))\sum_{i=1}^{n+1} i^r}\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1.1}$$

holds for $r > 0$ and $n \in \mathbb{N}$. We call the left-hand side of this inequality Alzer's inequality [1] and the right-hand side Martins' inequality [8].

Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \ge a_i^2$ for $i \ge 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \le a_i^2$ for $i \ge 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically concave sequence.

In [2], Martins' inequality was generalized as follows: let $\{a_i\}_{i\in\mathbb{N}}$ be an increasing, logarithmically concave, positive, and nonconstant sequence satisfying $(a_{\ell+1}/a_\ell)^\ell \ge (a_\ell/a_{\ell-1})^{\ell-1}$ for any positive integer $\ell > 1$, then

$$\left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{1.2}$$

where r is a positive number, $n, m \in \mathbb{N}$, and $a_i!$ denotes the sequence factorial $\prod_{i=1}^{n} a_i$. The upper bound is the best possible.

Recently, in [13], another generalization of Martins' inequality was obtained: let $n, m \in \mathbb{N}$ and let $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\{i[a_{i+1}/a_i - 1]\}_{i=1}^{n+m-1}$ is increasing. Then inequality (1.2) between ratios of the power means and of the geometric means holds. The upper bound is the best possible.

Alzer's inequality has invoked the interest of several mathematicians including, for example, Cerone et al. [3], Elezović and Pečarić [4], Guo [5, 16, 17], Kuang [6], Debnath [15], Liu [7], Luo [18], Ozeki [9], Sándor [19, 20], Ume [21], the first author [10, 11, 12, 14] of this note, and so on.

In [22], a general form of Alzer's inequality was obtained: let $\{a_i\}_{i=1}^{\infty}$ be a strictly increasing positive sequence and let m be a natural number. If $\{a_i\}_{i=1}^{\infty}$ is logarithmically concave and the sequence $\{(a_{i+1}/a_i)^i\}_{i=1}^{\infty}$ is increasing, then

$$\frac{a_n}{a_{n+m}} < \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \tag{1.3}$$

In this short note, utilizing the mathematical induction, we obtain the following theorem.

Theorem 1.1. *Let n and m be natural numbers. Suppose that $\{a_i\}_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by*

$$P_n(r) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{1/r}. \tag{1.4}$$

Then the sequence $\{P_i(r)/a_i\}_{i=1}^{n+m}$ is decreasing for any given positive real number r, that is,

$$\frac{P_n(r)}{P_{n+m}(r)} \geq \frac{a_n}{a_{n+m}}. \tag{1.5}$$

The lower bound in (1.5) is the best possible.

Considering that the exponential functions a^{x^α} and a^{α^x} for given constants $\alpha \geq 1$ and $a > 1$ are logarithmically convex on $[0,\infty)$, as a corollary of Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let $\alpha \geq 1$ and $a > 1$ be two constants. For any given real number r, the following inequalities hold:*

$$\begin{aligned} \frac{a^{(n+k)^\alpha}}{a^{(n+m+k)^\alpha}} &\leq \left(\frac{(1/n)\sum_{i=k+1}^{n+k} a^{i^\alpha r}}{(1/(n+m))\sum_{i=k+1}^{n+m+k} a^{i^\alpha r}}\right)^{1/r}, \\ \frac{a^{\alpha^{n+k}}}{a^{\alpha^{n+m+k}}} &\leq \left(\frac{(1/n)\sum_{i=k+1}^{n+k} a^{\alpha^i r}}{(1/(n+m))\sum_{i=k+1}^{n+m+k} a^{\alpha^i r}}\right)^{1/r}, \end{aligned} \tag{1.6}$$

where n and m are natural numbers and k is a nonnegative integer. The lower bounds above are the best possible.

2. Proof of Theorem 1.1. Inequality (1.5) is equivalent to

$$\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r} \geq \frac{a_n^r}{a_{n+m}^r}, \tag{2.1}$$

that is,

$$\frac{1}{(n+m)a_{n+m}^r}\sum_{i=1}^{n+m} a_i^r \le \frac{1}{na_n^r}\sum_{i=1}^{n} a_i^r. \tag{2.2}$$

This is also equivalent to

$$\frac{1}{(n+1)a_{n+1}^r}\sum_{i=1}^{n+1} a_i^r \le \frac{1}{na_n^r}\sum_{i=1}^{n} a_i^r. \tag{2.3}$$

Since

$$\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^{n} a_i^r + a_{n+1}^r, \tag{2.4}$$

inequality (2.3) reduces to

$$\sum_{i=1}^{n} a_i^r \ge \frac{na_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r}. \tag{2.5}$$

It is easy to see that inequality (2.5) holds for $n = 1$.

Assume that inequality (2.5) holds for some $n > 1$. Using the principle of mathematical induction and considering equality (2.4) and the inductive hypothesis, it is easy to show that the induction for inequality (2.5) on $n+1$ can be written as

$$\frac{(n+2)a_{n+2}^r - (n+1)a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r} \ge \left(\frac{a_{n+2}}{a_{n+1}}\right)^r, \tag{2.6}$$

which can be rearranged as

$$n\left[\left(\frac{a_{n+1}}{a_{n+2}}\right)^r - \left(\frac{a_n}{a_{n+1}}\right)^r\right] + \left(\frac{a_{n+1}}{a_{n+2}}\right)^r \le 1. \tag{2.7}$$

Since the sequence $\{a_i\}_{i=1}^{n+m}$ is increasing, we have $a_{n+1}/a_{n+2} \le 1$ and $(a_{n+1}/a_{n+2})^r \le 1$. From the logarithmical convexity of the sequence $\{a_i\}_{i=1}^{n+m}$, it follows that $a_{n+1}/a_{n+2} \le a_n/a_{n+1}$ and $(a_{n+1}/a_{n+2})^r - (a_n/a_{n+1})^r \le 0$. Therefore, inequality (2.7) is valid. Thus, inequality (1.5) holds.

It can be easily shown by L'Hospital rule that

$$\lim_{r\to\infty}\frac{P_n(r)}{P_{n+m}(r)} = \frac{a_n}{a_{n+m}}. \tag{2.8}$$

Hence, the lower bound in (1.5) is the best possible. The proof is complete.

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