IJMMS 2003:57, 3599–3607
PII. S0161171203301267
http://ijmms.hindawi.com

ON k-NEARLY UNIFORM CONVEX PROPERTY IN GENERALIZED CESÀRO SEQUENCE SPACES

WINATE SANHAN and SUTHEP SUANTAI

Received 19 January 2003

We define a generalized Cesàro sequence space $\text{ces}(p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers, and consider it equipped with the Luxemburg norm. The main purpose of this paper is to show that $\text{ces}(p)$ is k-nearly uniform convex (k-NUC) for $k \geq 2$ when $\lim_{n\to\infty} \inf p_n > 1$. Moreover, we also obtain that the Cesàro sequence space ces_p (where $1 < p < \infty$) is k-NUC, kR, NUC, and has a drop property.

2000 Mathematics Subject Classification: 46B20, 46B45.

1. Introduction. Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. For any subset A of X, we denote by $\text{conv}(A)$ (resp., $\overline{\text{conv}}(A)$) the convex hull (resp., the closed convex hull) of Clarkson [1] who introduced the concept of uniform convexity, and it is known that uniform convexity implies reflexivity of Banach spaces. There are different uniform geometric properties which have been defined between the uniform convexity and the reflexivity of Banach spaces. Huff [6] introduced the nearly uniform convexity of Banach spaces. He has proved that the class of nearly uniformly convexifiable spaces is strictly between superreflexive and reflexive Banach spaces.

A Banach space X is called *uniformly convex* (UC) if for each $\epsilon > 0$, there is $\delta > 0$ such that for $x, y \in S(X)$, the inequality $\|x - y\| > \epsilon$ implies that

$$\left\|\frac{1}{2}(x+y)\right\| < 1-\delta. \tag{1.1}$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \text{conv}\left(\{x\} \cup B(X)\right). \tag{1.2}$$

Rolewicz [12], basing on Daneš drop theorem [4], introduced the notion of drop property for Banach spaces.

A Banach space X has the *drop property* (D) if for every closed set C disjoint with $B(X)$, there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}. \tag{1.3}$$

A Banach space X is said to have the *Kadec-Klee property* (or *property* (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

In [13], Rolewicz proved that if the Banach space X has the drop property, then X is reflexive. Montesinos [11] extended this result by showing that X has the drop property if and only if X is reflexive and has the property (H).

Recall that a sequence $\{x_n\} \subset X$ is said to be ϵ-*separated sequence* for some $\epsilon > 0$ if

$$\operatorname{sep}(x_n) = \inf \{\|x_n - x_m\| : n \neq m\} > \epsilon. \tag{1.4}$$

A Banach space X is said to be *nearly uniformly convex* (NUC) if for every $\epsilon > 0$, there exists $\delta \in (0,1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $\operatorname{sep}(x_n) > \epsilon$, we have

$$\operatorname{conv}(x_n) \cap ((1-\delta)B(X)) \neq \emptyset. \tag{1.5}$$

Huff [6] proved that every NUC Banach space is reflexive and it has property (H).

Kutzarova [7] has defined k-nearly uniformly convex Banach spaces. Let $k \geq 2$ be an integer. A Banach space X is said to be k-*nearly uniformly convex* (k-NUC) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any sequence $(x_n) \subset B(X)$ with $\operatorname{sep}(x_n) \geq \epsilon$, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$\left\| \frac{x_{n_1} + x_{n_2} + x_{n_3} + \cdots + x_{n_k}}{k} \right\| < 1 - \delta. \tag{1.6}$$

Clearly, k-NUC Banach spaces are NUC but the opposite implication does not hold in general (see [7]).

Fan and Glicksberg [5] have introduced fully k-convex Banach spaces. A Banach space X is said to be *fully* k-*rotund* (kR) if for every sequence $(x_n) \subset B(X)$, $\|x_{n_1} + x_{n_2} + \cdots + x_{n_k}\| \to k$ as $n_1, n_2, \ldots, n_k \to \infty$ implies that (x_n) is convergent.

It is well known that UC implies kR and kR implies $(k+1)R$, and kR spaces are reflexive and rotund, and it is easy to see that k-NUC implies kR.

Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all natural and real numbers, respectively.

Let X be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X, we define

$$X_\varrho = \left\{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0\right\},$$
$$X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}. \tag{1.7}$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, for $x \in X_\varrho$, we define

$$\|x\| = \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \le 1\right\}. \tag{1.8}$$

Orlicz [10] proved that if ϱ is a convex modular on X, then $X_\varrho = X_\varrho^*$ and $\|\cdot\|$ is a norm on X_ϱ for which X_ϱ is a Banach space. The norm $\|\cdot\|$, defined as in (1.8), is called the Luxemburg norm.

A modular ϱ is said to satisfy the δ_2-condition ($\varrho \in \delta_2$) if for any $\epsilon > 0$, there exist constants $K \ge 2$ and $a > 0$ such that

$$\varrho(2u) \le K\varrho(u) + \epsilon \tag{1.9}$$

for all $u \in X_\varrho$ with $\varrho(u) \le a$.

If ϱ satisfies the δ_2-condition for any $a > 0$ with $K \ge 2$ dependent on a, we say that ϱ satisfies the strong δ_2-condition ($\varrho \in \delta_2^s$).

The following known results are very important for our consideration.

Theorem 1.1. *If $\varrho \in \delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|\varrho(u+v) - \varrho(u)| < \varepsilon \tag{1.10}$$

whenever $u, v \in X_\varrho$ with $\varrho(u) \le L$ and $\varrho(v) \le \delta$.

Proof. See [2, Lemma 2.1]. □

Theorem 1.2. (1) *If $\varrho \in \delta_2^s$, then for any $x \in X_\varrho$, $\|x\| = 1$ if and only if $\varrho(x) = 1$.*

(2) *If $\varrho \in \delta_2$, then for any sequence (x_n) in X_ϱ, $\|x_n\| \to 0$ if and only if $\varrho(x_n) \to 0$.*

Proof. See [2, Corollary 2.2 and Lemma 2.3]. □

Theorem 1.3. *If $\varrho \in \delta_2^s$, then for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\varrho(x) \le 1 - \epsilon$ implies $\|x\| \le 1 - \delta$.*

Proof. Suppose that the theorem does not hold, then there exist $\epsilon > 0$ and $x_n \in X_\varrho$ such that $\varrho(x_n) < 1 - \epsilon$ and $1/2 \le \|x_n\| \nearrow 1$. Let $a_n = 1/\|x_n\| - 1$. Then $a_n \to 0$ as $n \to \infty$. Let $L = \sup\{\varrho(2x_n);\ n \in \mathbb{N}\}$. Since $\varrho \in \delta_2^s$, there exists $K \ge 2$

such that

$$\varrho(2u) \le K\varrho(u) + 1 \tag{1.11}$$

for every $u \in X_\varrho$ with $\varrho(u) < 1$.

By (1.11), we have $\varrho(2x_n) \le K\varrho(x_n) + 1 < K + 1$ for all $n \in \mathbb{N}$. Hence, $0 < L < \infty$. By Theorem 1.2(1), we have

$$\begin{aligned} 1 = \varrho\left(\frac{x_n}{\|x_n\|}\right) &= \varrho(2a_n x_n + (1-a_n)x_n) \\ &\le a_n \varrho(2x_n) + (1-a_n)\varrho(x_n) \\ &\le a_n L + (1-\epsilon) \longrightarrow 1-\epsilon, \end{aligned} \tag{1.12}$$

which is a contradiction. □

Let l^0 be the space of all real sequences. For $1 < p < \infty$, the Cesàro sequence space (ces_p) is defined by

$$\text{ces}_p = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \infty \right\} \tag{1.13}$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{1/p}. \tag{1.14}$$

This space was first introduced by Shiue [14], which is useful in the theory of Matrix operator and others (see [8, 9]). Some geometric properties of the Cesàro sequence space ces_p were studied by many authors. It is known that $(\text{ces}_p, \|\cdot\|)$ is locally uniformly rotund (LUR) and has property (H) (see [9]). Cui and Meng [3] proved that $(\text{ces}_p, \|\cdot\|)$ has property (β).

Let $p = (p_n)$ be a sequences of positive real numbers with $p_n \ge 1$ for all $n \in \mathbb{N}$. The generalized Cesàro sequence space $\text{ces}(p)$ is defined by

$$\text{ces}(p) = \{x \in l^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}, \tag{1.15}$$

where

$$\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^{p_n} \tag{1.16}$$

is a convex modular on $\text{ces}(p)$.

We consider $\text{ces}(p)$ equipped with the Luxemburg norm:

$$\|x\| = \inf\left\{ \varepsilon > 0 : \rho\left(\frac{x}{\varepsilon}\right) \le 1 \right\}. \tag{1.17}$$

When $p_n = q$ for all $n \in \mathbb{N}$, we see that $\operatorname{ces}(p) = \operatorname{ces}_q$ and the Luxemburg norm on $\operatorname{ces}(p)$ given in (1.17) is equal to the norm $\|\cdot\|$ given in (1.14). In this paper, we show that $\operatorname{ces}(p)$ equipped with the Luxemburg norm is k-NUC for $k \geq 2$, so it is kR and (NUC).

Throughout this paper, we assume that $p = (p_n)$ is bounded with

$$\liminf_{n\to\infty} p_n > 1 \tag{1.18}$$

and that $M = \sup_n p_n$.

2. Main results

Proposition 2.1. *For $x \in \operatorname{ces}(p)$, the modular ρ on $\operatorname{ces}(p)$ satisfies the following properties:*

(1) *if $0 < a < 1$, then $a^M \rho(x/a) \leq \rho(x)$ and $\rho(ax) \leq a\rho(x)$,*
(2) *if $a \geq 1$, then $\rho(x) \leq a^M \rho(x/a)$,*
(3) *if $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.*

Proof. All assertions are clearly obtained by the definition and convexity of ρ. □

Proposition 2.2. *For any $x \in \operatorname{ces}(p)$,*

(1) *if $\|x\| \leq 1$, then $\rho(x) \leq \|x\|$,*
(2) *if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,*
(3) *$\|x\| = 1$ if and only if $\rho(x) = 1$.*

Proof. (1) Suppose that $\|x\| \leq 1$. If $x = 0$, then $\rho(x) = \|x\| = 0$. Suppose $x \neq 0$. By the definition of $\|\cdot\|$, there is a sequence (ϵ_n) with $\epsilon_n \downarrow \|x\|$ such that $\rho(x/\epsilon_n) \leq 1$. This implies that $\rho(x/\|x\|) \leq 1$. By Proposition 2.1(1), we have

$$\rho(x) = \rho\left(\frac{\|x\|\cdot x}{\|x\|}\right) \leq \|x\|\rho\left(\frac{x}{\|x\|}\right) \leq \|x\|. \tag{2.1}$$

(2) Suppose that $\|x\| > 1$. Then for $\epsilon \in (0, (\|x\|-1)/\|x\|)$, we have $(1-\epsilon)\|x\| > 1$. By Proposition 2.1(1), we have

$$1 < \rho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \leq \frac{\rho(x)}{(1-\epsilon)\|x\|}, \tag{2.2}$$

so that $(1-\epsilon)\|x\| < \rho(x)$. By taking $\epsilon \to 0$, we have $\rho(x) \geq \|x\|$.

(3) It follows from Theorem 1.2(1) because $\rho \in \delta_2^s$. □

Proposition 2.3. *For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left|\rho(u+v) - \rho(u)\right| < \varepsilon \tag{2.3}$$

whenever $u, v \in \operatorname{ces}(p)$ with $\rho(u) \leq L$ and $\rho(v) \leq \delta$.

Proof. Since $p = (p_n)$ is bounded, it is easy to see that $\rho \in \delta_2^s$. Hence, the proposition is obtained directly from Theorem 1.1. □

Proposition 2.4. *For every sequence* $(x_n) \in \operatorname{ces}(p)$, $\|x_n\| \to 0$ *if and only if* $\rho(x_n) \to 0$.

Proof. It follows directly from Theorem 1.2(2) because $\rho \in \delta_2^s$. □

Theorem 2.5. *For any* $x \in \operatorname{ces}(p)$ *and* $\epsilon \in (0,1)$, *there exists* $\delta \in (0,1)$ *such that* $\rho(x) \le 1-\epsilon$ *implies* $\|x\| \le 1-\delta$.

Proof. Since $\rho \in \delta_2^s$, the theorem is obtained directly from Theorem 1.3. □

Theorem 2.6. *The space* $\operatorname{ces}(p)$ *is* k*-NUC for any integer* $k \ge 2$.

Proof. Let $\epsilon > 0$ and $(x_n) \subset B(\operatorname{ces}(p))$ with $\operatorname{sep}(x_n) \ge \epsilon$. For each $m \in \mathbb{N}$, let

$$x_n^m = \Big(\underbrace{0,0,\dots,0}_{m-1}, x_n(m), x_n(m+1), \dots\Big). \tag{2.4}$$

Since for each $i \in \mathbb{N}$, $(x_n(i))_{n=1}^\infty$ is bounded, by using the diagonal method, we have that for each $m \in \mathbb{N}$, we can find a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$, $1 \le i \le m$. Therefore, there exists an increasing sequence of positive integer (t_m) such that $\operatorname{sep}((x_{n_j}^m)_{j>t_m}) \ge \epsilon$. Hence, there is a sequence of positive integers $(r_m)_{m=1}^\infty$ with $r_1 < r_2 < r_3 < \cdots$ such that $\|x_{r_m}^m\| \ge \epsilon/2$ for all $m \in \mathbb{N}$. Then by Proposition 2.4, we may assume that there exists $\eta > 0$ such that

$$\rho\left(x_{r_m}^m\right) \ge \eta \quad \forall m \in \mathbb{N}. \tag{2.5}$$

Let $\alpha > 0$ be such that $1 < \alpha < \lim_{n\to\infty} \inf p_n$. For fixed integer $k \ge 2$, let $\epsilon_1 = ((k^{\alpha-1}-1)/(k-1)k^\alpha)(\eta/2)$. Then by Proposition 2.3, there is a $\delta > 0$ such that

$$\left|\rho(u+v) - \rho(u)\right| < \epsilon_1 \tag{2.6}$$

whenever $\rho(u) \le 1$ and $\rho(v) \le \delta$.

Since by Proposition 2.2(1) $\rho(x_n) \le 1$ for all $n \in \mathbb{N}$, there exist positive integers m_i $(i = 1,2,\dots,k-1)$ with $m_1 < m_2 < \cdots < m_{k-1}$ such that $\rho(x_i^{m_i}) \le \delta$ and $\alpha \le p_j$ for all $j \ge m_{k-1}$. Define $m_k = m_{k-1}+1$. By (2.5), we have $\rho(x_{r_{m_k}}^{m_k}) \ge \eta$. Let $s_i = i$ for $1 \le i \le k-1$ and $s_k = r_{m_k}$.

Then in virtue of (2.5), (2.6), and convexity of function $f_i(u) = |u|^{p_i}$ $(i \in \mathbb{N})$, we have

$$\rho\left(\frac{x_{s_1}+x_{s_2}+\cdots+x_{s_k}}{k}\right) = \sum_{n=1}^\infty \left(\frac{1}{n}\sum_{i=1}^n \left|\frac{x_{s_1}(i)+x_{s_2}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n}$$

$$
\begin{aligned}
&= \sum_{n=1}^{m_1}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_1}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_1}(i)+x_{s_2}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\le \sum_{n=1}^{m_1}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_1}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_2}(i)+x_{s_3}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} + \epsilon_1 \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_2}(i)+x_{s_3}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_2}(i)+x_{s_3}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} + \epsilon_1 \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_2}(i)+x_{s_3}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_3}(i)+x_{s_4}(i)+\cdots+x_{s_k}(i)}{k}\right|\right)^{p_n} + 2\epsilon_1 \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{m_3}\frac{1}{k}\sum_{j=3}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \cdots + \sum_{n=m_{k-1}+1}^{m_k}\frac{1}{k}\sum_{j=k-1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\left|x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_k}(i)}{k}\right|\right)^{p_n} + (k-1)\epsilon_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\rho(x_{s_1})+\cdots+\rho(x_{s_{k-1}})}{k}+\frac{1}{k}\sum_{n=1}^{m_k}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{s_k}(i)|\right)^{p_n}\\
&\quad+\sum_{n=m_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x_{s_k}(i)}{k}\right|\right)^{p_n}+(k-1)\epsilon_1\\
&\leq \frac{k-1}{k}+\frac{1}{k}\sum_{n=1}^{m_k}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{s_k}(i)|\right)^{p_n}\\
&\quad+\frac{1}{k^{\alpha}}\sum_{n=m_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{s_k}(i)|\right)^{p_n}+(k-1)\epsilon_1\\
&\leq 1-\frac{1}{k}+\frac{1}{k}\left[1-\sum_{n=m_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{s_k}(i)|\right)^{p_n}\right]\\
&\quad+\frac{1}{k^{\alpha}}\sum_{n=m_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{s_k}(i)|\right)^{p_n}+(k-1)\epsilon_1\\
&\leq 1+(k-1)\epsilon_1-\left(\frac{k^{\alpha-1}-1}{k^{\alpha}}\right)\sum_{n=m_k+1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|x_{s_k}(i)|\right)^{p_n}\\
&\leq 1+(k-1)\epsilon_1-\left(\frac{k^{\alpha-1}-1}{k^{\alpha}}\right)\eta\\
&=1-\left(\frac{k^{\alpha-1}-1}{k^{\alpha}}\right)\left(\frac{\eta}{2}\right).
\end{aligned}
\tag{2.7}
$$

By Theorem 2.5, there exist $\gamma > 0$ such that $\|(x_{s_1}+x_{s_2}+\cdots+x_{s_k})/k\| < 1-\gamma$. Therefore, $\text{ces}(p)$ is k-NUC. □

Since k-NUC implies kR and kR implies R and reflexivity holds, and k-NUC implies NUC and NUC implies property (H) and reflexivity holds, by Theorem 2.6, the following results are obtained.

Corollary 2.7. *The space* $\text{ces}(p)$ *is* kR, *NUC, and has a drop property.*

Corollary 2.8. *For* $1 < p < \infty$, *the space* ces_p *is* k-*NUC.*

Corollary 2.9. *For* $1 < p < \infty$, *the space* ces_p *is* kR *and NUC.*

Corollary 2.10. *For* $1 < p < \infty$, *the space* ces_p *has the drop property.*

Acknowledgments. Suthep Suantai would like to thank the Thailand Research Fund for the financial support and the referee for pointing out the work of Cui and Hudzik [2]. Winate Sanhan was supported by The Royal Golden Jubilee Project.

References

[1] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.

[2] Y. Cui and H. Hudzik, *On the uniform Opial property in some modular sequence spaces*, Funct. Approx. Comment. Math. **26** (1998), 93–102.

[3] Y. Cui, C. Meng, and R. Płuciennik, *Banach-Saks property and property* (β) *in Cesàro sequence spaces*, Southeast Asian Bull. Math. **24** (2000), no. 2, 201–210.

[4] J. Daneš, *A geometric theorem useful in nonlinear functional analysis*, Boll. Un. Mat. Ital. (4) **6** (1972), 369–375.

[5] K. Fan and I. Glicksberg, *Fully convex normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 947–953.

[6] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), no. 4, 743–749.

[7] D. Kutzarova, *k-β and k-nearly uniformly convex Banach spaces*, J. Math. Anal. Appl. **162** (1991), no. 2, 322–338.

[8] P. Y. Lee, *Cesàro sequence spaces*, Math. Chronicle **13** (1984), 29–45.

[9] Y. Q. Lui, B. E. Wu, and P. Y. Lee, *Method of Sequence Spaces*, Guangdong of Science and Technology Press, 1996 (Chinese).

[10] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Mathematics, vol. 5, Polish Academy of Science, 1989.

[11] V. Montesinos, *Drop property equals reflexivity*, Studia Math. **87** (1987), no. 1, 93–100.

[12] S. Rolewicz, *On drop property*, Studia Math. **85** (1986), no. 1, 27–35 (1987).

[13] ———, *On Δ-uniform convexity and drop property*, Studia Math. **87** (1987), no. 2, 181–191.

[14] J.-s. Shiue, *On the Cesàro sequence spaces*, Tamkang J. Math. **1** (1970), no. 1, 19–25.

Winate Sanhan: Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

E-mail address: winate_s@yahoo.com

Suthep Suantai: Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

E-mail address: scmti005@chiangmai.ac.th