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COMPOSITION OPERATORS FROM THE BLOCH SPACE INTO THE SPACES Q_T

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Suppose that $\varphi(z)$ is an analytic self-map of the unit disk Δ. We consider the boundedness of the composition operator C_φ from Bloch space $\mathcal{B}$ into the spaces Q_T ($Q_{T,0}$) defined by a nonnegative, nondecreasing function $T(r)$ on $0 \le r < \infty$.

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1. Introduction. Let $\Delta = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and let $H(\Delta)$ be the space of all analytic functions in Δ. For $a \in \Delta$, Green's function with logarithmic singularity at $a \in \Delta$ is denoted by $g(z,a) = \log|(1-\bar{a}z)/(a-z)|$. For $0 < p < \infty$, the space Q_p consists of all functions f analytic in Δ for which

$$\sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 (g(z,a))^p \, dA(z) < \infty, \tag{1.1}$$

where $dA(z)$ is the Euclidean area element on Δ.

Q_p-spaces have been investigated by many authors (cf. [1, 2, 3, 9]). We know that $Q_1 = \mathrm{BMOA}$, the space of all analytic functions of bounded mean oscillation (cf. [4]). Further, the spaces Q_p are the same for each $p \in (1,\infty)$, and each space equals to the Bloch space $\mathcal{B}$, which is a Banach space with the norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \|f\|_b := |f(0)| + \sup_{z\in\Delta} (1-|z|^2)|f'(z)|. \tag{1.2}$$

Recently, we introduced a new space Q_T (cf. [5, 10]) by a nondecreasing function $T(r)$ on $0 \le r < \infty$ as follows.

Definition 1.1. Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \le r < \infty$. A function $f \in H(\Delta)$ is said to belong to Q_T if

$$\|f\|_{Q_T}^2 := \sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 T(g(z,a)) dA(z) < \infty. \tag{1.3}$$

If

$$\lim_{|a|\to 1} \iint_\Delta |f'(z)|^2 T(g(z,a)) dA(z) = 0, \tag{1.4}$$

then f is said to belong to $Q_{T,0}$.

For $0 < p < \infty$, if we take $T(r) = r^p$, the space Q_T coincides with the space Q_p. We note that $Q_T \subset \mathcal{B}$ for all nondecreasing functions T. We have previously shown that $Q_T = Q_p$ under certain growth conditions on $T(r)$ (cf. [10]).

In the present paper, first we give some basic properties of Q_T spaces, some of which are also new for the special case $Q_T = Q_p$. For example, Q_T is a Banach space with the norm $\|f\|_T$ defined by

$$\|f\|_T := |f(0)| + \|f\|_{Q_T}. \tag{1.5}$$

Then we investigate the boundedness of the composition operators from the Bloch space $\mathcal{B}$ into Q_T or $Q_{T,0}$. These results extend some previously known results (cf. [6, 8]).

2. Basic properties of Q_T spaces.

We give the following propositions.

Proposition 2.1. *The space Q_T is a subspace of the Bloch space $\mathcal{B}$.*

The proof of Proposition 2.1 can be found in [10].

Proposition 2.2. *The space Q_T is a Banach space with the norm defined in (1.5).*

Proof. For $f \in Q_T$ and $a \in \Delta$, define

$$I^2(f,a) := \iint_\Delta |f'(z)|^2 T(g(z,a))\,dA(z). \tag{2.1}$$

Let $f_1, f_2 \in Q_T$. It follows from Schwarz's inequality that

$$\iint_\Delta |f_1'(z) f_2'(z)| T(g(z,a))\,dA(z) \le I(f_1,a) I(f_2,a), \tag{2.2}$$

and then

$$\begin{aligned} I^2(f_1+f_2,a) &\le I^2(f_1,a) + 2I(f_1,a)I(f_2,a) + I^2(f_2,a) \\ &= (I(f_1,a) + I(f_2,a))^2. \end{aligned} \tag{2.3}$$

Thus, $I(f_1+f_2,a) \le I(f_1,a) + I(f_2,a)$ for all $a \in \Delta$. Hence

$$||f_1+f_2||_{Q_T} \le ||f_1||_{Q_T} + ||f_2||_{Q_T}. \tag{2.4}$$

Therefore,

$$\begin{aligned} ||f_1+f_2||_T^2 &= \left(|f_1(0)+f_2(0)| + ||f_1+f_2||_{Q_T}\right)^2 \\ &\le \left(|f_1(0)| + |f_2(0)| + ||f_1||_{Q_T} + ||f_2||_{Q_T}\right)^2 \\ &= \left(||f_1||_T + ||f_2||_T\right)^2, \end{aligned} \tag{2.5}$$

that is, $\|f_1+f_2\|_T \le \|f_1\|_T+\|f_2\|_T$. On the other hand, it is obvious that $\|f\|_T \ge 0$ for each $f \in Q_T$ and that $\|f\|_T = 0$ if and only if $f \equiv 0$. It is obvious that $\|cf\|_T = |c|\|f\|$ for any constant c. Thus, Q_T is a normed space.

Let $f \in Q_T$ and let $\phi_a(w) = (a-w)/(1-\bar{a}w)$, $a \in \Delta$. Then by changing a variable $w = \phi_a(z)$, we obtain

$$\begin{aligned} \|f\|_{Q_T}^2 &\ge \iint_\Delta |f'(z)|^2 T(g(z,a))\,dA(z) \\ &= \iint_\Delta |(f\circ\phi_a)'(w)|^2 T\left(\log\frac{1}{|w|}\right)dA(w) \\ &\ge T\left(\log\frac{1}{r}\right)\iint_{|w|<r} |(f\circ\phi_a)'(w)|^2 dA(w) \\ &\ge \pi r^2 T\left(\log\frac{1}{r}\right)(1-|a|^2)^2|f'(a)|^2. \end{aligned} \tag{2.6}$$

For r_0, $0<r_0<1$, such that $T(\log(1/r_0)) \ne 0$, we have

$$\|f\|_b \le \frac{\|f\|_{Q_T}}{r_0(\pi T(\log 1/r_0))^{1/2}}. \tag{2.7}$$

Since $f \in Q_T \subset \mathcal{B}$, we have for $z \in \Delta$,

$$\begin{aligned} |f(z)| &\le |f(0)| + \frac{\|f\|_b}{2}\log\frac{1+|z|}{1-|z|} \\ &\le |f(0)| + \frac{\|f\|_{Q_T}}{2r_0(\pi T(\log(1/r_0)))^{1/2}}\log\frac{1+|z|}{1-|z|} \\ &\le \|f\|_T\left(1+\frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}}\right)\log\frac{1+|z|}{1-|z|}. \end{aligned} \tag{2.8}$$

Suppose $\{f_n\}$ is a Cauchy sequence in Q_T. Then there is a constant $M>0$ such that

$$||f_n||_T \le M, \quad n = 1,2,\dots. \tag{2.9}$$

By the estimate (2.8) for a fixed $r_0 \in (0,1)$, we obtain that

$$|f_n(z)| \le M\left(1+\frac{1}{2r_0(\pi T(\log 1/r_0))^{1/2}}\right)\log\frac{1+|z|}{1-|z|} \tag{2.10}$$

holds for all integral numbers $n = 1,2,\dots$. Hence, there exist a subsequence $\{f_{n_j}(z)\}$ of $\{f_n(z)\}$ and an analytic function f defined on the unit disk Δ such that both $\{f_{n_j}(z)\}$ and $\{f'_{n_j}(z)\}$ converge uniformly to f and f', respectively. The conditions here are such that both the sequence of functions and the sequence of derivatives converge since we know that $\{f_n(z)\}$ is bounded on compact subsets of Δ by inequality (2.10). By Fatou's lemma, we get that

$$\begin{aligned}\iint_\Delta |f'(z)|^2 T(g(z,a))\,dA(z) &= \iint_\Delta \lim_{j\to\infty} |f'_{n_j}(z)|^2 T(g(z,a))\,dA(z)\\ &\le \liminf_{j\to\infty} \iint_\Delta |f'_{n_j}(z)|^2 T(g(z,a))\,dA(z)\\ &\le \liminf_{j\to\infty} \|f_{n_j}\|^2_{Q_T} \le M^2\end{aligned} \tag{2.11}$$

holds for all $a \in \Delta$, so that $f \in Q_T$. By a similar reasoning, we can prove that $\|f_n - f\|_T \to 0$ as $n \to \infty$. The proof of Proposition 2.2 is complete. □

3. Boundedness of composition operators. Let $\varphi(z)$ be an analytic self-map of the unit disk Δ. Let the composition operator C_φ induced by φ from $H(\Delta)$ to itself be defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(\Delta)$. The boundednesses of composition operators from $\mathcal{B}$ to itself and from $\mathcal{B}$ to Q_p have been studied in [6, 8], respectively. In this paper, we consider the same problems for the general spaces Q_T.

Theorem 3.1. *Let $T(r) \not\equiv 0$ be a nonnegative, nondecreasing function on $0 \le r < \infty$ and let φ be an analytic self-map of Δ. Then $C_\varphi : \mathcal{B} \to Q_T$ is bounded if and only if*

$$\sup_{a\in\Delta} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) < \infty. \tag{3.1}$$

Proof. Let (3.1) hold and let $K_1^2 (K_1 > 0)$ be the supremum in (3.1). If $f \in \mathcal{B}$, then for all $a \in \Delta$, we have

$$\begin{aligned}\iint_\Delta |(C_\varphi f)'(z)|^2 T(g(z,a))\,dA(z) &= \iint_\Delta |f'(\varphi(z))|^2 |\varphi'(z)|^2 T(g(z,a))\,dA(z)\\ &\le \|f\|_b^2 \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z)\\ &\le K_1^2 \|f\|_b^2.\end{aligned} \tag{3.2}$$

Consequently, $\|C_\varphi f\|_{Q_T} \le K_1 \|f\|_b$. Since $f(z) \in \mathcal{B}$, we obtain

$$\begin{aligned}\|C_\varphi f\|_T^2 &= \left(|f\circ\varphi(0)| + \|C_\varphi f\|_{Q_T}\right)^2\\ &\le \left(|f(0)| + \frac{\|f\|_b}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + K_1\|f\|_b\right)^2\\ &\le K^2\left(|f(0)| + \|f\|_b\right)^2 = K^2\|f\|_{\mathcal{B}}^2,\end{aligned} \tag{3.3}$$

where $K = \max\{1, K_1 + (1/2)\log(1+|\varphi(0)|)/(1-|\varphi(0)|)\}$. Thus, $\|C_\varphi f\|_T \le K\|f\|_{\mathcal{B}}$, which shows that $C_\varphi : \mathcal{B} \to Q_T$ is bounded.

Conversely, assume that $C_\varphi : \mathcal{B} \to Q_T$ is bounded, there exists a constant $K > 0$ such that for each $f \in \mathcal{B}$, we have

$$||C_\varphi f||_T \le K\|f\|_{\mathcal{B}}. \tag{3.4}$$

On the other hand, by a result in [7], there exist $f_1, f_2 \in \mathcal{B}$ such that

$$\frac{1}{1-|z|^2} \le |f_1'(z)| + |f_2'(z)| \tag{3.5}$$

holds for all $z \in \Delta$, so that

$$\frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} \le 2|(f_1\circ\varphi)'(z)|^2 + 2|(f_2\circ\varphi)'(z)|^2. \tag{3.6}$$

Thus, the inequalities

$$\begin{aligned}\iint_\Delta &\frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z)\\ &\le 2\iint_\Delta \left(|(f_1\circ\varphi)'(z)|^2 + |(f_2\circ\varphi)'(z)|^2\right)T(g(z,a))\,dA(z)\\ &\le 2K^2\left(||f_1||_{\mathcal{B}}^2 + ||f_2||_{\mathcal{B}}^2\right)\end{aligned} \tag{3.7}$$

hold for all $z, a \in \Delta$, which establishes (3.1). The proof of Theorem 3.1 is completed. □

Remark 3.2. Note that if $C_\varphi : \mathcal{B} \to \mathcal{B}$, then (3.1) holds for any increasing function T satisfying $Q_T = \mathcal{B}$. Indeed, we know that $Q_T = \mathcal{B}$ (see [5]) if and only if

$$\int_0^1 T\left(\log\left(\frac{1}{r}\right)\right)(1-r^2)^{-2} r\,dr < \infty. \tag{3.8}$$

The Schwarz-Pick lemma guarantees that $((1-|z|^2)/(1-|\varphi(z)|^2))|\varphi'(z)| \le 1$, so that (3.8) leads easily to (3.1). It means that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is always bounded (cf. [6]).

Remark 3.3. If one considers the composition operator C_φ from the Bloch space to the Dirichlet space

$$\mathcal{D} = \left\{f \in H(\Delta) : \iint_\Delta |f'(z)|^2 dA(z) < \infty\right\}, \tag{3.9}$$

then $C_\varphi : \mathcal{B} \to \mathcal{D}$ is bounded if and only if

$$\iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} dA(z) < \infty. \tag{3.10}$$

For the spaces $Q_{T,0}$, we have the following results.

Theorem 3.4. *Let $T(r)$ be a nonnegative, nondecreasing function on $0 \le r < \infty$ and let φ be an analytic self-map of Δ. Then $C_\varphi : \mathcal{B} \to Q_{T,0}$ is bounded if and only if*

$$\lim_{|a|\to 1} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) = 0. \tag{3.11}$$

Proof. Suppose $C_\varphi : \mathcal{B} \to Q_{T,0}$ is bounded. Using a way similar to the proof of Theorem 3.1, we choose functions $f_1, f_2 \in \mathcal{B}$ such that

$$\frac{1}{1-|z|^2} \le |f_1'(z)| + |f_2'(z)| \tag{3.12}$$

for all $z \in \Delta$. Then $C_\varphi f_1$ and $C_\varphi f_2$ belong to $Q_{T,0}$. Therefore,

$$\begin{aligned} &\lim_{|a|\to 1} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) \\ &\quad \le 2 \lim_{|a|\to 1} \iint_\Delta \left(|(f_1\circ\varphi)'(z)|^2 + |(f_2\circ\varphi)'(z)|^2\right) T(g(z,a))\,dA(z) = 0, \end{aligned} \tag{3.13}$$

which shows that (3.11) holds.

Conversely, by Theorem 3.1, we know that $C_\varphi : \mathcal{B} \to Q_T$ is bounded since condition (3.11) implies that

$$\sup_{a\in\Delta} \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z) < \infty. \tag{3.14}$$

We need only to prove that $C_\varphi f \in Q_{T,0}$ for each $f \in \mathcal{B}$, and this follows from the inequality

$$\begin{aligned} &\iint_\Delta |(C_\varphi f)'(z)|^2 T(g(z,a))\,dA(z) \\ &\quad = \iint_\Delta |f'(\varphi(z))|^2 |\varphi'(z)|^2 T(g(z,a))\,dA(z) \\ &\quad \le \|f\|_b^2 \iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^2} T(g(z,a))\,dA(z). \end{aligned} \tag{3.15}$$

The proof of Theorem 3.4 is completed. □

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