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ON THE LARGEST ANALYTIC SET FOR CYCLIC OPERATORS

A. BOURHIM

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We describe the set of analytic bounded point evaluations for an arbitrary cyclic bounded linear operator T on a Hilbert space $\mathcal{H}$; some related consequences are discussed. Furthermore, we show that two densely similar cyclic Banach-space operators possessing Bishop's property (β) have equal approximate point spectra.

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1. Introduction. In the present paper, all Banach spaces are complex. Let $\mathcal{X}$ be a Banach space and let $\mathcal{L}(\mathcal{X})$ denote the algebra of all linear bounded operators on $\mathcal{X}$. For an operator $T \in \mathcal{L}(\mathcal{X})$, let T^*, $\sigma(T)$, $\rho(T) := \mathbb{C}\backslash\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ap}}(T)$, $\Gamma(T)$, $\ker T$, and $\operatorname{ran} T$ denote the adjoint operator acting on the dual space $\mathcal{X}^*$, the spectrum, the resolvent set, the point spectrum, the approximate point spectrum, the compression spectrum, the kernel, and the range, respectively, of T. For an operator $T \in \mathcal{L}(\mathcal{X})$, let $\mathfrak{R}(T)$ denote the open set of complex numbers $\lambda \in \mathbb{C}$ for which there exists a nonzero analytic function $\phi : \mathcal{V} \to \mathcal{X}$ on some open disc $\mathcal{V}$ centered at λ such that

$$(T-\mu)\phi(\mu) = 0 \quad \forall \mu \in \mathcal{V}. \tag{1.1}$$

The operator T is said to have the *single-valued extension property* if $\mathfrak{R}(T)$ is empty. Equivalently if, for every open subset U of $\mathbb{C}$, the only analytic solution $\phi : U \to \mathcal{X}$ of the equation $(T-\lambda)\phi(\lambda) = 0$ $(\lambda \in U)$ is the identically zero function $\phi \equiv 0$ on U. Recall that the operator T is called *cyclic* with cyclic vector $x \in \mathcal{X}$ if the finite linear combinations of the vectors $x, Tx, T^2x, \ldots$ are dense in $\mathcal{X}$. For a subset F of $\mathbb{C}$, let $\overline{F} := \{\overline{z} : z \in F\}$ denote the conjugate set of F.

Let T be a cyclic linear bounded operator on a Hilbert space $\mathcal{H}$ with cyclic vector x. A point $\lambda \in \mathbb{C}$ is said to be a *bounded point evaluation* for T if there is a constant $M > 0$ such that

$$|p(\lambda)| \leq M\|p(T)x\| \tag{1.2}$$

for every polynomial p. The set of all bounded point evaluations for T will be denoted by $B(T)$. Note that it follows from Riesz Representation theorem that

$\lambda \in B(T)$ if and only if there exists a unique vector $k(\lambda) \in \mathcal{H}$ such that

$$p(\lambda) = \langle p(T)x, k(\lambda) \rangle \tag{1.3}$$

for every polynomial p. An open subset O of $\mathbb{C}$ is said to be an *analytic set* for T if it is contained in $B(T)$ and if for every $y \in \mathcal{H}$, the complex-valued function $\hat{y}$ defined on $B(T)$ by $\hat{y}(\lambda) = \langle y, k(\lambda) \rangle$ is analytic on O. The largest analytic set for T will be denoted by $B_a(T)$ and every point of it will be called an *analytic bounded point evaluation* for T.

This paper has been divided into three sections. In Section 2, we give a complete description of the largest analytic set for cyclic Hilbert-space operators and explain more about bounded point evaluations from the point of view of local spectral theory. In Section 3, we prove that $\mathfrak{R}(T^*) = \sigma(T) \backslash \sigma_{\text{ap}}(T)$ for every cyclic Banach-space operator possessing Bishop's property (β). Therefore, we use this result and show that densely similar cyclic Banach-space operators possessing Bishop's property (β) have the same approximate point spectra; this result generalizes [12, Theorem 4].

We will need to introduce some notions from the local spectral theory. Suppose that $\mathcal{X}$ is a Banach space. Let $T \in \mathcal{L}(\mathcal{X})$; the *local resolvent set* $\rho_T(x)$ of T at a point $x \in \mathcal{X}$ is the union of all open subsets $U \subset \mathbb{C}$ for which there is an analytic $\mathcal{X}$-valued function ϕ on U such that $(T - \lambda)\phi(\lambda) = x$ for every $\lambda \in U$. The complement in $\mathbb{C}$ of $\rho_T(x)$ is called the *local spectrum* of T at x and will be denoted by $\sigma_T(x)$; it is a closed subset contained in $\sigma(T)$. It is well known that T has the single-valued extension property if and only if zero is the only element x of $\mathcal{X}$ for which $\sigma_T(x) = \emptyset$. For a closed subset F of $\mathbb{C}$, let $\mathcal{X}_T(F) := \{x \in \mathcal{X} : \sigma_T(x) \subset F\}$ be the corresponding analytic spectral subspace; it is a T-hyperinvariant subspace, generally nonclosed in $\mathcal{X}$. The operator T is said to satisfy Dunford's condition (C) if for every closed subset F of $\mathbb{C}$, the linear subspace $\mathcal{X}_T(F)$ is closed. For every open subset U of $\mathbb{C}$, we let $\mathbb{O}(U, \mathcal{X})$ denote the space of analytic $\mathcal{X}$-valued functions defined on U. It is a Fréchet space when endowed with the topology of uniform convergence on compact subsets of U. Recall also that the operator T is said to possess Bishop's property (β) provided that for every open subset U of $\mathbb{C}$, the mapping $T_U : \mathbb{O}(U, \mathcal{X}) \to \mathbb{O}(U, \mathcal{X})$, given by $(T_U f)(\lambda) = (T - \lambda) f(\lambda)$ for every $f \in \mathbb{O}(U, \mathcal{X})$ and for $\lambda \in U$, is injective and has a closed range. It is known that the Bishop's property (β) implies the Dunford's condition (C) and it turns out that the single-valued extension property follows from the Dunford's condition (C). Stampfli [14] and Radjabalipour [10] have shown that hyponormal operators satisfy Dunford's condition (C), and Putinar [9] has shown that hyponormal operators, M-hyponormal operators, and more generally subscalar operators have Bishop's property (β). For thorough presentations of the local spectral theory, we refer to [5, 7].

2. Analytic bounded point evaluations for cyclic operators. Throughout this section, let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be a cyclic operator with cyclic vector $x \in \mathcal{H}$. For $\lambda \in B(T)$, let $k(\lambda)$ denote the vector of $\mathcal{H}$ given by (1.3).

It is well known that an open subset O of $\mathbb{C}$ which is contained in $B(T)$ is an analytic set for T if and only if the function $\lambda \mapsto \|k(\lambda)\|$ is bounded on compact subsets of O (see [6, Proposition II.7.6] and [15, Lemma 1.2]). Using a similar proof of [6, Proposition II.7.6], this result can be refined as follows; we include here the proof for completeness.

Proposition 2.1. *Let O be an open subset of $\mathbb{C}$. The following statements are equivalent:*

(i) *O is an analytic set for T;*

(ii) *$O \subset B(T)$ and the function $k : O \to \mathcal{H}$ is locally Lipschitz on O;*

(iii) *$O \subset B(T)$ and the function $k : O \to \mathcal{H}$ is continuous on O;*

(iv) *$O \subset B(T)$ and the function $\lambda \mapsto \|k(\lambda)\|$ is bounded on compact subsets of O;*

(v) *for every compact subset K of O, there is a constant $M > 0$ such that for every $\lambda \in K$,*

$$|p(\lambda)| \le M\|p(T)x\| \tag{2.1}$$

for every polynomial p.

Proof. It is clear that (iv) and (v) are equivalent since for every $\lambda \in B(T)$, we have

$$\|k(\lambda)\| = \sup_{p(T)x \neq 0} \frac{|p(\lambda)|}{\|p(T)x\|}. \tag{2.2}$$

On the other hand, the implications (ii)⇒(iii) and (iii)⇒(iv) are trivial. So, it suffices to establish the implications (i)⇒(ii) and (iv)⇒(i).

Assume that O is an analytic set for T. Let $\lambda_0 \in O$, then there is $\epsilon > 0$ such that $B := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \le \epsilon\} \subset O$. For every $y \in \mathcal{H}$, we have

$$\left\langle y, \frac{k(\mu) - k(\lambda)}{\overline{\mu} - \overline{\lambda}} \right\rangle = \frac{\hat{y}(\mu) - \hat{y}(\lambda)}{\mu - \lambda} \quad \text{for every } (\mu,\lambda) \in O \times O,\ \mu \neq \lambda. \tag{2.3}$$

Since the function ϕ_y defined on $O \times O$ by

$$\phi_y(\mu,\lambda) := \begin{cases} \dfrac{\hat{y}(\mu) - \hat{y}(\lambda)}{\mu - \lambda}, & \text{if } \mu \neq \lambda, \\ \hat{y}'(\lambda), & \text{if } \mu = \lambda, \end{cases} \tag{2.4}$$

is continuous, it follows that

$$\sup_{(\mu,\lambda) \in B \times B,\ \mu \neq \lambda} \left| \frac{\hat{y}(\mu) - \hat{y}(\lambda)}{\mu - \lambda} \right| \le \sup_{(\mu,\lambda) \in B \times B} |\phi_y(\mu,\lambda)| < +\infty. \tag{2.5}$$

By uniform boundedness principle,

$$\sup_{(\mu,\lambda)\in B\times B,\,\mu\neq\lambda}\left\|\frac{k(\mu)-k(\lambda)}{\overline{\mu}-\overline{\lambda}}\right\|<+\infty. \tag{2.6}$$

Hence, there is a constant $M>0$ such that

$$\|k(\mu)-k(\lambda)\|\le M|\mu-\lambda| \quad \text{for every } (\mu,\lambda)\in B\times B, \tag{2.7}$$

and the implication (i)⇒(ii) is proved.

Now, suppose that $O\subset B(T)$ and the function $\lambda\mapsto\|k(\lambda)\|$ is bounded on compact subsets of O. Let $y\in\mathcal{H}$, then there is a sequence of polynomials $(p_n)_n$ such that $\lim_{n\to+\infty}\|p_n(T)x-y\|=0$. It follows from the Cauchy-Schwartz inequality that for every compact subset K of O, we have

$$\sup_{\lambda\in K}|p_n(\lambda)-\hat{y}(\lambda)|\le\sup_{\lambda\in K}\|k(\lambda)\|\|p_n(T)x-y\|. \tag{2.8}$$

Hence, the function $\hat{y}$ is a uniform limit on O of a sequence of polynomials. By Montel's theorem, $\hat{y}$ is an analytic function on O. Therefore, O is an analytic set for T; so, the implication (iv)⇒(i) holds. □

The following result gives a complete description of $B_a(T)$; it is a simple but useful result from which one can derive many known results as immediate consequences.

Theorem 2.2. *The following identity hold: $B_a(T)=\overline{\Re(T^*)}$.*

Proof. First of all, note that if $(T-\lambda)^*u=0$ for some $u\in\mathcal{H}$, then for every polynomial p, we have

$$\langle p(T)x,u\rangle=p(\lambda)\langle x,u\rangle. \tag{2.9}$$

Let $\lambda\in\Re(T^*)$; there is a nonzero analytic $\mathcal{H}$-valued function $\phi:\mathcal{V}\to\mathcal{H}$ on some open disc $\mathcal{V}$ centered at λ such that

$$(T-\mu)\phi(\mu)=0 \quad \forall\mu\in\mathcal{V}. \tag{2.10}$$

Using the fact that a nonzero analytic $\mathcal{H}$-valued function has isolated zeros, one can assume that the function ϕ has no zeros in $\mathcal{V}$. Hence, $\mathcal{V}\subset\sigma_p(T^*)=\overline{B(T)}$; and therefore, it follows from (2.9) that

$$k(\overline{\mu})=\frac{\phi(\mu)}{\overline{\langle x,\phi(\mu)\rangle}} \quad \text{for every } \mu\in\mathcal{V}. \tag{2.11}$$

This shows that the function $k:\overline{\mathcal{V}}\to\mathcal{H}$ is continuous. By Proposition 2.1, $\overline{\mathcal{V}}\subset B_a(T)$; and so, $\overline{\Re(T^*)}\subset B_a(T)$.

Conversely, set $O=\overline{B_a(T)}$ and consider the $\mathcal{H}$-valued function defined on O by

$$\phi(\lambda):=k(\overline{\lambda}),\quad \lambda\in O. \tag{2.12}$$

We will show that the function ϕ is analytic on O. Indeed, for every $y\in\mathcal{H}$ and for every $\lambda_0\in O$, we have

$$\begin{aligned}\lim_{\lambda\to\lambda_0}\frac{\langle\phi(\lambda),y\rangle-\langle\phi(\lambda_0),y\rangle}{\lambda-\lambda_0}&=\lim_{\lambda\to\lambda_0}\frac{\langle k(\overline{\lambda}),y\rangle-\langle k(\overline{\lambda_0}),y\rangle}{\lambda-\lambda_0}\\&=\lim_{\lambda\to\lambda_0}\frac{\overline{\hat{y}(\overline{\lambda})}-\overline{\hat{y}(\overline{\lambda_0})}}{\lambda-\lambda_0}\\&=\overline{\left[\lim_{\lambda\to\lambda_0}\frac{\hat{y}(\overline{\lambda})-\hat{y}(\overline{\lambda_0})}{\overline{\lambda}-\overline{\lambda_0}}\right]}\\&=\overline{\hat{y}'(\overline{\lambda_0})}.\end{aligned} \tag{2.13}$$

Hence, for every $y\in\mathcal{H}$, the function $\lambda\mapsto\langle\phi(\lambda),y\rangle$ is analytic on O; therefore, the function ϕ is analytic on O. On the other hand, the function ϕ is without zeros on O and satisfies the following equation:

$$(T^*-\lambda)\phi(\lambda)=0\quad\text{for every }\lambda\in O. \tag{2.14}$$

This gives $O=\overline{B_a(T)}\subset\mathfrak{R}(T^*)$, and the proof is completed. □

Corollary 2.3. *The following identities hold:*

$$\begin{aligned}B_a(T)&=\{\lambda\in B(T):\sigma_{T^*-\overline{\lambda}}(k(\lambda))=\varnothing\}\\&=\{\lambda\in B(T):\sigma_{T^*}(k(\lambda))=\varnothing\}.\end{aligned} \tag{2.15}$$

Proof. Since for every $\lambda\in B_a(T)$, $\overline{\lambda}$ is a simple eigenvalue for T^* with corresponding eigenvector $k(\lambda)$, the proof follows by combining Theorem 2.2 and [1, Theorem 1.9]. □

Remark 2.4. In view of Theorem 2.2, the following are immediate consequences:

(i) $B_a(T)$ is independent of the choice of cyclic vector for T (see [15, Proposition 1.4]);

(ii) $B_a(T)=\varnothing$ if and only if T^* has the single-valued extension property. In particular, if T is a cyclic normal operator, then $B_a(T)=\varnothing$.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Recall that the two operators $R\in\mathcal{L}(\mathcal{X})$ and $S\in\mathcal{L}(\mathcal{Y})$ are said to be *densely similar* (*quasisimilar*) if there exist two bounded linear transformations $X:\mathcal{X}\to\mathcal{Y}$ and $Y:\mathcal{Y}\to\mathcal{X}$ having dense range (having dense range and injectives) such that

$$XR=SX,\qquad RY=YS. \tag{2.16}$$

In 1982, Raphael showed that quasisimilar cyclic subnormal operators have the same analytic bounded point evaluations (see [12]). In 1994, Williams proved that general quasisimilar cyclic Hilbert-space operators have the same analytic bounded point evaluations (see [15, Theorem 1.5]). In view of Theorem 2.2, one can see immediately that general densely similar cyclic Hilbert-space operators have the same analytic bounded point evaluations.

To end this section, we will be mainly concerned with two interesting open problems related to the bounded point evaluations for cyclic hyponormal operators. Recall that an operator $R \in \mathcal{L}(\mathcal{H})$ is said to be *subnormal* if it has a normal extension. The operator R is said to be *hyponormal* if $\|R^*y\| \le \|Ry\|$ for every $y \in \mathcal{H}$. Note that every subnormal operator is hyponormal with converse false (see [6]). Recall also that the operator R is said to be *pure* if $\{0\}$ is the only reducing subspace M such that $R_{|M}$ is normal.

Combining Theorem 2.2 and [6, Theorem VIII. 4.3], we see that the cyclic operator T is normal if and only if T is a subnormal operator and T^* has the single-valued extension property. So, one may ask if this result remains valid for noncyclic subnormal cases. Unfortunately, this result is no longer valid; an example of a nonnormal, decomposable, subnormal operator is constructed by Radjabalipour (see [11]). However, we do not know if a similar result remains valid for the case of cyclic hyponormal operators; this suggests the following question.

Question 2.5. Suppose that T is a cyclic hyponormal operator and T^* has the single-valued extension property. Is T a normal operator?

The next problem is of some interest in view of the fact that if it has a positive answer, then one can deduce immediately that every hyponormal operator has a proper closed invariant subspace.

Question 2.6. Suppose that T is a pure cyclic hyponormal operator. Do we have that $B_a(T) \neq \emptyset$?

3. Densely similarity and approximate point spectra for cyclic operators possessing Bishop's property (β). We first need to give some notations and definitions. Let $\mathcal{X}$ be a Banach space; recall that an operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *semi-Fredholm* if $\operatorname{ran} T$ is closed and $\dim(\ker T) < +\infty$ or $\operatorname{codim}(\operatorname{ran} T) < +\infty$. Moreover, if $\operatorname{ran} T$ is closed and both $\dim(\ker T)$ and $\operatorname{codim}(\operatorname{ran} T)$ are finite, then the operator T is said to be *Fredholm*. If T is semi-Fredholm, then the *index* of T is defined by $\operatorname{ind}(T) := \dim(\ker T) - \operatorname{codim}(\operatorname{ran} T)$. For an operator $T \in \mathcal{L}(\mathcal{X})$, define

$$\sigma_{lre}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\},$$
$$\rho_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is Fredholm}\}.$$

These are called the *Wolf spectrum* and *Fredholm domain*, respectively, of T.

Let $T \in \mathcal{L}(\mathcal{H})$ be a cyclic operator on a Hilbert space $\mathcal{H}$. It is shown in [3] that if T possesses Bishop's property (β), then $B_a(T) = \Gamma(T) \backslash \sigma_{\mathrm{ap}}(T)$ if and only if

$B_a(T)\cap\sigma_p(T) = \varnothing$ and was derived from this result that if T is hyponormal, M-hyponormal, or p-hyponormal operator, then $B_a(T) = \Gamma(T)\backslash\sigma_{\mathrm{ap}}(T)$ (see also [2]). However, using generalized spectral theory, it is proved in [8] that $B_a(T)\backslash\sigma_{lre}(T) = \Gamma(T)\backslash\sigma_{\mathrm{g}}(T)$, where $\sigma_{\mathrm{g}}(T)$ denotes the generalized spectrum of T. Therefore, the localized version of Bishop's property (β) allowed to show that

$$B_a(T)\backslash\sigma_\beta(T) = \Gamma(T)\backslash\sigma_{\mathrm{ap}}(T), \tag{3.1}$$

where $\sigma_\beta(T)$ is the set of points $\lambda\in\mathbb{C}$ on which T fails to have Bishop's property (β). As a consequence, it is obtained that if T possesses Bishop's property (β), then $B_a(T) = \Gamma(T)\backslash\sigma_{\mathrm{ap}}(T)$. In view of Theorem 2.2, one may ask whether a similar description of $\mathfrak{R}(T^*)$ can be obtained if T is a cyclic Banach-space operator possessing Bishop's property (β). In fact, we will prove that $\mathfrak{R}(T^*) = \sigma(T)\backslash\sigma_{\mathrm{ap}}(T)$ for every cyclic Banach-space operator T possessing Bishop's property (β). The idea behind a part of our proof comes from the proof of [2, Theorem 4.1].

Theorem 3.1. *Suppose that $T\in\mathcal{L}(\mathcal{X})$ is a cyclic operator on a Banach space $\mathcal{X}$. If T possesses Bishop's property (β), then*

$$\mathfrak{R}(T^*) = \sigma(T)\backslash\sigma_{\mathrm{ap}}(T). \tag{3.2}$$

Proof. A straightforward proof of [6, Lemma II.7.8] shows that

$$\sigma(T)\backslash\sigma_{\mathrm{ap}}(T)\subset\mathfrak{R}(T^*). \tag{3.3}$$

In order to prove the reverse inclusion of (3.3), it suffices to show that $T-\lambda$ is injective and has a closed range for every $\lambda\in\mathfrak{R}(T^*)$.

First, we prove that for every $\lambda\in\mathfrak{R}(T^*)$, we have $\operatorname{codim}(\operatorname{ran}(T-\lambda)) = 1$; in particular, $\operatorname{ran}(T-\lambda)$ is closed (see [7, Lemma 3.1.2]). Indeed, let $\lambda_0\in\mathfrak{R}(T^*)$; there is an analytic function without zeros, $\Lambda:\mathcal{V}\to\mathcal{X}^*$, on some open disc $\mathcal{V}$ centered at λ_0 such that

$$(T^*-\lambda)\Lambda(\lambda) = 0 \quad \forall\lambda\in\mathcal{V}. \tag{3.4}$$

Set $\Phi(\lambda) = \Lambda(\lambda)/\langle x,\Lambda(\lambda)\rangle$, $\lambda\in\mathcal{V}$, where $x\in\mathcal{X}$ is a cyclic vector for T, and the symbol $\langle\cdot,\cdot\rangle$ designs the duality map between $\mathcal{X}$ and $\mathcal{X}^*$. Note that for every polynomial p, we have

$$p(\lambda) = \langle p(T)x,\Phi(\lambda)\rangle \quad \forall\lambda\in\mathcal{V}. \tag{3.5}$$

Let $y \in \mathscr{X}$; there is a sequence of polynomials $(p_n)_{n\geq 0}$ such that $(p_n(T)x)_{n\geq 0}$ converges to y in $\mathscr{X}$. Define analytic $\mathscr{X}$-valued functions on $\mathscr{V}$ by

$$f(\lambda) := y - \langle y, \Phi(\lambda)\rangle x, \qquad f_n(\lambda) := p_n(T)x - p_n(\lambda)x \quad (n \geq 0). \tag{3.6}$$

For every compact subset K of $\mathscr{V}$, we have

$$\begin{aligned} \sup_{\lambda\in K} \|f_n(\lambda) - f(\lambda)\| &\leq \|p_n(T)x - y\| + \sup_{\lambda\in K}\|[p_n(\lambda) - \langle y, \Phi(\lambda)\rangle]x\| \\ &\leq \|p_n(T)x - y\| + \|x\| \sup_{\lambda\in K} |p_n(\lambda) - \langle y, \Phi(\lambda)\rangle| \\ &\leq \Big[1 + \|x\| \sup_{\lambda\in K}\|\Phi(\lambda)\|\Big]\|p_n(T)x - y\|. \end{aligned} \tag{3.7}$$

Therefore, $f_n \to f$ in $\mathbb{O}(\mathscr{V},\mathscr{X})$. As $\operatorname{ran}(T_{\mathscr{V}})$ is closed and each $f_n \in \operatorname{ran}(T_{\mathscr{V}})$, there is $g \in \mathbb{O}(\mathscr{V},\mathscr{X})$ such that $f(\lambda) = (T-\lambda)g(\lambda)$ for all $\lambda \in \mathscr{V}$. In particular, we have

$$y = (T-\lambda_0)g(\lambda_0) + \langle y, \Phi(\lambda_0)\rangle x. \tag{3.8}$$

This shows that $\operatorname{codim}(\operatorname{ran}(T-\lambda_0)) = 1$.

Next, suppose for the sake of contradiction that there is $\lambda_0 \in \Re(T^*)$ such that $T-\lambda_0$ is not injective. As $\operatorname{codim}(\operatorname{ran}(T-\lambda_0)) = 1$, $T-\lambda_0$ is a semi-Fredholm operator with

$$\operatorname{ind}(T-\lambda_0) = \dim(\ker(T-\lambda_0)) - \operatorname{codim}(\operatorname{ran}(T-\lambda_0)) \geq 0. \tag{3.9}$$

By [1, Corollary 2.7], we deduce that $\operatorname{ind}(T-\lambda_0) = 0$; and so, $T-\lambda_0$ is a Fredholm operator for which $\dim(\ker(T-\lambda_0)) = \operatorname{codim}(\operatorname{ran}(T-\lambda_0)) = 1$. Since $\rho_e(T)$ is an open set and the index is a constant function on the components of $\rho_e(T)$, there is $\delta > 0$ such that

$$\begin{aligned} B(\lambda_0,\delta) := \{\lambda \in \mathbb{C} : |\lambda-\lambda_0| < \delta\} \subset \Re(T^*) \cap \rho_e(T), \\ \operatorname{ind}(T-\lambda) = \operatorname{ind}(T-\lambda_0) = 0 \quad \text{for every } \lambda \in B(\lambda_0,\delta). \end{aligned} \tag{3.10}$$

As $\operatorname{codim}(\operatorname{ran}(T-\lambda)) = 1$, for every $\lambda \in \Re(T^*)$, we have

$$\dim(\ker(T-\lambda)) = 1 \quad \text{for every } \lambda \in B(\lambda_0,\delta); \tag{3.11}$$

in particular, $B(\lambda_0,\delta) \subset \sigma_p(T)$. We have a contradiction to [1, Theorem 2.6] since T has the single-valued extension property. Thus, $T-\lambda_0$ is injective, and the proof is completed. □

Note that (3.3) holds for arbitrary Banach-space operator T not necessarily cyclic. The following example shows that this inclusion may not be reversed in general even if the operator T possesses Bishop's property (β).

Example 3.2. Let $(e_n)_{n\geq 0}$ be the canonical basis of the Banach space $\mathscr{X} = l^1$ of all complex sequences $x := (x_n)_{n\geq 0}$ such that $\|x\| = \sum_{n\geq 0} |x_n| < +\infty$. Let

$$Ue_n = e_{n+1} \quad (n \geq 0) \tag{3.12}$$

be the unweighted unilateral shift on $\mathscr{X}$ and let $T = U \oplus 2U$. Note that T possesses Bishop's property (β) since U is an isometry [7, Proposition 1.6.7]. On the other hand, it follows from [13] that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 2\}$, $\sigma_{\text{ap}}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 2\}$, and $\mathfrak{R}(T^*) = \{\lambda \in \mathbb{C} : |\lambda| < 2\}$. This shows that $\sigma(T) \backslash \sigma_{\text{ap}}(T) \subsetneqq \mathfrak{R}(T^*)$.

Note that in general two densely similar Banach-space operators which possess Bishop's property (β) have equal spectra, compression spectra, and essential spectra (see, e.g., [7]), but may have unequal approximate point spectra as shown by an example in [4]. Here, we show that general densely similar cyclic Banach-space operators possessing Bishop's property (β) have the same approximate point spectra.

Proposition 3.3. *Two densely similar cyclic operators possessing Bishop's property (β) have equal approximate point spectra.*

Before proving this result, we need the following lemma. This is a special case of much more general results from [7]; we give here a direct proof.

Lemma 3.4. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces. Let $R \in \mathscr{L}(\mathscr{X})$, $S \in \mathscr{L}(\mathscr{Y})$, and let $X : \mathscr{X} \to \mathscr{Y}$ be a bounded linear transformation with dense range such that $XR = SX$. If S satisfies Dunford's condition (C), then $\sigma(S) \subset \sigma(R)$; in particular, if both R and S satisfy Dunford's condition (C) and are densely similar, then $\sigma(S) = \sigma(R)$.*

Proof. We will first show that $\sigma_S(Xy) \subset \sigma_R(y)$ for every $y \in \mathscr{X}$. Indeed, let $y \in \mathscr{X}$. If $Xy = 0$, then clearly $\sigma_S(Xy) = \varnothing \subset \sigma_R(y)$. Thus, we may suppose that $Xy \neq 0$. Let $\lambda_0 \in \rho_R(y)$; so, there is an open neighborhood $\mathscr{V}$ of λ_0 and a nonzero analytic $\mathscr{X}$-valued function $\phi : \mathscr{V} \to \mathscr{X}$ such that

$$(R - \lambda)\phi(\lambda) = y \quad \text{for every } \lambda \in \mathscr{V}. \tag{3.13}$$

Since $X(R - \lambda) = (S - \lambda)X$ for every $\lambda \in \mathbb{C}$, we have

$$(S - \lambda)X\phi(\lambda) = Xy \quad \text{for every } \lambda \in \mathscr{V}. \tag{3.14}$$

It is clear that the analytic $\mathscr{Y}$-valued function $X\phi : \mathscr{V} \to \mathscr{Y}$ is without zeros since $Xy \neq 0$. Hence, $\mathscr{V} \subset \rho_S(Xy)$; thus, $\sigma_S(Xy) \subset \sigma_R(y)$ for every $y \in \mathscr{X}$. Since $\mathscr{Y}_S(\sigma(R))$ is a closed linear subspace and X has dense range, we have

$\mathcal{Y}_S(\sigma(R)) = \mathcal{Y}$. This shows that $\sigma_S(y) \subset \sigma(R)$ for every $y \in \mathcal{Y}$. As $\sigma(S) = \bigcup_{y\in\mathcal{Y}} \sigma_S(y)$ (see [7, Proposition 1.3.2]), we have $\sigma(S) \subset \sigma(R)$, and the proof is completed. □

Proof of Proposition 3.3. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and let $T_1 \in \mathcal{L}(\mathcal{X})$ and $T_2 \in \mathcal{L}(\mathcal{Y})$ be two densely similar cyclic operators possessing Bishop's property (β). We have $\mathfrak{R}(T_1^*) = \mathfrak{R}(T_2^*)$; so, in view of Theorem 3.1, we have

$$\sigma(T_1)\backslash\sigma_{\text{ap}}(T_1) = \sigma(T_2)\backslash\sigma_{\text{ap}}(T_2). \tag{3.15}$$

As $\sigma(T_1) = \sigma(T_2)$ (see Lemma 3.4), we have $\sigma_{\text{ap}}(T_1) = \sigma_{\text{ap}}(T_2)$. □

We conclude this paper by mentioning that one can show with no extra effort that Theorem 3.1 and Proposition 3.3 remain valid for rationally cyclic Banach-space operators.

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A. Bourhim: The Abdus Salam International Centre for Theoretical Physics, Mathematics Section, Strada Costiera 11, 34100 Trieste, Italy
E-mail address: bourhim@ictp.trieste.it