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ON A CERTAIN CLASS OF NONSTATIONARY SEQUENCES IN HILBERT SPACE

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To my Professor A. A. Yansevitch

We study the functions of correlation $K(n,m) = \langle X(n), X(m)\rangle$ of certain sequences: $X(n) = T^n x_0$, $x_0 \in H$ where T is a contraction in Hilbert space H. By using the spectral methods of the nonunitary operators, we give the general form of $K(n,m)$ and its asymptotic behaviour $\lim_{p\to+\infty} \mathbf{K}(n+p, m+p)$.

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1. Introduction. Let $X(n)$ $(n \in A = IN$ or $Z)$ be a sequence of elements of a separable Hilbert space H. The function of correlation of $X(n)$ is given by formula

$$K(n,m) = \langle X(n), X(m)\rangle. \tag{1.1}$$

If the function of correlation depends only on the difference of arguments, that is, $K(n,m) = K(n-m)$, one calls that $X(n)$ is stationary. Kolmogorov (see [4]) showed that if $X(n)$ is stationary and $A = Z$, then

$$X(n) = U^n x_0, \quad x_0 = X(0), \tag{1.2}$$

where U is a unitary operator acting in the subspace H_X which is defined as the closed linear envelope of $X = \{X(n);\ n \in Z\}$. This representation as well as the spectral theory of the monoparametric groups of unitary operators allowed to find the general form of the function $K(n,m)$ in the stationary case. More exactly, one has (see [4])

$$K(n,m) = \int_{-\pi}^{+\pi} e^{i(n-m)\lambda}\, dF_X(\lambda), \tag{1.3}$$

where F_X is real function, continuous on the left and nondecreasing on $[-\pi; +\pi]$ such that $F_X(-\pi) = 0$. This function is called spectral function of $X(n)$.

In this paper, we are interested in some sequences of the form

$$X(n) = T^n x_0, \quad x_0 \in H, \tag{1.4}$$

where T is a linear contraction ($\|T\| \le 1$) in H. Such sequences are called linearly representable and were introduced by Yansevitch [8, 9]. They represent a natural generalization of the sequences of the form (1.2). But they were especially introduced like the analogue of the processes of the form

$$Y(t) = e^{itA}y_0, \tag{1.5}$$

where A is a dissipative ($(A-A^*)/i \ge 0$) operator in H. The correlation theory of these processes constituted a remarkable field of application for the spectral theory of nonselfadjoint operators [2, 3, 5, 10].

Necessary and sufficient criteria in terms of function of correlation for linear representability (1.4) are established by the following theorem [8].

Theorem 1.1. *A given function $K(n,m)$ is the function of correlation of a certain sequence $X(n) = T^n x_0$ if and only if there exists a constant C $(0 \prec C \prec +\infty)$ such that*

$$\begin{gathered} \sum_{n,m=0}^{N} K(n,m)\lambda_n\overline{\lambda}_m \ge 0, \\ \left| \sum_{n=0}^{N}\sum_{m=0}^{M} \big(K(n+1,m)-K(n,m)\big)\lambda_n\overline{\mu}_m \right|^2 \\ \le C \cdot \sum_{n,p=0}^{N} K(n,p)\lambda_n\overline{\lambda}_p \cdot \sum_{m,q=0}^{M} K(m,q)\mu_m\overline{\mu}_q \end{gathered} \tag{1.6}$$

for every $(\lambda_n)_{n=0}^{N}$ and $(\mu_m)_{m=0}^{M}$ in the field of complex numbers.

Definition 1.2. Let $X(n) = T^n x_0$ be a linearly representable sequence. The difference of correlation of $X(n)$ is the function

$$W(n,m) = K(n,m) - K(n+1,m+1). \tag{1.7}$$

It is clear that in the stationary case, $W(n,m) = 0$.

Formula (1.7) implies that, for every natural $p \ge 1$,

$$K(n,m) = K(n+p,m+p) + \sum_{j=0}^{p-1} W(n+j,m+j), \tag{1.8}$$

what gives, for $p \to +\infty$,

$$K(n,m) = \lim_{p\to+\infty} K(n+p,m+p) + \sum_{j=0}^{+\infty} W(n+j,m+j). \tag{1.9}$$

Furthermore, since $(I-T^*T)$ is selfadjoint, then

$$(I-T^*T)=\sum_{k=1}^{r}\langle\cdot;g_k\rangle\cdot g_k,\quad g_k\in(I-T^*T)H,\ r=\dim(I-T^*T)H. \tag{1.10}$$

Hence,

$$\begin{aligned}
K(n,m)&=\lim_{p\to+\infty}K(n+p,m+p)+\sum_{j=0}^{+\infty}W(n+j,m+j)\\
&=\lim_{p\to+\infty}K(n+p,m+p)+\sum_{j=0}^{+\infty}\langle(I-T^*T)X(n+j);X(m+j)\rangle\\
&=\lim_{p\to+\infty}K(n+p,m+p)+\sum_{j=0}^{+\infty}\sum_{k=1}^{r}\langle X(n+j);g_k\rangle\cdot\langle g_k;X(m+j)\rangle\\
&=\lim_{p\to+\infty}K(n+p,m+p)+\sum_{j=0}^{+\infty}\sum_{k=1}^{r}\Phi_k(n+j)\cdot\overline{\Phi_k(m+j)},\\
\Phi_k(n)&=\langle X(n);g_k\rangle=\langle T^n x_0;g_k\rangle.
\end{aligned} \tag{1.11}$$

Consequently, the study of linearly representable sequences can be carried out in two stages.

(a) To find the limit $\lim_{p\to+\infty}K(n+p,m+p)$.

(b) To give the explicit expression of the quantity $\Phi_k(n)$.

In [8], the case when $\dim(I-T^*T)H=1$ was considered and the spectrum of T is made up only of eigenvalues $\{\lambda_k\}_{k=1}^{+\infty}$ such that $|\lambda_k|\prec 1$, $k\geq 1$. In this case, one has [8]

$$\begin{gathered}
\lim_{p\to+\infty}K(n+p,m+p)=0,\\
K(n,m)=\sum_{j=0}^{+\infty}\Phi(n+j)\cdot\overline{\Phi(m+j)},\\
\Phi(n)=\sum_{k=1}^{+\infty}f_{0k}\cdot\frac{-1}{2\pi i}\sqrt{1-|\lambda_k|^2}\oint_\Gamma\frac{\lambda^n}{\lambda-\lambda_k}\prod_{j=1}^{k-1}\frac{1-\lambda\cdot\lambda_j}{\lambda-\lambda_j}\cdot\frac{|\lambda_j|}{\overline{\lambda_j}}\cdot d\lambda,\\
\sum_{k=1}^{+\infty}|f_{0k}|^2\prec+\infty,
\end{gathered} \tag{1.12}$$

where Γ is a closed contour containing all the spectrum of T.

Let T be a simple contraction (i.e., there is no invariant for T and T^* subspace in which, T induces a unitary operator) with spectrum $\sigma(T)$ on the circle unit. It is known [1] that there exists an increasing function α on the interval $[0,l]$ $(l\succ 0)$ such that

$$\sigma(T)=\{e^{-i\alpha(x)}:x\in[0,l]\}. \tag{1.13}$$

Definition 1.3. Say that $X(n) = T^n x_0$ belongs to the class $D^{(r)}[\alpha]$ if T is a contraction such that (1.13) holds and $\dim(I - T^*T)H \le r$.

Throughout this paper, we will suppose that α is a continuous function. In this case, we will prove the following results.

Theorem 1.4. *Let $T^n x_0$ be an element of class $D^{(r)}[\alpha]$. Assume that T is simple. Then,*

$$K(n,m) = \widetilde{\mathbf{K}}_\infty(n-m) + F(n-m) + \sum_{j=0}^{+\infty}\sum_{k=1}^{r} \Phi_k(n+j)\cdot\overline{\Phi_k(m+j)},$$

$$\Phi_k(n) = -\frac{1}{2\pi i}\oint_\Gamma \lambda^n \cdot \left\{ \frac{\sqrt{2}e^{-x}}{e^{-i\alpha(x)} - \lambda} \int_0^l \Psi_{0k}(x)\cdot e^{2\int_0^x (e^{-i\alpha(t)}/e^{-i\alpha(t)} - \lambda)dt} dx \right\} d\lambda, \tag{1.14}$$

where $\Psi_{0k} \in L^2_{[0;l]}$, Γ is any closed contour containing all the spectrum of T, $F(n-m)$ is a Hermitian nonnegative function which equals zero in the case when $\dim(I - T^*T)H = 1$, and $\widetilde{\mathbf{K}}_\infty(n-m)$ is defined by the spectrum of T. Moreover, if T has a singular spectrum or the measurement of the intersection of its spectrum with the unit circle is null, then $\widetilde{\mathbf{K}}_\infty(n-m) = F(n-m) = 0$.

Theorem 1.5. *If a function $K(n,m)$ admits the representation (1.14), then there exists a linearly representable sequence $X(n) = T^n x_0$ such that $X(n) \in D^{(r)}[\alpha]$ and the function of correlation of $X(n)$ equals $K(n,m)$.*

Throughout this paper, H is a separable Hilbert space and $\oplus$ denotes orthogonal sum.

2. On the structure of $\lim_{p\to+\infty}\mathbf{K}(n+p, m+p)$

Proposition 2.1. *If T is a contraction in H, then the sequence $A_n = T^{*n}T^n$ admits a positive strong limit $R = s\cdot\lim_{n\to+\infty} T^{*n}T^n$ which verifies the relation*

$$T^{*n}RT^m = RT^{m-n} \quad (n \ge m). \tag{2.1}$$

Moreover, if T is invertible, then

$$T^{*n}R = RT^{-n}. \tag{2.2}$$

Proof. The existence and positivity of R are a consequence of the fact that the sequence A_n is a decreasing and bounded sequence of positive operators. Formulas (2.1) and (2.2) are verified easily. □

Corollary 2.2. *If $X(n) = T^n x_0 \in D^{(r)}[\alpha]$, then*

$$\lim_{p\to+\infty}\mathbf{K}(n+p, m+p) = \mathbf{K}_\infty(n-m) = \langle RT^{n-m}x_0, x_0\rangle,$$
$$\lim_{p\to+\infty}\mathbf{W}(n+p, m+p) = \mathbf{0}. \tag{2.3}$$

Consider now the sequence

$$\hat{\Psi}(x,n) = \hat{T}^n\Psi_0(x), \quad \Psi_0 \in L^2_{[0;l]} \ (l \prec \infty), \tag{2.4}$$

$$(\hat{T}f)(x) = e^{-i\alpha(x)}f(x) - 2e^{-i\alpha(x)+x}\int_x^l e^{-t}f(t)\,dt. \tag{2.5}$$

A direct calculation shows that $\sigma(T) = \{e^{-i\alpha(x)} : x \in [0,l]\}$ and

$$(I - \hat{T}^*\hat{T}) = \langle \cdot ; g\rangle \cdot g \quad (g(x) = \sqrt{2}e^{-x}). \tag{2.6}$$

Hence, the sequence $\hat{\Psi}(x,n)$ is an element of the class $D^{(1)}[\alpha]$.

For every $u \in [0,l]$, let

$$L^2_{[u;l]} = \{f \in L^2_{[0;l]} : f(x) = 0 \text{ for } x \in [0,l]\}. \tag{2.7}$$

Let also P_u be the orthoprojector of $L^2_{[0;l]}$ on $L^2_{[u;l]}$.

Proposition 2.3. *The sequence $A_n(u) = T^{*n}P_uT^n$ admits a positive strong limit R_u which verifies the relation $T^{*n}R_uT^m = R_uT^{m-n}$ $(n \ge m)$. Moreover, if T is invertible, then $T^{*n}R_u = R_uT^{-n}$.*

Pose that

$$\begin{gathered} L_0(n,u) = \langle P_u(\hat{\Psi}(x,n)), \hat{\Psi}(x,n)\rangle = \int_u^l |\hat{\Psi}(t,n)|^2\,dt, \\ \hat{K}(n,m,u) = \langle P_u(\hat{\Psi}(x,n)), \hat{\Psi}(x,m)\rangle, \\ \widehat{W}(n,m,u) = \hat{K}(n,m,u) - \hat{K}(n+1,m+1,u) = y(u,n)\cdot\overline{y(u,m)}, \\ y(u,n) = \sqrt{2}e^u\int_u^l e^{-t}\cdot\hat{\Psi}(t,n)\,dt. \end{gathered} \tag{2.8}$$

For $n \ge m$,

$$\begin{aligned} \hat{K}(n,m,u) &= \hat{K}(n-m,0,u) - \sum_{j=1}^{m}\widehat{W}(n-j,m-j,u) \\ &= \hat{K}(n-m,0,u) - \sum_{j=0}^{m-1}\widehat{W}(n-m+j,j,u). \end{aligned} \tag{2.9}$$

Thus, for $p \ge 1$,

$$\hat{K}(n+p,m+p,u) = \hat{K}(n-m,0,u) - \sum_{j=0}^{m+p-1}\widehat{W}(n-m+j,j,u). \tag{2.10}$$

Let $\tau = n-m$ and $\widehat{\mathbf{K}}_\infty(n-m,u) = \lim_{p\to+\infty}\widehat{\mathbf{K}}(n+p,m+p,u)$. Then,

$$\widehat{\mathbf{K}}_\infty(n-m) = \widehat{K}(\tau,0,u) - \sum_{j=0}^{+\infty} y(u,\tau+j)\overline{y(u,j)}. \tag{2.11}$$

Let

$$\mathbf{L}_p(\tau,u) = \widehat{K}(\tau,0,u) - \sum_{j=0}^{p-1} y(u,\tau+j)\overline{y(u,j)} \quad (p \geq 1). \tag{2.12}$$

Then,

$$\begin{aligned}
\widehat{\mathbf{K}}_\infty(\tau,u) &= \lim_{p\to+\infty}\mathbf{L}_p(\tau,u),\\
\mathbf{L}_p(0,u) = \widehat{K}(0,0,u) - \sum_{j=0}^{p-1} y(u,j)\overline{y(u,j)} &= \widehat{K}(p,p,u) = L_0(p,u),\\
K_0(u) = \lim_{p\to+\infty}\mathbf{L}_p(0,u) &= \lim_{p\to+\infty}\mathbf{L}_0(p,u) = \langle R_u\Psi_0,\Psi_0\rangle.
\end{aligned} \tag{2.13}$$

Theorem 2.4. *The function $\widehat{\mathbf{K}}_\infty(\tau,u)$ admits the following representation:*

$$\widehat{\mathbf{K}}_\infty(\tau,u) = -\int_u^l e^{i\tau\alpha(x)}\,dK_0(x). \tag{2.14}$$

In particular,

$$\widehat{\mathbf{K}}_\infty(n-m) = -\int_0^l e^{i(n-m)\alpha(x)}\,dK_0(x). \tag{2.15}$$

Proof. Remark that K_0 is a decreasing function. Thus integrals in (2.14) and (2.15) exist. A direct but long calculation makes it possible to affirm that

$$\begin{aligned}
\frac{d}{du}\left(\mathbf{L}_p(\tau+1,u)\right) = e^{i\alpha(u)}\left(\frac{d}{du}\mathbf{L}_p(\tau,u)\right) &+ \sqrt{2}y(u,\tau+p)\overline{y(u,p-1)}\\
&+ 2y(u,\tau+p)\overline{\Psi(u,p-1)}.
\end{aligned} \tag{2.16}$$

Hence,

$$\begin{aligned}
\frac{d}{du}\left(\mathbf{L}_p(\tau,u)\right) = e^{i\tau\alpha(u)}&\left(\frac{d}{du}\mathbf{L}_0(p,u)\right)\\
&+ \sqrt{2}\,\overline{y(u,p-1)}\sum_{j=1}^{\tau} e^{i(1-j)\alpha(u)}\cdot y(u,\tau+p-j)\\
&+ 2\overline{\Psi(u,p-1)}\sum_{j=1}^{\tau} e^{i(1-j)\alpha(u)}\cdot y(u,\tau+p-j).
\end{aligned} \tag{2.17}$$

Let

$$
\begin{aligned}
I_1 &= -\int_u^l e^{i\tau\alpha(x)}\left(\frac{d}{dx}\mathbf{L}_0(p,x)\right)dx,\\
I_2 &= -\sqrt{2}\int_u^l \overline{y(x,p-1)}\sum_{j=1}^{\tau} e^{i(1-j)\alpha(x)}\cdot y(x,\tau+p-j)\,dx,\\
I_3 &= -2\int_u^l \overline{\Psi(x,p-1)}\sum_{j=1}^{\tau} e^{i(1-j)\alpha(x)}\cdot y(x,\tau+p-j)\,dx,
\end{aligned}
\tag{2.18}
$$

then

$$
\mathbf{L}_p(\tau,u) = I_1 + I_2 + I_3. \tag{2.19}
$$

By using the theorem of Lebesgue about dominated convergence, one can show that $I_2 = I_3 = 0$. Thus,

$$
\mathbf{L}_p(\tau,u) = -\int_u^l e^{i\tau\alpha(x)}\,d\big(\mathbf{L}_0(p,x)\big). \tag{2.20}
$$

Furthermore,

$$
\mathbf{L}_0(p,x) = \int_x^l \left|\hat{\Psi}(t,p)\right|^2 dt \tag{2.21}
$$

is an absolutely continuous function in x. Moreover, since operator $\hat{T}$ is a contraction, then

$$
\mathbf{L}_0(p,x) = \int_x^l \left|\hat{\Psi}(t,p)\right|^2 dt \le \int_0^l \left|\hat{\Psi}(t,p)\right|^2 dt \le \|\hat{\Psi}_0\|^2. \tag{2.22}
$$

That means that the sequence V_p $(p \ge 1)$ of total variation of $\mathbf{L}_0(p,x)$ on $[0,l]$ is bounded. Moreover, function $e^{i\tau\alpha(x)}$ is continuous. Thus,

$$
\begin{aligned}
\lim_{p\to+\infty}\mathbf{L}_p(\tau,u) &= -\int_u^l e^{i\tau\alpha(x)}\,d\Big(\lim_{p\to+\infty}\mathbf{L}_0(p,x)\Big)\\
&= -\int_u^l e^{i\tau\alpha(x)}\,d\mathbf{K}_0(x).
\end{aligned}
\tag{2.23}
$$

□

It is known [6] that if the $X(n) = T^n x_0 \in D^{(1)}[\alpha]$ and T is simple, then $T = U^{-1}\hat{T}U$ where U is a unitary operator from H into $L^2_{[0;l]}$. Hence, from Theorem 2.4, the following theorem follows.

Theorem 2.5. *Let $X(n) = T^n x_0 \in D^{(1)}[\alpha]$. Suppose that T is simple. Then, there exists an increasing function β on $[0,l]$ such that*

$$
\mathbf{K}_\infty(n-m) = \lim_{p\to+\infty}\mathbf{K}(n+p,m+p) = \int_0^l e^{i(n-m)\alpha(x)}\,d\beta(x). \tag{2.24}
$$

Consider now the space $L_2^r = L^2_{[0;l]} \oplus \cdots \oplus L^2_{[0;l]}$ (r times), with scalar product:

$$\langle f;g\rangle_r = \sum_{j=1}^{r} \int_0^l f_j(x)\cdot \overline{g_j(x)}dx, \quad f = (f_1,\dots,f_r),\ g = (g_1,\dots,g_r). \tag{2.25}$$

In this space, define the operator $\overline{T}(r) = \hat{T} \oplus \cdots \oplus \hat{T}$ as follows:

$$(\overline{T}(r))(f_1,\dots,f_r) = (\hat{T}f_1,\dots,\hat{T}f_r). \tag{2.26}$$

Every sequence of the form

$$\begin{gathered}\widetilde{\Psi}(x,n) = (\overline{T}(r))^n(\widetilde{\Psi}_0(x)) = (\hat{T}^n(\widetilde{\Psi}_{01}),\dots,\hat{T}^n(\widetilde{\Psi}_{0r})),\\ \widetilde{\Psi}_0 = (\widetilde{\Psi}_{01},\dots,\widetilde{\Psi}_{0r}) \in L_2^r,\end{gathered} \tag{2.27}$$

is an element of class $D^{(r)}[\alpha]$ (see [1]). The following relations hold immediately:

$$\begin{gathered}\widetilde{\mathbf{K}}_\infty(n-m) = \lim_{p\to+\infty} \widetilde{\mathbf{K}}(n+p,m+p) = \sum_{j=1}^{r} \hat{\mathbf{K}}_\infty^{(j)}(n-m),\\ \hat{\mathbf{K}}_\infty^{(j)}(n-m) = -\int_0^l e^{i(n-m)\alpha(x)}dK_0^{(j)}(x),\\ K_0^{(j)}(x) = \langle R_x\Psi_{0j},\Psi_{0j}\rangle, \quad (j=1,\dots,r).\end{gathered} \tag{2.28}$$

Theorem 2.6. *Let $X(n) = T^n x_0 \in D^{(r)}[\alpha]$. Suppose that T is simple. Then, there exists r increasing functions $\{\beta_j\}_{j=1}^r$ on $[0,l]$ and there exists a Hermitian nonnegative function $F(n-m)$ such that*

$$\mathbf{K}_\infty(n-m) = \sum_{j=1}^{r}\int_0^l e^{i(n-m)\alpha(x)}d\beta_j(x) + F(n-m). \tag{2.29}$$

Proof. According to [1], there exists a unitary operator B defined in a Hilbert space M such that operator T is unitarily equivalent to the restriction of operator $\overline{B}(r) = \overline{T}(r)\oplus B$ on a certain invariant subspace $\Theta \subset L_2^r \oplus M$, that is, $T = U^{-1}\overline{B}(r)U$ where U is a unitary operator from H into Θ. Thus,

$$\begin{aligned}K(n+p,m+p) &= \langle T^{n+p}x_0;T^{m+p}x_0\rangle\\ &= \langle U^{-1}\overline{B}(r)^{n+p}U(x_0);U^{-1}\overline{B}(r)^{m+p}U(x_0)\rangle\\ &= \langle \overline{B}(r)^{n+p}(f_0);\overline{B}(r)^{m+p}(f_0)\rangle,\end{aligned} \tag{2.30}$$

where $f_0 = U(x_0) = \widetilde{\Psi}_0 + x_M \in \Theta$ ($\widetilde{\Psi}_0 \in L_2^r,\ x_M \in M$).

Since $\overline{B}(r)^n = \overline{T}(r)^n \oplus B^n$, then

$$\begin{aligned} K(n+p,m+p) &= \langle \overline{T}(r)^{n+p}(\widetilde{\Psi}_0);\overline{T}(r)^{m+p}(\widetilde{\Psi}_0)\rangle \\ &\quad + \langle B^{n+p}(x_M);B^{m+p}(x_M)\rangle \\ &= \langle \overline{T}(r)^{n+p}(\widetilde{\Psi}_0);\overline{T}(r)^{m+p}(\widetilde{\Psi}_0)\rangle \\ &\quad + \langle B^{n-m}(x_M);x_M\rangle. \end{aligned} \tag{2.31}$$

Let $F(n-m) = \langle B^{n-m}(x_M);x_M\rangle$. It is clear that function $F(n-m)$ satisfies all conditions of Theorem 2.6. Finally, one has

$$\begin{aligned} \mathbf{K}_\infty(n-m) &= \lim_{p\to+\infty} K(n+p,m+p) \\ &= \lim_{p\to+\infty} \langle (\overline{T}(r))^{n+p}(\widetilde{\Psi}_0);(\overline{T}(r))^{m+p}(\widetilde{\Psi}_0)\rangle + F(n-m) \\ &= \widetilde{\mathbf{K}}_\infty(n-m)+F(n-m). \end{aligned} \tag{2.32}$$

To complete the demonstration, it is enough to notice that

$$\widetilde{\mathbf{K}}_\infty(n-m) = \sum_{j=1}^{r} \widehat{\mathbf{K}}_\infty^{(j)}(n-m) = \sum_{j=1}^{r}\int_0^l e^{i(n-m)\alpha(x)}\,d\beta_j(x). \tag{2.33}$$

□

We now will see two situations where $\mathbf{K}_\infty(n-m) = 0$.

Proposition 2.7. *Let $X(n) = T^n x_0 \in D^{(r)}[\alpha]$. If T is simple and the measurement of the intersection of its spectrum with the circle unit is null, then $\mathbf{K}_\infty(n-m) = 0$.*

Proof. One has

$$\begin{aligned} \left|K(n+p,m+p)\right|^2 &= \left|\langle T^{n+p}x_0;T^{m+p}x_0\rangle\right|^2 \\ &\le \left\|T^{n+p}x_0\right\|^2\cdot\left\|T^{m+p}x_0\right\|^2. \end{aligned} \tag{2.34}$$

Under these assumptions, one has according to [7]

$$\lim_{p\to+\infty}\left\|T^{n+p}x_0\right\|^2 = \lim_{p\to+\infty}\left\|T^{m+p}x_0\right\|^2 = 0. \tag{2.35}$$

□

Theorem 2.8. *Let $T^n x_0$ be an element of class $D^{(r)}[\alpha]$ and let $\sigma(t) = \text{mes}\cdot\{x\in[0,l]:\alpha(x)\prec t\}$, $t\in[\alpha(0);\alpha(l)]$, be the repartition function of α. If T is simple and σ singular, then $\mathbf{K}_\infty(n-m) = 0$.*

Proof. Under these assumptions, operator T is unitarily equivalent to operator $\overline{T}(r)$ (see [1]). Thus, $\mathbf{K}_\infty(n-m) = \widetilde{\mathbf{K}}_\infty(n-m)$. But in this case, the characteristical function $\widetilde{S}(\lambda)$ of operator $\overline{T}(r)$ satisfies the following relations (see [1, 7]):

$$\begin{aligned}\det\widetilde{S}(\lambda) &= \exp\left\{\int_0^l \left(e^{i\alpha(t)}+\lambda\right)\left(e^{i\alpha(t)}-\lambda\right)^{-1}dt\right\}\\ &= \exp\left\{-\int_{\alpha(0)}^{\alpha(l)} \left(e^{it}+\lambda\right)\left(e^{it}-\lambda\right)^{-1}d\sigma(t)\right\}\\ &= \exp\left\{-\int_0^{2\pi} \left(e^{it}+\lambda\right)\left(e^{it}-\lambda\right)^{-1}d\nu(t)\right\},\end{aligned} \tag{2.36}$$

where

$$\nu(t)=\begin{cases}\sigma(t), & t\in[0,l],\\ 0, & t\notin[0,l]\end{cases} \tag{2.37}$$

is a singular function. Thus $\det\widetilde{S}(\lambda)$ is an interior function and according to [1], for every $\widetilde{\Psi}=(\widetilde{\Psi}_1,\dots,\widetilde{\Psi}_r)\in L_2^r$, $\lim_{p\to+\infty}\|T^{n+p}x_0\|^2=0$. By using the same reasoning that in Proposition 2.7, one shows that $\mathbf{K}_\infty(n-m)=\widetilde{\mathbf{K}}_\infty(n-m)=0$. □

3. General form of $\mathbf{K}(n,m)$

Theorem 3.1. *Let $T^n x_0$ be an element of class $D^{(r)}[\alpha]$. Assume that T is simple. Then,*

$$\begin{gathered}K(n,m)=\widetilde{\mathbf{K}}_\infty(n-m)+F(n-m)+\sum_{j=0}^{+\infty}\sum_{k=1}^{r}\Phi_k(n+j)\cdot\overline{\Phi_k(m+j)},\\ \Phi_k(n)=\frac{-1}{2\pi i}\oint_\Gamma\lambda^n\cdot\left\{\frac{\sqrt{2}e^{-x}}{e^{-i\alpha(x)}-\lambda}\int_0^l\Psi_{0k}(x)\cdot e^{2\int_0^x(e^{-i\alpha(t)}/e^{-i\alpha(t)}-\lambda)\,dt}dx\right\}d\lambda,\end{gathered} \tag{3.1}$$

where $\Psi_{0k}\in L^2_{[0;l]}$ and $F(n-m)$ is a Hermitian nonnegative function.

Proof. Using the same reasoning that in Theorem 2.6, one can affirm that

$$K(n,m)=\widetilde{K}(n,m)+F(n-m), \tag{3.2}$$

where $F(n-m)$ satisfies the conditions of Theorem 3.1. According to (1.11),

$$\begin{gathered}\widetilde{K}(n,m)=\lim_{p\to+\infty}K(n+p,m+p)+\sum_{j=0}^{+\infty}\sum_{k=1}^{r}\Phi_k(n+j)\cdot\overline{\Phi_k(m+j)},\\ \Phi_k(n)=\langle(\overline{T}(r))^n(\widetilde{\Psi}_0);g_k\rangle,\qquad (I-(\overline{T}(r))^*(\overline{T}(r)))=\sum_{k=1}^{r}\langle\cdot;g_k\rangle\cdot g_k.\end{gathered} \tag{3.3}$$

One has (see [1])

$$g_k = h_k \cdot e_k, \qquad h_k(x) = \sqrt{2}e^{-x}, \tag{3.4}$$

where e_k $(k = 1,\dots,r)$ is the canonical basic in C^r. Thus,

$$\Phi_k(n) = \langle (\overline{T}(r))^n(\widetilde{\Psi}_0); g_k\rangle = \Phi_k(n) = \langle \hat{T}^n(\widetilde{\Psi}_{0k}); h_k\rangle. \tag{3.5}$$

Since $\hat{T}$ is bounded, then

$$\hat{T}^n = \frac{-1}{2\pi i}\oint_\Gamma \lambda^n(\hat{T} - \lambda\cdot I)^{-1}\,d\lambda, \tag{3.6}$$

where Γ is a closed contour containing all the spectrum of $\hat{T}$.

Consequently,

$$\begin{aligned}\Phi_k(n) &= \frac{-1}{2\pi i}\oint_\Gamma \langle \lambda^n(\hat{T} - \lambda\cdot I)^{-1}(\widetilde{\Psi}_{0k}); h_k\rangle d\lambda \\ &= \frac{-1}{2\pi i}\oint_\Gamma \lambda^n\langle \widetilde{\Psi}_{0k}; (\hat{T}^* - \overline{\lambda}\cdot I)^{-1}(h_k)\rangle.\end{aligned} \tag{3.7}$$

A direct calculation shows that the form $\Phi_k(n)$ is as in (3.1). □

Theorem 3.1 admits the following reciprocal.

Theorem 3.2. *If a function $K(n,m)$ admits the representation (3.1), then there exists a linearly representable sequence $X(n) = T^n x_0$ such that $X(n) \in D^{(r)}[\alpha]$ and the function of correlation of $X(n)$ equals $K(n,m)$.*

Proof. Since $F(n-m)$ is a Hermitian nonnegative function, there exists (see [4]) a unitary operator S defined in a Hilbert space M such that

$$F(n-m) = \langle S^n x_M, S^m x_M\rangle, \quad (x_M \in M). \tag{3.8}$$

By the functions α and $\{\widetilde{\Psi}_{0k}\}_{k=1}^r$ appearing in representation (3.1), construct, in the space L_2^r, the sequence

$$\widetilde{\Psi}(x,n) = (\overline{T}(r))^n(\widetilde{\Psi}_0(x)) = (\hat{T}^n(\widetilde{\Psi}_{01}),\dots,\hat{T}^n(\widetilde{\Psi}_{0r})), \tag{3.9}$$

where operator T is defined in $L^2_{[0;l]}$ by formula (2.5). Let H denotes the Hilbert space $L_r^2 \oplus M$ with scalar product:

$$\langle g + Y_M, g' + Y'_M\rangle = \langle g, g'\rangle_{L_r^2} + \langle Y_M, Y'_M\rangle_M. \tag{3.10}$$

In this space, define the operator $T = \overline{T}(r) \oplus S$ by $T(g + y_M) = \overline{T}(r)(g) + S(y_M)$. Operator T is a contraction and $\dim(I - T^*T)H = r$. Thus, the sequence $X(n) = T^n(f + x_M) = \overline{X}(n) + S^n x_M$ is an element of class $D^{(r)}[\alpha]$ whose function of correlation equals the given function $K(n,m)$. □

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