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MODULUS OF SMOOTHNESS AND THEOREMS CONCERNING APPROXIMATION ON COMPACT GROUPS

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We consider the generalized shift operator defined by $(\mathrm{Sh}_u f)(g) = \int_G f(tut^{-1}g)dt$ on a compact group G, and by using this operator, we define "spherical" modulus of smoothness. So, we prove Stechkin and Jackson-type theorems.

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1. Introduction. In this paper, we prove some theorems on absolutely convergent Fourier series in the metric space $L_2(G)$, where G is a compact group. The algebra of absolutely convergent Fourier series is a subject matter about which a good deal, although far from everything, is known (see [5, page 328]). Like many branches of harmonic analysis on T and R, the theory of absolutely convergent Fourier series is a fruitful source of questions about the corresponding entity for compact groups. By using some absolute convergence theorems of the classical Fourier series, (see [1, 11]), a generalized form of Stechkin [6] and Szasz theorem [1, 11] of the Fourier series on compact groups is obtained. Thus, we solve open problems formulated in [5, page 366] (see also [3, Chapter I, page 9]).

2. Preliminaries and notation. Now, we explain some of the notation and terminologies used throughout the paper.

Let G be a compact group with a dual space $\hat{G}$, dg denote the Haar measure on G normalized by the condition $\int_G dg = 1$, and $\int_G f(g)dg$ denote the Haar integral of a function f on G. Let U_α, $\alpha \in \hat{G}$ denotes the irreducible unitary representation of G in the finite dimensional Hilbert space V_α. We reserve the symbol d_α for the dimension of U_α. Thus, d_α is a positive integer. Also, we denote by χ_α and t_{ij}^{α} $(i,j = 1,2,\dots,d_\alpha)$, $\alpha \in \hat{G}$ the character and matrix elements (coordinate functions) of U_α, respectively.

Let $L_p(G)$ be the space of all functions f equipped with the norm

$$\|f\|_p = \left\{ \int_G |f(g)|^p dg \right\}^{1/p}. \tag{2.1}$$

We write $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(G)}$, and $L_\infty = C$ is the corresponding space of continuous functions, and $\|f\| = \max\{|f(g)| : g \in G\}$. As it is known (see [4] or [10, page 99]), the space $L_2(G)$ can be decomposed into the sum

$$L_2(G) = \sum_{\alpha\in\hat{G}} \oplus H_\alpha, \tag{2.2}$$

where

$$H_\alpha = \{f \in C(G) : f(g) = \operatorname{tr}(U_\alpha(g)C),\ C = \operatorname{Hom}(V_\alpha, V_\alpha)\}. \tag{2.3}$$

This theorem is one of the most important results of the harmonic analysis on compact groups. The orthogonal projection $Y_\alpha : L_2(G) \to H_\alpha$ is given by

$$(Y_\alpha f)(g) = d_\alpha \int_G f(h)\chi_\alpha(gh^{-1})dh, \tag{2.4}$$

where $(Y_\alpha f)(g)$ does not depend on the choice of a basis in L_2. Carrying out this construction for every space H_α, $\alpha \in \hat{G}$, we obtain an orthonormal basis in L_2 consisting of the functions $\sqrt{d_\alpha}t_{ij}^\alpha$, $\alpha \in \hat{G}$, $1 \le i,j \le d_\alpha$. Any function $f \in L_2(G)$ can be expanded into a Fourier series with respect to this basis

$$f(g) = \sum_{\alpha\in\hat{G}} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g), \tag{2.5}$$

where the Fourier coefficients a_{ij}^α are defined by the following relations:

$$a_{ij}^\alpha = d_\alpha \int_G f(g)\overline{t_{ij}^\alpha(g)}dg, \tag{2.6}$$

such that $\overline{t_{ij}^\alpha(g)} = t_{ij}^\alpha(g^{-1})$, where g^{-1} is the inverse of g. Note that (2.5) is a convergent series in the mean and that the Parseval's equality

$$\int_G |f(g)|^2 dg = \sum_{\alpha\in\hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left|a_{ij}^\alpha\right|^2 \tag{2.7}$$

holds. The aforementioned result of harmonic analysis on a compact group can be found, for example, in [4, 5, 7, 10].

We denote by Sh_u the generalized translation operator on compact group G defined by

$$\begin{gathered}(\operatorname{Sh}_u f)(g) = \int_G f(tut^{-1}g)dt,\\ (\triangle_u f)(g) = f(g) - (\operatorname{Sh}_u f)(g) = (E - \operatorname{Sh}_u)f,\end{gathered} \tag{2.8}$$

where $u, g \in G$ and E is the identity operator. We set

$$\triangle_u^k f = \triangle_u\left(\triangle_u^{k-1} f\right) = (E - \mathrm{Sh}_u)^k f = \sum_{i=0}^{k} (-1)^{k+i} C_k^i \,\mathrm{Sh}_u^i f, \tag{2.9}$$

in which $\mathrm{Sh}_u^0 f = f$ and $\mathrm{Sh}_u(\mathrm{Sh}_u^{i-1} f) = \mathrm{Sh}_u^i f$, $i = 1, 2, \ldots, k$ and $k \in N$.

We note that α is a complicated index. Since $\hat{G}$ is a countable set, there are only countably many $\alpha \in \hat{G}$ for which $\alpha_{ij}^{\alpha} \neq 0$ for some i and j; enumerate them as $\{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$. So, $d_{\alpha_0} < d_{\alpha_1} < d_{\alpha_2} < \cdots < d_{\alpha_n} < \cdots$. Because of that, the symbol "$\alpha < n$" is interpreted as $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \subset \hat{G}$, and $\alpha \geq n$ denotes the set $\hat{G} \setminus (\alpha < n)$. Let d_α, as usual, be the dimension of U_α. For typographical convenience, we write d_n for the dimension of the representation U^{α_n}, $n = 1, 2, \ldots$. (See [5, page 458].)

We denote by $E_n(f)_p$ the approximation of the function $f \in L_p(G)$ by "Spherical" polynomials of degree not greater than n:

$$E_n(f)_p = \inf\left\{ \|f - T_n\|_p : T_n \in \sum_{\alpha < n, \alpha \in \hat{G}} \oplus H_\alpha \right\}. \tag{2.10}$$

The sequence of best approximations $\{E_n(f)_p\}_{n=0}^{\infty}$ is a constructive characteristic of the function f. In the capacity of structural characteristic of the function f on a compact group G, we define its Spherical modulus of smoothness of order k by

$$\omega_k(f;\tau)_p = \sup\left\{ \|(E - \mathrm{Sh}_u)^k f\|_p : u \in W_\tau \right\}, \tag{2.11}$$

where W_τ is a neighborhood of e in G. In other words,

$$W_\tau = \{u : \rho(u, e) < \tau,\ u \in G\}, \tag{2.12}$$

where ρ is a pseudometric on G and τ is any positive real number. It is easy to show the following properties of $\omega_k(f, \tau)_p$:

(a) $\lim_{\tau \to 0} \omega_k(f, \tau)_p = 0$;

(b) $\omega_k(f, \tau)_p$ is a continuous monotonically increasing function with respect to τ;

(c) $\omega_k(f_1 + f_2, \tau)_p \leq \omega_k(f_1, \tau)_p + \omega_k(f_2, \tau)_p$;

(d) $\omega_{k+l}(f, \tau)_p \leq 2^l \omega_k(f, \tau)_p$, $l = 1, 2, \ldots$.

3. Main results. We need the following simple but useful lemma.

Lemma 3.1. *The following equality holds for all $u, g \in G$:*

$$\left(\mathrm{Sh}_u t_{ij}^{\alpha}\right)(g) = \frac{\chi_\alpha(u)}{d_\alpha} t_{ij}^{\alpha}(g). \tag{3.1}$$

Proof. Using the orthogonality relations and other formulas for matrix elements $t_{ij}^{\alpha}(g)$ (see [7, page 189]), we have

$$\begin{aligned} \int_G t_{ij}^{\alpha}(tut^{-1}g)dt &= \sum_{p=1}^{d_\alpha}\sum_{q=1}^{d_\alpha} t_{qp}^{\alpha}(u)t_{ij}^{\alpha}(g)\int_G t_{iq}^{\alpha}(t)\overline{t_{qp}^{\alpha}(t)}dt \\ &= \frac{1}{d_\alpha}\sum_{p=1}^{d_\alpha} t_{pp}^{\alpha}(u)t_{ij}^{\alpha}(g) = \frac{1}{d_\alpha}\chi_\alpha(u)t_{ij}^{\alpha}(g). \end{aligned} \tag{3.2}$$

This proves the lemma. □

The following formula is the particular event of the above lemma:

$$\int_G \chi_\alpha(tut^{-1}g)dt = \frac{\chi_\alpha(u)\chi_\alpha(g)}{d_\alpha}. \tag{3.3}$$

It can be called a Weyl formula.

We note that the expansion (2.5) is connected with the expansion

$$f(g) = \sum_{\alpha\in G} Y_\alpha(f)(g), \quad Y_\alpha \in H_\alpha, \tag{3.4}$$

which is defined by (2.4), that is, by the equality

$$Y_\alpha(f)(g) = \sum_{i,j=1}^{d_\alpha} a_{ij}^{\alpha} t_{ij}^{\alpha}(g). \tag{3.5}$$

Thus, the coefficients a_{ij}^{α} are defined by (2.6). Using Lemma 3.1 and the definition of Y_α, we obtain

$$\begin{aligned} Y_\alpha(\mathrm{Sh}_u f)(g) &= \sum_{i,j=1}^{d_\alpha} a_{ij}^{\alpha}\int_G t_{ij}^{\alpha}(tut^{-1}g)dt \\ &= \sum_{i,j=1}^{d_\alpha} a_{ij}^{\alpha}\frac{\chi_\alpha(u)}{d_\alpha}t_{ij}^{\alpha}(g) \\ &= \frac{\chi_\alpha(u)}{d_\alpha}Y_\alpha(f)(g). \end{aligned} \tag{3.6}$$

The following are simple facts with frequent usage: if $f \in L_p$, then

(1) $\|\mathrm{Sh}_u f\|_p \le \|f\|_p$;

(2) $\|f - \mathrm{Sh}_u f\|_p \to 0$ as $u \to e$;

(3) $(Y_\alpha(\mathrm{Sh}_u f))(g) = (\chi_\alpha(u)/\chi_\alpha(e))(Y_\alpha f)(g)$ for all $\alpha \in \hat{G}$.

We note that $\chi_\alpha(e) = d_\alpha$.

Theorem 3.2. *If $f \in L_2$ and f is not constant, then*

$$E_n(f)_2 \le \sqrt{\frac{d_n}{d_n - 2k}}\,\omega_k\left(f;\frac{1}{n}\right)_2, \quad n = 1,2,\ldots. \tag{3.7}$$

Proof. Let $f \in L_2$ and $S_n(f,g)$ denote the nth partial sum of the Fourier series (2.5), that is,

$$S_n(f,g) = \sum_{\alpha<n}\sum_{i,j=1}^{d_\alpha} a_{ij}^{\alpha} t_{ij}^{\alpha}(g) = \sum_{p=0}^{n}\sum_{i,j=1}^{d_{\alpha p}} a_{ij}^{\alpha_p} t_{ij}^{\alpha_p}(g). \tag{3.8}$$

Using Parseval's equality for the compact group G, we have

$$E_n^2(f)_2 = \|f - S_n(f)\|_2^2 = \sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left|a_{ij}^{\alpha}\right|^2. \tag{3.9}$$

Using (3), it is not hard to see that

$$(Y_\alpha(\triangle^k f))(g) = \left(1 - \frac{\chi_\alpha(u)}{d_\alpha}\right)^k (Y_\alpha f)(g), \quad \alpha \in \hat{G}. \tag{3.10}$$

Consequently, $(\triangle^k f)(g) = \sum_{\alpha\in\hat{G}}(1-\chi_\alpha(u)/d_\alpha)^k a_{ij}^{\alpha} t_{ij}^{\alpha}$. By another application of Parseval's equality, we obtain

$$\begin{aligned}\|\triangle_u^k f\|_2^2 &= \sum_{\alpha\in\hat{G}}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left|1-\frac{\chi_\alpha(u)}{d_\alpha}\right|^{2k}\left|a_{ij}^{\alpha}\right|^2 \ge \sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left|1-\frac{\chi_\alpha(u)}{d_\alpha}\right|^{2k}\left|a_{ij}^{\alpha}\right|^2 \\ &= \sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(1-\frac{2\operatorname{Re}\chi_\alpha(u)}{d_\alpha}+\frac{|\chi_\alpha(u)|^2}{d_\alpha^2}\right)^k\left|a_{ij}^{\alpha}\right|^2.\end{aligned} \tag{3.11}$$

Now, using Bernolly's inequality $(1+x)^k \ge 1+kx$ for $x \ge -1$, we obtain

$$\left\|\triangle_u^k f\right\|_2^2 \ge \sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left(1-\frac{2k\operatorname{Re}\chi_\alpha(u)}{d_\alpha}+\frac{k|\chi_\alpha(u)|^2}{d_\alpha^2}\right)\left|a_{ij}^{\alpha}\right|^2. \tag{3.12}$$

Consequently,

$$\left\|\triangle_u^k f\right\|_2^2 \ge \sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\left|a_{ij}^{\alpha}\right|^2 - \sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\frac{2k\operatorname{Re}\chi_\alpha(u)}{d_\alpha}\left|a_{ij}^{\alpha}\right|^2; \tag{3.13}$$

therefore,

$$E_n^2(f)_2 \le \left\|\triangle_u^k f\right\|_2^2 + 2k\sum_{\alpha\ge n}\frac{1}{d_\alpha}\sum_{i,j=1}^{d_\alpha}\frac{\operatorname{Re}\chi_\alpha(u)}{d_\alpha}\left|a_{ij}^{\alpha}\right|^2. \tag{3.14}$$

Let Φ_{W_τ} be a nonnegative integrable function vanishing outside W_τ and satisfying the condition $\int_G \Phi_{W_\tau}(g)dg = 1$. For example, we can take $\Phi_{W_\tau} = \xi_{W_\tau}/\mu(W_\tau)$, where $\mu(W_\tau)$ is the Haar measure of W_τ and ξ_{W_τ} is the characteristic function of W_τ. Multiplying both sides of (3.14) by $\Phi_{W_{1/n}}$, and integrating with respect to u on G, and using the equality $\int_G |\chi_\alpha|^2 dg = 1$ (see [7, page 195]), we obtain

$$\begin{aligned}\int_G E_n^2(f)_2 \Phi_{W_{1/n}}(u)du &\le \int_G \left\|\triangle_u^k f\right\|_2^2 \Phi_{W_{1/n}}\,du \\ &\quad + 2k \sum_{\alpha\ge n} \frac{1}{d_\alpha^2} \sum_{i,j=1}^{d_\alpha} \left|a_{ij}^\alpha\right|^2 \int_G |\chi_\alpha(u)|\Phi_{W_{1/n}}(u)du \\ &\le \sup\left\|\triangle_u^k f\right\|_2^2 + \frac{2k}{d_n} \sum_{\alpha\ge n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left|a_{ij}^\alpha\right|^2.\end{aligned} \tag{3.15}$$

Therefore, it is not hard to see that

$$E_n^2(f)_2 \le \omega_k^2\left(f, \frac{1}{n}\right)_2 + \frac{2k}{d_n} E_n^2(f)_2. \tag{3.16}$$

Finally, we obtain

$$E_n(f)_2 \le \sqrt{\frac{d_n}{d_n - 2k}}\,\omega_k\left(f, \frac{1}{n}\right)_2, \tag{3.17}$$

which proves the theorem. □

This theorem is given without proof in [8] for the case where $k = 1$.

We note that the matrix elements of unitary representations $t_{ij}^\alpha(g)$ satisfy the relations

$$\sum_{j=1}^{d_\alpha} t_{ij}^\alpha(g)\overline{t_{kj}^\alpha(g)} = \sum_{j=1}^{d_\alpha} t_{ij}^\alpha(g)\overline{t_{jk}^\alpha(g)} = \begin{cases} 0 & \text{if } i \ne k, \\ 1 & \text{if } i = k. \end{cases} \tag{3.18}$$

In particular, we have

$$\sum_{j=1}^{d_\alpha} \left|t_{ij}^\alpha\right|^2 = 1 \Longrightarrow \left|t_{ij}^\alpha(g)\right| \le 1 \tag{3.19}$$

for all $\alpha \in \hat{G}$ and $i, j = 1, 2, \ldots, d_\alpha$. Furthermore, it is obvious that $|a_{ij}^\alpha t_{ij}^\alpha(g)| \le |a_{ij}^\alpha|$; therefore, according to the sufficient condition for absolutely convergent Fourier series on the group G, the series $\sum_{\alpha\in\hat{G}} \sum_{i,j=1}^{\alpha} |a_{ij}^\alpha|$ is convergent. Let $A(G) := \{f : \sum_{\alpha\in\hat{G}} \sum_{i,j=1}^{\alpha} |a_{ij}^\alpha| < +\infty\}$. Using Theorem 3.2, and repeating the proof of analogous theorems (see [1, Chapter IX] or [6, Chapter II]) with some changes, we obtain the following theorems.

Theorem 3.3. *If* $f(g) \in L_2(G)$*, then*

$$\sum_{n=1}^{\infty} \frac{\omega_k(f,1/n)_2}{\sqrt{n}} < +\infty \Longrightarrow f(g) \in A(G). \tag{3.20}$$

This theorem is analogous to the Szasz theorem of the classical Fourier series in the case where $k = 1$ and $G = T$.

Theorem 3.4. *If* $f(g) \in L_2(G)$*, then*

$$\sum_{n=1}^{\infty} \frac{E_n(f)_2}{\sqrt{n}} < +\infty \Longrightarrow f(g) \in A(G). \tag{3.21}$$

This theorem is also analogous to a theorem in trigonometric case proved by Stechkin [9].

4. Applications to compact group SU(2)**.** The group SU(2) consists of unimodular unitary matrices of the second order, that is, matrices of the form

$$u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{4.1}$$

Therefore, each element u of SU(2) is uniquely determined by a pair of complex numbers α and β such that $|\alpha|^2 + |\beta|^2 = 1$. We have (see [5]) the relation "$(\alpha,\beta) \mapsto (\phi,\theta,\psi)$," where $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$, and the parameters ϕ, θ, and ψ are called Euler angles defined by

$$|\alpha| = \cos\frac{\theta}{2}; \qquad \operatorname{Arg}\alpha = \frac{\phi+\psi}{2}; \qquad \operatorname{Arg}\beta = \frac{\phi-\psi}{2}. \tag{4.2}$$

Let ϕ, θ, and ψ satisfy the conditions

$$0 \le \phi < 2\pi, \qquad 0 \le \theta < \pi, \qquad -2\pi \le \psi < 2\pi. \tag{4.3}$$

Also, we know that the dimension of the representation T^l of SU(2) is equal to $2l+1$, where $l = 0, 1/2, 1, \dots$ and the matrix elements of T^l for group SU(2) are defined by

$$t^l_{mn}(u) = e^{-(n\psi+m\phi)} P^l_{mn}(\cos\theta) i^{(m-n)}. \tag{4.4}$$

Expressing $t^l_{mn}(u)$ in terms of $P^l_{mn}(\cos\theta)$, we arrive at the following conclusion:

Any function $f(\phi,\theta,\psi)$, $0 \le \phi < 2\pi$, $0 \le \theta < \pi$, and $-2\pi \le \psi < 2\pi$ belonging to the space $L^2(\mathrm{SU}(2))$ such that

$$\int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi} |f(\phi,\theta,\psi)|^2 \sin\theta \, d\theta \, d\phi \, d\psi < \infty \tag{4.5}$$

can be expanded into the mean-convergent series

$$f(\phi,\theta,\psi) = \sum_{l}\sum_{m=-l}^{l}\sum_{n=-l}^{l} \alpha^{l}_{mn} e^{-i(m\phi+n\psi)} P^{l}_{mn}(\cos\theta), \tag{4.6}$$

where

$$\alpha^{l}_{mn} = \frac{2l+1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}\int_{0}^{\pi} f(\phi,\theta,\psi) e^{i(m\phi+n\psi)} P^{l}_{mn}(\cos\theta)\sin\theta\, d\theta d\phi d\psi. \tag{4.7}$$

In addition, we obtain from Parseval's equality that

$$\sum_{l}\sum_{m=-l}^{l}\sum_{n=-l}^{l}\frac{1}{2l+1}\left|\alpha^{l}_{mn}\right|^2 = \frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}\int_{0}^{\pi}\left|f(\phi,\theta,\psi)\right|^2\sin\theta\, d\theta d\phi d\psi. \tag{4.8}$$

Using Theorem 3.2, we obtain the following theorem.

Theorem 4.1. *If $f(\phi,\theta,\psi) \in L_2(\mathrm{SU}(2))$, then*

$$\begin{gathered} E_n(f)_2 \le \sqrt{1+\frac{2}{n-1}}\,\omega_k\left(f,\frac{1}{n}\right)_2, \\ \left\{\sum_{l\ge n}\sum_{m=-l}^{l}\sum_{n=-l}^{l}\frac{1}{2l+1}\left|\alpha^{l}_{mn}\right|^2\right\}^{1/2} \le \sqrt{1+\frac{2}{n-1}}\,\omega_k\left(f,\frac{1}{n}\right)_2. \end{gathered} \tag{4.9}$$

Using the relation between the polynomial $P^{(\alpha,\beta)}_n(z)$ and $P^{l}_{mn}(z)$, we conclude that

$$P^{l}_{mn}(z) = 2^{-m}\left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}\right]^{1/2}(1-z)^{(m-n)/2}(1+z)^{(m+n)/2}P^{(m-n,m+n)}_{l-m}. \tag{4.10}$$

The Jacobi polynomials obtained here are characterized by the condition that α and β are integers and $n+\alpha+\beta \in Z_+$.

Now, we consider the following case.

Let $L^{(\alpha,\beta)}_2[-1,1]$ be the Hilbert space of the functions f defined on the segment $[-1,1]$ with the scalar product

$$(f_1,f_2) = \int_{-1}^{1} f_1(x)\overline{f_2(x)}(1-x)^{\alpha}(1+x)^{\beta}dx; \tag{4.11}$$

then, any function f in this space is expanded into the mean-convergent series

$$f(x) = \sum_{n=0}^{\infty}\alpha_n \hat{P}^{(\alpha,\beta)}_n(x), \tag{4.12}$$

where the polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ are given by

$$\hat{P}_k^{(\alpha,\beta)}(x) = 2^{-(\alpha+\beta+1)/2}\left[\frac{k!(k+\alpha+\beta)!(\alpha+\beta+2k+1)}{(k+\alpha)!(k+\beta)!}\right]^{1/2} P_k^{(\alpha,\beta)}(x), \tag{4.13}$$

$$\alpha_n = \int_{-1}^{1} f(x)\hat{P}_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx. \tag{4.14}$$

The Parseval's equality

$$\int_{-1}^{1} |f(x)|^2(1-x)^{\alpha}(1+x)^{\beta}dx = \sum_{n=0}^{\infty}|\alpha|^2 \tag{4.15}$$

holds. The formulas (4.12), (4.14), and (4.15) are proved for integral nonnegative values of α and β. We can show that they are valid for arbitrary real values of α and β exceeding -1. Finally, we reach the following theorem.

Theorem 4.2. *If $f(x) \in L_2[-1,1]$, then the following hold for Jacobi series:*

$$\begin{gathered} E_n(f)_2 \le \sqrt{1+\frac{2}{n-1}}\,\omega_k\left(f,\frac{1}{n}\right)_2, \\ \left\{\sum_{l=n}^{\infty}|\alpha_l|^2\right\}^{1/2} \le \sqrt{1+\frac{2}{n-1}}\,\omega_k\left(f,\frac{1}{n}\right)_2. \end{gathered} \tag{4.16}$$

Note. For the ideas similar to this paper we refer to [2] and its references.

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