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GENERALIZED JOIN-HEMIMORPHISMS ON BOOLEAN ALGEBRAS

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We introduce the notions of generalized join-hemimorphism and generalized Boolean relation as an extension of the notions of join-hemimorphism and Boolean relation, respectively. We prove a duality between these two notions. We will also define a generalization of the notion of Boolean algebra with operators by considering a finite family of Boolean algebras endowed with a generalized join-hemimorphism. Finally, we define suitable notions of subalgebra, congruences, Boolean equivalence, and open filters.

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1. Introduction. In [4], Halmos generalizes the notion of Boolean homomorphism introducing the notion of join-hemimorphism between two Boolean algebras. A join-hemimorphism is a mapping between two Boolean algebras preserving 0 and $\vee$. As it is shown by Halmos in [4] and by Wright in [10], there exists a duality between join-hemimorphism and Boolean relations. On the other hand, Jónsson and Tarski in [5, 6] introduce the class of Boolean algebras with operators (BAO). They showed that a Boolean algebra endowed with a family of operators can be represented as a subalgebra of a power algebra $\mathcal{P}(X)$, where the operators of $\mathcal{P}(X)$ are defined by means of certain finitary relations on X. This class of algebras plays a key role in modal logic, and has very important applications in theoretical computer science (see, e.g., [1, 2]). The Halmos-Wright duality can be extended to a duality between BAO and Boolean spaces endowed with a set of finitary relations, which are a generalization of the Boolean relations. The aim of this paper is the study of an extension of these dualities.

In Section 2, we recall some notions on Boolean duality. In Section 3, we define the notion of generalized join-hemimorphism as a mapping between a finite product of a family of Boolean algebras $\{B_1,\dots,B_n\}$ into a Boolean algebras B_0 such that it preserves 0 and $\vee$ in each coordinate. We will prove that there exists a duality between generalized join-hemimorphism and certain $(n+1)$-ary relations called generalized Boolean relations. This duality extends the duality given by Halmos and Wright. In Section 4, we define the generalized modal algebra as a pair $\langle\{B_0,B_1,\dots,B_n\},\Diamond\rangle$, where $B_0,B_1,\dots,B_n$ are Boolean algebras and $\Diamond:\prod_{i=1}^{n}B_i\to B_0$ is a generalized join-hemimorphism.

In Section 5, we define the notions of generalized subalgebra and generalized Boolean equivalence and we prove that these notions are duals. Similarly, in Section 6, we introduce the generalized congruences and generalized open filters and we will prove that there exists a bijective correspondence between them.

2. Preliminaries. A topological space is a pair $\langle X,\mathcal{O}(X)\rangle$, or X, for short, where $\mathcal{O}(X)$ is a subset of $\mathcal{P}(X)$ that is closed under finite intersections and arbitrary unions. The set $\mathcal{O}(X)$ is called the set of open sets of the topological space. The collection of all closed sets of a topological space $\langle X,\mathcal{O}(X)\rangle$ is denoted by $\mathcal{C}(X)$. The set $\text{Clop}(X)$ is the set of closed and open sets of $\langle X,\mathcal{O}(X)\rangle$. A *Boolean space* $\langle X,\mathcal{O}(X)\rangle$ is a topological space that is *compact and totally disconnected*, that is, given distinct points $x,y\in X$, there is a clopen subset U of X such that $x\in U$ and $y\notin U$. If $\langle X,\mathcal{O}(X)\rangle$ is a Boolean space, then $\text{Clop}(X)$ is a basis for X and is a Boolean algebra under set-theoretical complement and intersection. Also, the application

$$H_X : X \longrightarrow Ul\big(\text{Clop}(X)\big), \tag{2.1}$$

given by $H_X(x)=\{U\in\text{Clop}(X) : x\in U\}$, is a bijective and continuous function. To each Boolean algebra A, we can associate a Boolean space $\text{Spec}(A)$ whose points are the elements of $Ul(A)$ with the topology determined by the clopen basis $\beta_A(A)=\{\beta_A(a) : a\in A\}$, where $\beta_A : A\to\mathcal{P}(Ul(A))$ is the Stone mapping defined by

$$\beta_A(a)=\big\{P\in Ul(A) : a\in P\big\}. \tag{2.2}$$

By the above considerations, we have that, if X is a Boolean space, then $X\cong\text{Spec}(\text{Clop}(X))$, and if A is a Boolean algebra, then $A\cong\text{Clop}(\text{Spec}(A))$.

Let B be a Boolean algebra. The filter (ideal) generated by a set $H\subseteq A$ will be denoted by $F(H)$ $(I(H))$. The lattice of all filters (ideals) of B is denoted by $\text{Fi}(A)$ $(\text{Id}(A))$.

Let Y be a subset of a set X. The theoretical complement of Y is denoted by $Y^c=X-Y$.

3. Generalized join-hemimorphisms

Definition 3.1. Let $B_0,B_1,\dots,B_n$ be Boolean algebras. A *generalized join-hemimorphism* is a function $\Diamond:\prod_{i=1}^n B_i\to B_0$ such that

(1) $\Diamond(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_n)=0$,

(2) $\Diamond(a_1,\dots,a_{i-1},x\vee y,a_{i+1},\dots,a_n) = \Diamond(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)\vee \Diamond(a_1,\dots,a_{i-1},y,a_{i+1},\dots,a_n)$.

It is easy to see that each generalized join-hemimorphism $h : \prod_{i=1}^{n} B_i \to B_0$ is monotonic in each variable, that is, $x, y \in B_i$, $x \leq y$, then

$$\Diamond(a_1,\dots,x,\dots,a_n) \leq \Diamond(a_1,\dots,y,\dots,a_n). \tag{3.1}$$

A generalized join-hemimorphism $h : \prod_{i=1}^{n} B_i \to B_0$ defines a *generalized meet-hemimorphism* $\Box : \prod_{i=1}^{n} B_i \to B_0$ as follows:

$$\Box(a_1,a_2,\dots,a_n) = \neg\Diamond(\neg a_1,\neg a_2,\dots,\neg a_n). \tag{3.2}$$

It is clear that $\Box$ preserves 1 and $\wedge$, and is monotonic in each variable.

Let $B_0, B_1, \dots, B_n$ be Boolean algebras and let $\Diamond : \prod_{i=1}^{n} B_i \to B_0$ be a generalized join-hemimorphism. Let $\vec{F} = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a filter of B_i, for $1 \leq i \leq n$. We consider the set

$$\Diamond(\vec{F}) = \{y \in B_0 : (\exists x_i \in F_i)\ (\Diamond(x_1,x_2,\dots,x_n) \leq y)\}. \tag{3.3}$$

Theorem 3.2. *Let $B_0, B_1, \dots, B_n$ be Boolean algebras and let $\Diamond : \prod_{i=1}^{n} B_i \to B_0$ be a generalized join-hemimorphism. Let F_i be a proper filter of B_i, for $1 \leq i \leq n$. Then*

(1) *$\Diamond(\vec{F})$ is a proper filter of B_0,*

(2) *if $P \in Ul(B_0)$ and $\Diamond(\vec{F}) \subseteq P$, then there exist $Q_i \in Ul(B_i)$, for $1 \leq i \leq n$ such that*

$$F_i \subseteq Q_i, \qquad Q_1 \times Q_2 \times \cdots \times Q_n \subseteq \Diamond^{-1}(P). \tag{3.4}$$

Proof. (1) It is easy to take into account that the function $\Diamond$ is monotonic in each variable.

(2) Consider the family

$$\mathcal{F}_1 = \{Q_1 \in F_i(B_1) : F_1 \subseteq Q_1,\ \Diamond(Q_1 \times F_2 \times \cdots \times F_n) \subseteq P\}. \tag{3.5}$$

We note that $\mathcal{F}_1 \neq \varnothing$ because $F_1 \in \mathcal{F}_1$. By Zorn's lemma, there exists a maximal element Q_1 in $\mathcal{F}_1$. We prove that $Q_1 \in Ul(B_1)$. Let $a \in B_1$ and suppose that a, $\neg a \notin Q_1$. Consider the filter $F_a = F(Q_1 \cup \{a\})$ and $F_{\neg a} = F(Q_1 \cup \{\neg a\})$. Since q_1 is maximal in $\mathcal{F}_1$, then F_a, $F_{\neg a} \notin \mathcal{F}_1$. So, there exist $(x_1,x_2,\dots,x_n) \in F_a \times F_2 \times \cdots \times F_n$ and $(y_1,y_2,\dots,y_n) \in F_{\neg a} \times F_2 \times \cdots \times F_n$ such that $\Diamond(x_1,x_2,\dots,x_n) \notin P$ and $\Diamond(y_1,y_2,\dots,y_n) \notin P$. Since $x_1 \in F_a$ and $y_1 \in F_{\neg a}$, then $q_1 \wedge a \leq x_1$ and $q_2 \wedge \neg a \leq y_1$, for some $q_1, q_2 \in Q_1$. As Q_1 is a filter of B_1, $q = q_1 \wedge q_2 \in Q_1$, $q \wedge a \leq x_1$, and $q \wedge \neg a \leq y_1$, and as each F_i is a filter, we get $z_i = x_i \wedge y_i \in F_i$, for $i = 2,\dots,n$. Then

$$\begin{aligned}\Diamond(q,z_2,\dots,z_n) &= \Diamond(q \wedge (a \vee \neg a),z_2,\dots,z_n)\\ &= \Diamond((q \wedge a) \vee (q \wedge \neg a),z_2,\dots,z_n)\\ &= \Diamond(q \wedge a,z_2,\dots,z_n) \vee \Diamond(q \wedge \neg a,z_2,\dots,z_n) \in P.\end{aligned} \tag{3.6}$$

If $\Diamond(q\wedge a,z_2,\dots,z_n)\in P$, then

$$\Diamond(q\wedge a,z_2,\dots,z_n)\le\Diamond(q_1\wedge a,x_2,\dots,x_n)\le\Diamond(x_1,x_2,\dots,x_n)\in P, \tag{3.7}$$

which is a contradiction. If $\Diamond(q\wedge\neg a,z_2,\dots,z_n)\in P$, then we deduce that $\Diamond(y_1,y_2,\dots,y_n)\in P$, which also is a contradiction. Thus, $a\in Q_1$ or $\neg a\in Q_1$.

Suppose that we have determinate ultrafilters $Q_1,\dots,Q_k$ in $B_1,\dots,B_k$, respectively, such that $F_i\subseteq Q_i$, for $1\le i\le k$, and $\Diamond(Q_1\times\cdots\times Q_k\times\cdots\times F_n)\subseteq P$. Consider the set

$$\mathcal{F}_{k+1}=\{Q_{k+1}\in B_{k+1}:F_i\subseteq Q_i,\ \Diamond(Q_1\times\cdots\times Q_{k+1}\times\cdots\times F_n)\subseteq P\}. \tag{3.8}$$

We note that $\mathcal{F}_{k+1}\neq\emptyset$ because $F_{k+1}\in\mathcal{F}_{k+1}$. By the Zorn's lemma, there exists a maximal element Q_{k+1} in $\mathcal{F}_{k+1}$. As in the above case, we can prove that $Q_{k+1}\in Ul(B_{k+1})$.

Therefore, we have ultrafilters $Q_1,\dots,Q_n$ in $B_1,\dots,B_n$, respectively, such that $F_i\subseteq Q_i$ and $\Diamond(Q_1\times Q_2\times\cdots\times Q_n)\subseteq P$. It is easy to check that the last inclusion implies that $Q_1\times Q_2\times\cdots\times Q_n\subseteq\Diamond^{-1}(P)$. ☐

Example 3.3. Let $X_0,\dots,X_n$ be sets. Let $R\subseteq\prod_{i=0}^n X_i$. Then, the function $\Diamond_R:\mathcal{P}(X_1)\times\cdots\times\mathcal{P}(X_n)\to\mathcal{P}(X_0)$, defined by

$$\Diamond_R(U_1,U_2,\dots,U_n)=\{x_0\in X_0:R(x_0)\cap(U_1\times U_2\times\cdots\times U_n)\neq\emptyset\}, \tag{3.9}$$

where $R(x_0)=\{(x_1,\dots,x_n)\in\prod_{i=1}^n X_i:(x_0,x_1,\dots,x_n)\in R\}$, is a generalized join-homomorphism.

Definition 3.4. Let $X_0,\dots,X_n$ be Boolean spaces. Consider a relation $R\subseteq\prod_{i=0}^n X_i$. Then R is a *generalized Boolean relation* if

(1) $R(x)$ is a closed subset in the product topology of $X_1\times\cdots\times X_n$, for each $x\in X_0$;
(2) for all $U_i\in\mathrm{Clop}(X_i)$, with $1\le i\le n$, $\Diamond_R(U_1,\dots,U_n)\in\mathrm{Clop}(X_0)$.

We note that if $1\le i\le 2$, then we have the notion of Boolean relation as defined in [4].

Theorem 3.5. *Let $B_0,B_1,\dots,B_n$ be Boolean algebras and let $\Diamond:\prod_{i=1}^n B_i\to B_0$ be a generalized join-hemimorphism. Then the relation $R_\Diamond\subseteq\prod_{i=0}^n Ul(B_i)$, defined by*

$$(P,P_1,\dots,P_n)\in R_\Diamond\Longleftrightarrow P_1\times\cdots\times P_n\subseteq\Diamond^{-1}(P), \tag{3.10}$$

is a generalized Boolean relation such that

$$\beta_{B_0}(\Diamond(a_1,\dots,a_n)) = \Diamond_{R_\Diamond}((\beta_{B_1}(a_1),\dots,\beta_{B_n}(a_n))), \tag{3.11}$$

for all $(a_1,\dots,a_n) \in \prod_{i=1}^n B_i$.

Proof. We prove that for $P \in Ul(B_0)$,

$$R_\Diamond(P) = \bigcap\{(\beta_{B_1}(a_1),\dots,\beta_{B_n}(a_n))^c : \Diamond(a_1,\dots,a_n) \notin P\}. \tag{3.12}$$

If $(P_1,\dots,P_n) \in R_\Diamond(P)$ and

$$(P_1,\dots,P_n) \notin \bigcap\{(\beta_{B_1}(a_1),\dots,\beta_{B_n}(a_n))^c : \Diamond(a_1,\dots,a_n) \notin P\}, \tag{3.13}$$

then, for some $\Diamond(a_1,\dots,a_n) \notin P$, we get $(P_1,\dots,P_n) \in (\beta_{B_1}(a_1),\dots,\beta_{B_n}(a_n))$, that is, $a_i \in P_i$ for $1 \le i \le n$. It follows that $\Diamond(a_1,\dots,a_n) \in P$, which is a contradiction. The other direction is similar. Thus, $R_\Diamond(P)$ is a closed subset of $\prod_{i=1}^n Ul(B_i)$.

Equality (3.11) follows by Theorem 3.2. □

We note that the relation $R_\Diamond \subseteq \prod_{i=0}^n Ul(B_i)$ defined in Theorem 3.5 also can be defined using the notion of generalized meet-hemimorphism $\Box$ in the following way:

$$(P,P_1,\dots,P_n) \in R_\Diamond \iff \Box^{-1}(P) \subseteq P_1 + \cdots + P_n, \tag{3.14}$$

where $P_1 + \cdots + P_n = \{(a_1,\dots,a_n) \in \prod_{i=1}^n B_i : a_i \in P_i, \text{ for some } 1 \le i \le n\}$.

Lemma 3.6. *Let* $X_0,\dots,X_n$ *be Boolean spaces. Consider a relation* $R \subseteq \prod_{i=0}^n X_i$. *Suppose that for all* $U_i \in \operatorname{Clop}(X_i)$, *with* $1 \le i \le n$, $\Diamond_R(U_1,\dots,U_n) \in \operatorname{Clop}(X_0)$. *Then the following conditions are equivalent:*

(1) *R is a generalized Boolean relation;*

(2) *if* $(H_{X_0}(x_0),\dots,H_{X_n}(x_n)) \in R_{\Diamond_R}$, *then* $(x_0,\dots,x_n) \in R$.

Proof. (1)⇒(2). Suppose that $(H_{X_0}(x_0),\dots,H_{X_n}(x_n)) \in R_{\Diamond_R}$ and $(x_0,\dots,x_n) \notin R$. Since $R(x_0)$ is a closed subset of $\prod_{i=1}^n X_i$, there exist $U_i \in \operatorname{Clop}(X_i)$ such that $R(x_0) \cap (U_1,\dots,U_n) = \emptyset$ and $x_i \in U_i$. Then, $x_0 \notin \Diamond_R((U_1,\dots,U_n))$, that is, $(U_1,\dots,U_n) \notin H_{X_1}(x_1) \times \cdots \times H_{X_n}(x_n)$, which is a contradiction.

(2)⇒(1). We have to prove that $R(x)$ is a closed subset of $X_0 \times \cdots \times X_1$. Suppose that $(x_1,\dots,x_n) \notin R(x)$. Then, $(H_{X_0}(x),\dots,H_{X_n}(x_n)) \notin R_{\Diamond_R}$, that is, for each $1 \le i \le n$, there exist $U_i \in D_i$ such that $R(x) \cap (U_1,\dots,U_n) = \emptyset$ and $x_i \in U_i$. Thus, $R(x)$ is a closed subset. □

By the above results, we deduce that there exists a duality between generalized Boolean relations and generalized join-hemimorphisms.

Theorem 3.7. *Let $X_0,\dots,X_n$ be Boolean spaces. Let $R \subseteq \prod_{i=0}^{n} X_i$ be a generalized Boolean relation. Then the mapping $\Diamond_R : \mathrm{Clop}(X_1)\times\cdots\times\mathrm{Clop}(X_n) \to \mathrm{Clop}(X_0)$, defined as in Example 3.3, is a generalized join-hemimorphism such that $(H_{X_0}(x_0),\dots,H_{X_n}(x_n)) \in R_{\Diamond_R}$ if and only if $(x_0,\dots,x_n) \in R$, for all $(x_0,\dots,x_n) \in \prod_{i=0}^{n} X_i$.*

Proof. It is clear that if $(x_0,\dots,x_n) \in R$, then $(H_{X_0}(x_0),\dots,H_{X_n}(x_n)) \in R_{\Diamond_R}$. The other direction follows by Lemma 3.6. □

As an application of the above duality, we prove a generalization of the result that asserts that the Boolean homomorphisms are the minimal elements in the set of all join-hemimorphisms between two Boolean algebras (see [3]).

Let $\{B_i\} = \{B_0,\dots,B_n\}$ be a family of Boolean algebras. Let $\mathrm{GJH}(\prod_{i=1}^{n} B_i, B_0)$ be the set of all generalized join-hemimorphisms between $\prod_{i=1}^{n} B_i$ and B_0 endowed with the pointwise order. Similarly, let $\mathrm{GBR}(\prod_{i=0}^{n} X_i)$ be the set of all generalized Boolean relations defined in $\prod_{i=0}^{n} X_i$ endowed with the pointwise order. Let $\Diamond_1$ and $\Diamond_2 \in \mathrm{GJH}(\prod_{i=1}^{n} B_i, B_0)$ and let $R_{\Diamond_1}$ and $R_{\Diamond_2} \in \mathrm{GBR}(\prod_{i=0}^{n} Ul(B_i))$ be the associate generalized Boolean relations. It is clear that $\Diamond_1 \le \Diamond_2$ if and only if $R_{\Diamond_1} \subseteq R_{\Diamond_2}$.

Theorem 3.8. *An element of $\mathrm{GBR}(\prod_{i=0}^{n} X_i)$ is minimal if and only if it is a continuous function.*

Proof. Let $R \subseteq \prod_{i=0}^{n} X_i$ be a minimal element in $\mathrm{GBR}(\prod_{i=0}^{n} X_i)$. We prove that it is a function. Let $x \in X_0$ and let $\vec{x},\vec{y} \in \prod_{i=1}^{n} X_i$ such that $\vec{x},\vec{y} \in R(x)$. Suppose that $\vec{x} \ne \vec{y}$. Then, $x_i \ne y_i$ for some $1 \le i \le n$. Then, there exist $U_i \in \mathrm{Clop}(X_i)$ such that $x_i \in U_i$ and $y_i \notin U_i$. Consider the sequence $\vec{U} = (X_1,\dots,U_i,\dots,X_n)$. Then, $\vec{x} \in \vec{U}$ and $\vec{y} \notin \vec{U}$. We define an auxiliary relation $R_{\vec{U}} \subseteq \prod_{i=0}^{n} X_i$ by

$$R_{\vec{U}}(x) = \begin{cases} R(x), & \text{if } x \notin \Diamond_R(\vec{U}), \\ R(x)\cap\vec{U}, & \text{if } x \in \Diamond_R(\vec{U}). \end{cases} \tag{3.15}$$

We prove that $R_{\vec{U}}$ is a generalized Boolean relation. It is clear that $R_{\vec{U}}$ is closed. Let $\vec{V} = (V_1,\dots,V_n) \in \mathrm{Clop}(X_1)\times\cdots\times\mathrm{Clop}(X_n)$. Then

$$\begin{aligned} \Diamond_{R_{\vec{U}}}(\vec{V}) &= \{x \in X_0 : R_{\vec{U}}(x)\cap\vec{V} \ne \varnothing\} \\ &= \{x \in X_0 : x \in \Diamond_R(\vec{U}),\ R(x)\cap\vec{U}\cap\vec{V} \ne \varnothing\} \\ &\quad \cup \{x \in X_0 : x \notin \Diamond_R(\vec{U}),\ R(x)\cap\vec{V} \ne \varnothing\} \\ &= \Diamond_R(\vec{U}\cap\vec{V}) \cup \Diamond_R(\vec{V}) \cap \Diamond_R(\vec{U})^c \in \mathrm{Clop}(X_0). \end{aligned} \tag{3.16}$$

Thus, $R_{\vec{U}}$ is a generalized Boolean relation. It is clear that $R_{\vec{U}} \subset R$, that is, R is not a minimal element in $\mathrm{GBR}(\prod_{i=0}^{n} X_i)$, which is a contradiction. Thus, R is a continuous function.

If R is continuous function, then it is easy to see that R is a minimal element in $\mathrm{GBR}(\prod_{i=0}^{n} X_i)$. □

4. Generalized modal algebras. Now we consider a finite family of Boolean algebras endowed with a generalized join-hemimorphism. This class of structures is a generalization of the notion of Boolean algebra with operators. In the sequel, we will write $\{X_i\}$ to denote the family of sets $\{X_i : 1 \leq i \leq n\}$.

Definition 4.1. Let $\{B_i\}$ be a family of Boolean algebras. *A generalized modal algebra* is a pair $B = \langle \{B_i\}, \Diamond \rangle$, where $\Diamond : \prod_{i=1}^{n} B_i \to B_0$ is a *generalized join-hemimorphism*.

Definition 4.2. *A generalized modal space* is a structure $\mathcal{F} = \langle \{X_i\}, R \rangle$, where $X_0, X_1, \ldots, X_n$ are Boolean spaces and R is generalized Boolean relation.

Definition 4.3. Let $B = \langle \{B_i\}, \Diamond_B \rangle$ and $A = \langle \{A_i\}, \Diamond_A \rangle$ be two generalized modal algebras. A *generalized homomorphism* between B and A is a finite sequence $h = (h_0, h_1, \ldots, h_n)$ such that

(1) $h_i : B_i \to A_i$ is a Boolean homomorphism for each $1 \leq i \leq n$,
(2) $h_0(\Diamond_B(a_1, \ldots, a_n)) = \Diamond_A(h_1(a_1), \ldots, h_n(a_n))$.

We write $h : B \to A$ to denote that there exists a generalized homomorphism h between the generalized modal algebras B and A. We say that a generalized homomorphism h between two generalized modal algebras A and B is *injective* if each Boolean homomorphism h_i, $1 \leq i \leq n$, is injective, and h is *surjective* if each h_i surjective. Finally, h is a *generalized isomorphism* if h is bijective generalized homomorphism.

Theorem 4.4. *Let $B = \langle \{B_i\}, \Diamond \rangle$ be a generalized modal algebra. Then the structure $\mathcal{F}(B) = \langle \{Ul(B_i)\}, R_\Diamond \rangle$ is a generalized modal space such that B is isomorphic to the generalized modal algebra $A(\mathcal{F}(B)) = \langle \{\mathrm{Clop}(Ul(B_i))\}, R_{\Diamond_R} \rangle$.*

Proof. It is clear that $R_\Diamond$ is a generalized join-hemimorphism. By Theorem 3.5 we have that $\beta = (\beta_{B_0}, \beta_{B_1}, \ldots, \beta_{B_n})$ is a generalized homomorphism, and since each $\beta_{B_i} : B_i \to \mathrm{Clop}(Ul(B_i))$, for $1 \leq i \leq n$, is a Boolean isomorphism, then β is an generalized isomorphism. □

Definition 4.5. Let $\mathcal{F} = \langle \{X_i\}, R \rangle$ and $\mathcal{G}_g = \langle \{Y_i\}, S \rangle$ be two generalized modal spaces. A *generalized morphism* between $\mathcal{F}$ and $\mathcal{G}$ is a sequence $f = (f_0, f_1, \ldots, f_n)$ such that

(1) $f_i : X_i \to Y_i$ are continuous functions,
(2) if $(x_0, x_1, \ldots, x_n) \in R$, then $(f_0(x_0), f_i(x_1), \ldots, f_n(x_n)) \in S$,

(3) if $(f_0(x_0), y_1, \dots, y_n) \in S$, then there exist $x_i \in X_i$, with $1 \le i \le n$, such that $(x_0, x_1, \dots, x_n) \in R$ and $f_i(x_i) = y_i$ with $1 \le i \le n$.

We write $f : \mathcal{F} \to \mathcal{G}$ to denote that there exists a generalized morphism f between the generalized modal spaces $\mathcal{F}$ and $\mathcal{G}$.

Proposition 4.6. *Let $\mathcal{F} = \langle \{X_i\}, R\rangle$ and $\mathcal{G}_g = \langle \{Y_i\}, S\rangle$ be two generalized modal spaces. If $f : \mathcal{F} \to \mathcal{G}$ is a generalized morphism, then the map $A(f)$: $A(\mathcal{G}) \to A(\mathcal{F})$, defined by $A(f) = (f_0^{-1}, \dots, f_n^{-1})$ with $f_i^{-1} : \mathrm{Clop}(Y_i) \to \mathrm{Clop}(X_i)$, is a generalized homomorphism.*

Proof. It is clear that each map $f_i^{-1} : \mathrm{Clop}(Y_i) \to \mathrm{Clop}(X_i)$ is a Boolean homomorphism. Let $U_i \in \mathrm{Clop}(X_i)$, $1 \le i \le n$, and $x_0 \in X_0$ such that $x_0 \in f_0^{-1}(\Diamond_S(U_1, \dots, U_n))$. Then $S(f_0(x_0)) \cap (U_1, \dots, U_n) \neq \emptyset$. It follows that there exist $y_i \in Y_i$, with $i = 1, \dots, n$, such that $(f_0(x_0), y_1, \dots, y_n) \in S$. Since f is a generalized morphism, there exist $x_i \in X_i$, with $i = 1, \dots, n$, such that $(x_0, x_1, \dots, x_n)$ $\in R$ and $f_i(x_i) = y_i$. So, $x_i \in f_i^{-1}(U_i)$, with $1 \le i \le n$ and this implies that

$$R(x_0) \cap (f_1^{-1}(U_1), \dots, f_n^{-1}(U_n)) \neq \emptyset, \tag{4.1}$$

that is, $x_0 \in \Diamond_R(f_1^{-1}(U_1), \dots, f_n^{-1}(U_n))$.

The other direction is easy and left to the reader. □

Theorem 4.7. *Let $B = \langle \{B_i\}, \Diamond_B\rangle$ and $A = \langle \{A_i\}, \Diamond_A\rangle$ be two generalized modal algebras and let $h = (h_0, \dots, h_n)$ be a generalized homomorphism. Then the sequence $\mathcal{F}(f) = (h_0^{-1}, \dots, h_n^{-1})$ is a generalized morphism between the dual spaces $\mathcal{A}(\mathcal{F}(A))$ and $\mathcal{A}(\mathcal{F}(B))$.*

Proof. It is clear that each $h_i^{-1} : Ul(B_i) \to Ul(A_i)$ is a continuous function. We prove conditions (2) and (3) of Definition 4.5.

(2) Let $(P_0, \dots, P_n) \in R_{\Diamond_B}$ and let $(a_1, \dots, a_n) \in h_1^{-1}(P_1) \times \cdots \times h_n^{-1}(P_n)$. Since $P \times \cdots \times P_n \subseteq \Diamond_B^{-1}(P_0)$,

$$\Diamond_A(h_1(a_1), \dots, h_n(a_n)) = h_0(\Diamond_B(a_1, \dots, a_n)) \in P_0. \tag{4.2}$$

So, $(a_1, \dots, a_n) \in \Diamond_A(h_0^{-1}(P_0))$.

(3) Let $(h_1^{-1}(P_0), Q_1, \dots, Q_n) \in R_{\Diamond_B}$. We prove that

$$\Diamond_A(h_1(Q_1) \times \cdots \times h_n(Q_n)) \subseteq P_0. \tag{4.3}$$

Let $q_i \in Q_i$, with $1 \le i \le n$, such that $(h_1(q_1), \dots, h_n(q_n)) \notin \Diamond_A^{-1}(P_0)$. Then

$$\Diamond_A(h_1(q_1), \dots, h_n(q_n)) = h_0(\Diamond_B(q_1, \dots, q_n)) \notin P_0, \tag{4.4}$$

that is, $(q_1,\dots,q_n) \notin \Diamond_B^{-1}(h_0^{-1}(P_0))$, which is a contradiction. Thus,

$$\Diamond_A(h_1(Q_1)\times\cdots\times h_n(Q_n)) \subseteq P_0. \tag{4.5}$$

We consider the filter $F_i = F(h_i(Q_i))$, for $1 \le i \le n$. Then it is clear that

$$F_1 \times F_2 \times\cdots\times F_n \subseteq \Diamond_B^{-1}(P_0). \tag{4.6}$$

By Theorem 3.2, there exist ultrafilters $P_i \in Ul(A_i)$, for $1 \le i \le n$, such that $F_i \subseteq P_i$ and $P_1 \times P_2 \times\cdots\times P_n \subseteq \Diamond_A^{-1}(P_0)$. Since $h_i(Q_i) \subseteq F_i \subseteq P_i$, we get $Q_i = h_i^{-1}(P_i)$, $1 \le i \le n$. □

We denote by $\mathcal{GMA}$ the class of generalized modal algebras with generalized homomorphisms and denote by $\mathcal{GME}$ the class of generalized modal spaces with generalized morphism. By the above results and by the Boolean duality, we can say that the classes $\mathcal{GMA}$ and $\mathcal{GME}$ are dually equivalents.

5. Generalized subalgebras. It is known that there exists a duality between Boolean subalgebras of a Boolean algebra A and equivalence relations defined on the dual space $Ul(A)$ [7]. The duality is given as follows. Let X be a Boolean space and let E be an equivalence relation on X. A subset $U \subseteq X$ is said to be *E-closed* if for any $x,y \in X$, such that $(x,y) \in E$ and $x \in U$, then $y \in U$, that is,

$$U_E = \{y \in X : (x,y) \in E,\ x \in U\} \subseteq U. \tag{5.1}$$

A *Boolean equivalence* is an equivalence E defined on X such that, for any $x,y \in X$, if $(x,y) \notin E$, there exists an E-closed $U \in \text{Clop}(X)$ such that $x \in U$ and $y \notin U$. The Boolean subalgebra of $\text{Clop}(X)$ associated with a Boolean equivalence E is defined by

$$B(E) = \{U \in \text{Clop}(X) : U_E = U\}. \tag{5.2}$$

If A is a Boolean algebra and B is a Boolean subalgebra of A, then the relation $E(B) \subseteq Ul(A)^2$, given by

$$(P,Q) \in E(B) \Longleftrightarrow P\cap B = Q \cap B, \tag{5.3}$$

is a Boolean equivalence.

Theorem 5.1 [7]. *Let A be a Boolean algebra. Then there exists a dual order-isomorphism between Boolean subalgebras of A and Boolean equivalences defined on $Ul(A)$.*

Definition 5.2. Let $B = \langle\{B_i\},\Diamond\rangle$ be a generalized modal algebra. A *subalgebra* of B is a sequence $A = (A_0,\dots,A_n)$ such that, for each $0 \le i \le n$, A_i is a subalgebra of B_i, and for each $\vec{a} \in \prod_{i=1}^n B_i$, we get $\Diamond(\vec{a}) \in B_0$.

Definition 5.3. Let $\mathcal{F} = \langle \{X_i\}, R\rangle$ be a generalized modal spaces. A *generalized Boolean equivalence* is a sequence $E = (E_0, E_1, \dots, E_n)$ such that, for each $0 \le i \le n$, E_i is a Boolean equivalence of X_i, and if $(a,b) \in E_0$ and $(a, x_1, \dots, x_n) \in R$, then there exists $(y_1, \dots, y_n) \in \prod_{i=1}^{n} X_i$, such that $(b, y_1, \dots, y_n) \in R$ and $(x_i, y_i) \in E_i$, for each $1 \le i \le n$.

Theorem 5.4. *Let $B = \langle \{B_i\}, \Diamond\rangle$ be a generalized modal algebra. Then the following conditions are equivalent:*

(1) *$A = \langle \{A_i\}, \Diamond\rangle$ is a subalgebra of B;*

(2) *the sequence $E_A = (E_{B_0}, E_{B_1}, \dots, E_{B_n})$ is a generalized Boolean equivalence.*

Proof. (1)⇒(2). Let $A = \langle \{A_i\}, \Diamond\rangle$ be a subalgebra of B. Let $P_0, Q_0 \in Ul(B_0)$ such that $P_0 \cap A_0 = Q_0 \cap A_0$ and let $(P_0, P_1, \dots, P_n) \in R_B$. We consider the set $\Diamond(F(P_1 \cap A_1), \dots, F(P_n \cap A_n))$, where $F(P_i \cap A_i)$ is the filter generated by the set $P_i \cap A_i$. We prove that

$$\Diamond(F(P_1 \cap A_1), \dots, F(P_n \cap A_n)) \subseteq Q_0. \tag{5.4}$$

Let $\vec{a} = (a_1, \dots, a_n)$ with $a_i \in P_i \cap A_i$, for $1 \le i \le n$. Since $\vec{a} \in P_1 \times \cdots \times P_n \subseteq \Diamond^{-1}(P_0)$, $\Diamond\vec{a} \in P_0$. As $\vec{a} \in \prod_{i=1}^{n} A_i$ and A is a subalgebra of B, then $\Diamond\vec{a} \in P_0 \in A_0 = Q_0 \cap A_0$. Thus, $\Diamond\vec{a} \in Q_0$ and (5.4) is valid. By Theorem 3.2, there exist ultrafilters $Q_i \in Ul(B_i)$, for $1 \le i \le n$, such that $P_i \cap A_i = Q_i \cap A_i$. Therefore, $(Q_0, Q_1, \dots, Q_n) \in R_B$ and $P_i \cap A_i = Q_i \cap A_i$ for each $1 \le i \le n$.

(2)⇒(1). Suppose that $E_A = (E_{B_0}, E_{B_1}, \dots, E_{B_n})$ is a generalized Boolean equivalence. Let $\vec{a} \in \prod_{i=1}^{n} A_i$ and suppose that $\Diamond\vec{a} \notin A_0$. Consider the set in B_0,

$$(F(\Diamond\vec{a}) \cap A_0) \cup \{\neg\Diamond\vec{a}\}. \tag{5.5}$$

This set has the finite intersection property. Suppose the contrary. Then, there exists $x \in F(\Diamond\vec{a}) \cap A_0$ such that $x \wedge \neg\Diamond\vec{a} = 0$. It follows that $\Diamond\vec{a} \le x \le \Diamond\vec{a}$, that is, $x = \Diamond\vec{a} \in A_0$, which is a contradiction. Thus, the set (5.5) has the finite intersection property. So, there exists an ultrafilter $P_0 \in Ul(B_0)$ such that

$$F(\Diamond\vec{a}) \cap A_0 \subseteq P_0, \qquad \neg\Diamond\vec{a} \in P_0. \tag{5.6}$$

Consider the set

$$\{\Diamond\vec{a}\} \cup P_0 \cap A_0. \tag{5.7}$$

This set has the finite intersection property, because in contrary case there exists $p \in P_0 \cap A_0$ such that $\Diamond\vec{a} \le \neg p$. This implies that $\neg p \in F(\Diamond\vec{a}) \cap A_0 \subseteq P_0$, which is a contradiction. Thus, there exists $Q_0 \in Ul(B_0)$ such that

$$\Diamond\vec{a} \in Q_0, \qquad P_0 \cap A_0 = Q_0 \cap A_0. \tag{5.8}$$

Since $\Diamond\vec{a} \in Q_0$, there exists $(Q_1,\dots,Q_n) \in \prod_{i=1}^{n} Ul(B_i)$ such that

$$(Q_0,Q_1,\dots,Q_n) \in R_B, \qquad \vec{a} \in (Q_1,\dots,Q_n). \tag{5.9}$$

By hypothesis, there exists $(P_1,\dots,P_n) \in R_B(P_0)$ such that $P_i \cap A_0 = Q_i \cap A_0$, for $1 \le i \le n$. Hence, $a_i \in Q_i \cap A_0$, we have $\Diamond\vec{a} \in P_0$, which is a contradiction by (5.6). Thus, $\Diamond\vec{a} \in A_0$. □

6. Generalized congruences. Recall that, given a modal algebra B, there exists a bijective correspondence between congruences of B and filters F of B closed under $\Box$, that is, $\Box a \in F$ when $a \in F$ (see, e.g., [8, 9]). This class of filters are called open filters. In this section, we introduce a generalization of the notion of congruences and open filter.

Let B be a Boolean algebra. Recall that if F is a filter of B, the relation

$$\theta(F) = \{(x,y) : \exists f : x \wedge f = y \wedge f\} \tag{6.1}$$

is a Boolean congruence. On the other hand, if θ is a Boolean congruence, then

$$F(\theta) = \{x \in B : (x,1) \in \theta\} \tag{6.2}$$

is a filter of B such that $\theta(F(\theta)) = \theta$ and $F(\theta(F)) = F$.

Let $B = \langle \{B_i\}, \Diamond\rangle$ be a generalized modal algebra, let F_i be a filter of B_i, $1 \le i \le n$, and let $F_1 + \cdots + F_n = \{(a_1,\dots,a_n) \in \prod_{i=1}^{n} B_i : a_i \in F_i,$ for some $1 \le i \le n\}$.

Definition 6.1. Let $B = \langle \{B_i\}, \Diamond\rangle$ be a generalized modal algebra. A *generalized modal filter* of B is a sequence $\vec{F} = (F_0,F_1,\dots,F_n)$ such that

(1) F_i is a filter of B_i, $0 \le i \le n$;

(2) for any $\vec{a} \in F_1 + \cdots + F_n$, $\Box\vec{a} \in F_0$.

Definition 6.2. Let $B = \langle \{B_i\}, \Diamond\rangle$ be a generalized modal algebra. A *generalized modal congruence* of B is a finite sequence

$$\theta = (\theta_0,\dots,\theta_n) \tag{6.3}$$

such that

(1) θ_i is a Boolean congruence of B_i, for each $0 \le i \le n$;

(2) if $(a_i,b_i) \in \theta_i$ with $1 \le i \le n$, then $(\Diamond(a_1,\dots,a_n), \Diamond(b_1,\dots,b_n)) \in \theta_0$.

Theorem 6.3. *Let $B = \langle \{B_i\}, \Diamond\rangle$ be a generalized modal algebra. There exists a bijective correspondence between congruences of B and the generalized modal filter of B.*

Proof. Let $\theta = (\theta_0, \ldots, \theta_n)$ be a generalized congruence of B. Define the sequence $F(\theta) = (F(\theta_0), \ldots, F(\theta_n))$. Let $(a_1, a_2, \ldots, a_n) \in F(\theta_1) + \cdots + F(\theta_n)$. Then there exist some $1 \le j \le n$ such that $(a_j, 1) \in \theta_j$. Since $(a_i, a_i) \in \theta_i$ for every $1 \le i \le n$, then

$$(\Box(a_1, \ldots, a_j, \ldots, a_n), \Box(a_1, \ldots, 1, \ldots, a_n)) = (\Box(a_1, \ldots, a_j, \ldots, a_n), 1) \in \theta_0. \tag{6.4}$$

Thus, $\Box(a_1, \ldots, a_j, \ldots, a_n) \in F(\theta_0)$.

Let $\vec{F} = (F_0, F_1, \ldots, F_n)$ be a generalized modal filter. Consider the sequence $\theta(\vec{F}) = (\theta(F_0), \ldots, \theta(F_n))$. Let $(a_i, b_i) \in \theta(F_i)$, for $1 \le i \le n$. Then, for each $1 \le i \le n$, there exist $f_i \in F_i$ such that $a_i \wedge f_i = b_i \wedge f_i$. We prove that there exists $f_0 \in F_0$ such that $\Box(a_1, \ldots, a_n) \wedge f_0 = \Box(b_1, \ldots, b_n) \wedge f_0$. Suppose the contrary, that is, for every $f_0 \in F_0$,

$$\Box(a_1, \ldots, a_n) \wedge f_0 \neq \Box(b_1, \ldots, b_n) \wedge f_0. \tag{6.5}$$

Then there exists $P_0 \in Ul(A_0)$ such that

$$F_0 \subseteq P_0, \qquad \Box(a_1, \ldots, a_n) \in P_0, \qquad \Box(b_1, \ldots, b_n) \notin P_0. \tag{6.6}$$

So, there exists $(P_1, \ldots, P_n) \in R_B(P_0)$ such that

$$a_i \notin P_i, \quad \forall 1 \le i \le n. \tag{6.7}$$

Since $\Box(a_1, \ldots, a_n) \in P_0$, then $a_j \in P_j$, for some $1 \le j \le n$, and as $(0, \ldots, f_j, \ldots, 0) \in F_1 + \cdots + F_n$, we get $\Box(0, \ldots, f_j, \ldots, 0) \in F_0 \subseteq P_0$. It follows that $f_j \in P_j$ and since $a_j \wedge f_j = b_j \wedge f_j$, $b_j \in P_j$, which is a contradiction by (6.7). Thus, there exists $f_0 \in F_0$ such that $\Box(a_1, \ldots, a_n) \wedge f_0 = \Box(b_1, \ldots, b_n) \wedge f_0$. $\Box$

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