IJMMS 2003:11, 661–672
PII. S0161171203206293
http://ijmms.hindawi.com

A GENERAL LEMMA FOR FIXED-POINT THEOREMS INVOLVING MORE THAN TWO MAPS IN D-METRIC SPACES WITH APPLICATIONS

B. C. DHAGE, SMRATI ARYA, and JEONG SHEOK UME

Received 26 June 2002

A general procedural lemma for fixed-point theorems for three and four maps in a D-metric space is proved, and it is further applied for proving the common fixed-point theorems of three and four maps in a D-metric space satisfying certain contractive conditions.

2000 Mathematics Subject Classification: 47H10.

1. Introduction. The concept of a D-metric space is introduced by the first author in [2]. A nonempty set X, together with a function $\rho : X \times X \times X \to [0,\infty)$, is called a D-metric space with D-metric ρ if ρ satisfies the following properties:

(i) $\rho(x,y,z) = 0 \Leftrightarrow x = y = z$ (coincidence),

(ii) $\rho(x,y,z) = \rho(p\{x,z,y\})$ (symmetry), where p is a permutation function,

(iii) $\rho(x,y,z) \le \rho(x,y,a)+\rho(x,a,z)+\rho(a,y,z)$ for all $x,y,z,a \in X$ (tetrahedral inequality).

A few details along with some specific examples of a D-metric space appear in [3].

A sequence $\{x_n\} \subset X$ is said to be convergent to a point $x \in X$ if

$$\lim_{m,n\to\infty} \rho(x_m,x_n,x) = 0. \tag{1.1}$$

Similarly, a sequence $\{x_n\} \subset X$ is called D-Cauchy if

$$\lim_{m,n,p\to\infty} \rho(x_m,x_n,x_p) = 0. \tag{1.2}$$

A complete D-metric space is one in which every D-Cauchy sequence converges to a point. Further, a subset S of a D-metric space X is called bounded if there exists a constant $k > 0$ such that $\rho(x,y,z) \le k$ for all $x,y,z \in S$, and the constant k is called a D-bound of S. The infinimum of all such D-bounds k is called the diameter of S, and it is denoted by $\delta(S)$. Finally, it is known that a mapping $f : X \to X$ is continuous if and only if, for any sequence $\{x_n\} \subset X$, $x_n \to x^*$ implies $fx_n \to fx^*$.

It has been shown in [5] that the D-metric ρ is continuous on X^3 in the topology of D-metric convergence which is Hausdorff. For details of a D-metric space, the reader is referred to Dhage [5].

The general existence principles for the fixed-point theorems for a single and a pair of maps in D-metric space have been established in Dhage and Rhoades [6] and Dhage [1], respectively. However, the extension of these existence principles to three or four maps is not possible.

In this paper, we just formulate the general procedure for common fixed-point theorems for more than two selfmaps of a D-metric space and discuss some of its applications.

We need the following auxiliary results in the sequence.

Proposition 1.1. *Let $\{x_n\}$ be a sequence in a D-metric space X satisfying*

$$\rho(x_n,x_{n+1},z) \le \lambda\rho(x_{n-1},x_n,z) \tag{1.3}$$

for all $n \in \mathbb{N}$ and $z \in \{x_n\}$, where $0 \le \lambda < 1$. Then,

$$\rho(x_n,x_{n+1},x_m) \le \lambda^n k \tag{1.4}$$

for all $m > n$, where $k = (2/(1-\lambda))\max\{\rho(x_0,x_0,x_1),\rho(x_0,x_1,x_1)\}$.

Proof. From (1.3),

$$\rho(x_n,x_{n+1},x_m) \le \lambda\rho(x_{n-1},x_n,x_m) \tag{1.5}$$

for each $m > n \in \mathbb{N}$. By induction,

$$\rho(x_n,x_{n+1},x_m) \le \lambda^n\rho(x_0,x_1,x_m). \tag{1.6}$$

Let $q = \max\{\rho(x_0,x_0,x_1),\rho(x_0,x_1,x_1)\}$. Using the tetrahedral inequality,

$$\begin{aligned}
\rho(x_0,x_1,x_m) &\le \rho(x_0,x_1,x_{m-1}) + \rho(x_{m-1},x_0,x_m) + \rho(x_{m-1},x_m,x_1) \\
&= \rho(x_0,x_1,x_{m-1}) + \rho(x_{m-1},x_m,x_0) + \rho(x_{m-1},x_m,x_1) \\
&\le \rho(x_0,x_1,x_{m-1}) + \lambda^{m-1}\rho(x_0,x_1,x_0) + \lambda^{m-1}\rho(x_0,x_1,x_1) \\
&\le \rho(x_0,x_1,x_{m-1}) + 2\lambda^{m-1}q \\
&\le \rho(x_0,x_1,x_{m-2}) + 2\lambda^{m-1}q + 2\lambda^{m-2}q \\
&\;\;\vdots \\
&\le \rho(x_0,x_1,x_1) + 2(\lambda+\lambda^2+\cdots+\lambda^{m-1})q \\
&\le 2(1+\lambda+\lambda^2+\cdots+\lambda^{m-1})q < \frac{2}{1-\lambda}q = k.
\end{aligned} \tag{1.7}$$

Substituting (1.7) into (1.6) yields the desired inequality (1.4). □

Proposition 1.2. *Every sequence* $\{x_n\} \subset X$ *satisfying (1.3) is bounded with a D-bound* $k = (2/(1-\lambda)) \max\{\rho(x_0,x_0,x_1), \rho(x_0,x_1,x_1)\}$.

Proof. Let $q = \max\{\rho(x_0,x_0,x_1), \rho(x_0,x_1,x_1)\}$. Then, for any integers $r \geq s \geq n$, there exists positive integer p and t such that

$$\begin{aligned}
\rho(x_n,x_r,x_s) &= \rho(x_n,x_{n+p},x_{n+t}) \\
&\leq \rho(x_n,x_{n+1},x_{n+t}) + \rho(x_n,x_{n+1},x_{n+p}) + \rho(x_{n+1},x_{n+p},x_{n+t}) \\
&\leq 2\lambda^n q + \rho(x_{n+1},x_{n+p},x_{n+q}) \\
&\leq 2\lambda^n q + 2\lambda^{n+1} q + \rho(x_{n+2},x_{n+p},x_{n+t}) \\
&\;\;\vdots \\
&\leq 2\left(\sum_{j=1}^{n+p-2} \lambda^j\right) q + \rho(x_{n+p-1},x_{n+p},x_{n+r}) \\
&\leq 2\left(\sum_{j=1}^{n+p-1} \lambda^j\right) q < 2\left(\sum_{j=1}^{\infty} \lambda^j\right) q < \frac{2}{1-\lambda} q = k.
\end{aligned} \tag{1.8}$$

Then, $\{x_n\}$ is bounded and the proof is complete. □

2. Main results. Before going to the main results of this paper, we state a lemma proved in Dhage [4].

Lemma 2.1 (*D*-Cauchy principle)**.** *Let* $\{y_n\}$ *be a bounded sequence in D-metric space with D-bound k satisfying*

$$\rho(y_n,y_{n+1},y_m) \leq \lambda^n k \tag{2.1}$$

for all $m > n \in \mathbb{N}$. *Then,* $\{y_n\}$ *is D-Cauchy.*

Let $A,B,S,T : X \to X$ be four maps such that

$$A(X) \subseteq T(X), \qquad B(X) \subseteq S(X). \tag{2.2}$$

Condition (2.2) ensures that it is possible to define a sequence $\{y_n\}$ in X as follows. Let $x \in X$ be arbitrary. Then, in view of condition (2.2), there exists a sequence $\{x_n\}$ such that

$$x_0 = x, \qquad Ax_{2n} = Tx_{2n+1}, \quad Bx_{2n+1} = Sx_{2n+2}, \quad n \geq 0. \tag{2.3}$$

Now, define $\{y_n\}$ in X by

$$y_0 = Sx_0, \qquad y_{2n+1} = Tx_{2n+1}, \quad y_{2n+2} = Sx_{2n+2}, \quad n \geq 0. \tag{2.4}$$

A point $x \in X$ is called a coincidence point of two maps $A,B : X \to X$ if $Ax = Bx$, and in this case, the mappings A and B are called coincident on X.

Similarly, a coincidence point of three or four maps on a D-metric space is defined.

Lemma 2.2. *Let $A,B,S,T : X \to X$ satisfy (2.2), and let $\{y_n\} \subset X$ be defined by (2.4). Further, assume that $\{y_n\}$ is complete. Suppose that there exists a $\lambda \in [0,1)$ such that*

$$\rho(y_n, y_{n+1}, z) \le \lambda \rho(y_{n-1}, y_n, z) \tag{2.5}$$

for all $n \in \mathbb{N}$ and $z \in \{y_n\}$. Then, either
- (a) *A and S have a coincidence point,*
- (b) *B and T have a coincidence point,*
- (c) *A, S, and T have a coincidence point,*
- (d) *B, S, and T have a coincidence point, or*
- (e) *$\{y_n\}$ converges to a point $u \in X$ and, for all $m > n \in \mathbb{N}$,*

$$\rho(y_n, y_m, u) \le 2\sum_{j=n}^{m} \lambda^j k, \qquad \rho(y_n, u, u) \le 2\frac{\lambda^n}{1-\lambda} k, \tag{2.6}$$

where $k = \delta(\{y_n\})$.

Proof. Suppose that $y_{2n} = y_{2n+1}$ for some n. Then, $Sx_{2n} = Tx_{2n+1} = Ax_{2n}$ and (a) holds. Also, if $x_{2n} = x_{2n+1}$, then $Tx_{2n} = Tx_{2n+1}$ and so (c) holds. Similarly, if $y_{2n+1} = y_{2n+2}$ for some n, then it is shown analogously that (b) and (d) hold.

Suppose now that $y_n \ne y_{n+1}$ for each n. Then, from Proposition 1.1, it follows that

$$\rho(y_n, y_{n+1}, y_m) \le \lambda^n \rho(y_0, y_1, y_m) \le \lambda^n k \tag{2.7}$$

for all $m > n \in \mathbb{N}$. Now, an application of Lemma 2.1 yields that $\{y_n\}$ is D-Cauchy. Since $\{y_n\}$ is complete, there exists a point $u \in X$ such that $\lim_n y_n = u$.

Now, for any positive integers m and n, $m > n$, by repeated application of the tetrahedral inequality,

$$\begin{aligned}
\rho(y_n, y_m, u) &\le \rho(y_n, y_{n+1}, y_m) + \rho(y_n, y_{n+1}, u) + \rho(y_{n+1}, y_m, u) \\
&\le \lambda^n \rho(y_0, y_1, y_m) + \lambda^n \rho(y_0, y_1, u) + \rho(y_{n+1}, y_m, u) \\
&\le 2\lambda^n k + \rho(y_{n+1}, y_m, u) \\
&\le 2\lambda^n k + 2\lambda^{n+1} k + \rho(y_{n+2}, y_m, u) \\
&\vdots \\
&\le 2(\lambda^n + \lambda^{n+1} + \cdots + \lambda^m) k \\
&= 2\sum_{j=n}^{m} \lambda^j k.
\end{aligned} \tag{2.8}$$

The above inequality further gives that

$$\begin{aligned}\rho(y_n, y_m, u) &\le 2\sum_{j=n}^{m}\lambda^j k\\ &= 2\lambda^n(1+\lambda+\cdots+\lambda^{m-n})k \\ &= 2\lambda^n\left(\frac{1-\lambda^{m-n}}{1-\lambda}\right)k.\end{aligned} \tag{2.9}$$

Taking the limit as $m \to \infty$ in the above inequality,

$$\rho(y_n, u, u) \le 2\frac{\lambda^n}{1-\lambda}k. \tag{2.10}$$

The proof of Lemma 2.2 is complete. □

The three-maps version of Lemma 2.2 is obtained in two ways: one by setting $S = T$ and the other by setting $A = B$. In the situation when $S = T$, condition (2.2) reduces to

$$A(X) \subseteq S(X), \qquad B(X) \subseteq S(X). \tag{2.11}$$

Then, it is possible to choose a sequence $\{x_n\} \subset X$ such that

$$x_0 = x, \qquad Ax_{2n} = Sx_{2n+1}, \quad Bx_{2n+1} = Sx_{2n+2}, \quad n \ge 0. \tag{2.12}$$

Now, define a sequence $\{y_n\}$ in X as follows:

$$y_0 = Sx_0, \qquad y_{2n} = Sx_{2n}, \quad y_{2n+1} = Sx_{2n+1}, \quad n \in \mathbb{N}. \tag{2.13}$$

Lemma 2.3. *Let $A, B, S : X \to X$ satisfy (2.11). Suppose that there exists an $x \in X$ such that the sequence $\{y_n\} \subset X$ defined by (2.13) is complete. Further, suppose that*

$$\rho(y_n, y_{n+1}, z) \le \lambda\rho(y_{n-1}, y_n, z) \tag{2.14}$$

for all $n \in \mathbb{N}$ and $z \in \{y_n\}$, where $0 \le \lambda < 1$. Then, either

(a) *A and S have a coincidence point,*

(b) *B and S have a coincidence point, or*

(c) *$\{y_n\}$ converges to a point $u \in X$ and, for all positive integers m and n, $m > n$,*

$$\rho(y_n, y_m, u) \le 2\sum_{j=n}^{m}\lambda^j k, \qquad \rho(y_n, u, u) \le 2\frac{\lambda^n}{1-\lambda}k, \tag{2.15}$$

where $k = \delta(\{y_n\})$.

It is known that the fixed-point theorems for more than two maps require some sort of commutativity condition on the mappings under consideration.

Below, we will apply Lemma 2.2 for proving the common fixed-point theorem for four maps on a D-metric space under a suitable commutativity condition.

A sequence $\{x_n\} \subset X$ is called a sequence of coincidence for the maps $A,B : X \to X$ if $\lim_n Ax_n = \lim_n Bx_n$. In this case, the mappings A and B are called limit coincident on X. Similarly, two maps $A,B : X \to X$ are called commuting or commutative if $(AB)(x) = (BA)(x)$ for all $x \in X$ and limit commuting if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_n (AB)(x_n) = \lim_n (BA)(x_n). \tag{2.16}$$

Finally, two maps $A,B : X \to X$ are called limit coincidentally commuting if their limit coincidence implies the limit commuting on X, that is, for any sequence $\{x_n\} \subset X$ if

$$\lim_n Ax_n = \lim_n Bx_n \Longrightarrow \lim_n (AB)(x_n) = \lim_n (BA)(x_n). \tag{2.17}$$

It is known that the limit coincidentally commuting mappings commute at their coincidence points. See, for details, Dhage [4].

Now, we are ready to give some applications of Lemma 2.2 for proving the existence of a common fixed point of four maps on a D-metric space X.

An orbit of four selfmaps A, B, S, and T of a D-metric space X at a point $x \in X$ is a set $O_{A,B}(S,T : x)$ in X defined by

$$\begin{aligned} O_{A,B}(S,T:x) = \\ \{y_0 = Sx_0,\ y_{2n+1} = Ax_{2n} = Tx_{2n+1},\ y_{2n+2} = Bx_{2n+1} = Sx_{2n+1} : n \geq 0\}. \end{aligned} \tag{2.18}$$

Clearly, the orbit $O_{A,B}(S,T : x)$ is well defined if A, B, S, and T satisfy condition (2.2). By $\overline{O_{A,B}(S,T : x)}$, we denote the closure of the orbit $O_{A,B}(S,T : x)$ in X.

Theorem 2.4. *Let $A,B,S,T : X \to X$ be four selfmaps of a D-metric space X satisfying (2.2) and*

$$\rho(Ax,By,z) \leq \lambda \max\{\rho(Sx,Ty,z),\rho(Sx,Ax,z),\rho(Ty,By,z)\} \tag{2.19}$$

for all $x,y,z \in X$, where $0 \leq \lambda < 1$. Assume further that
(a) *$\overline{O_{A,B}(S,T : x)}$ is complete for each $x \in X$,*
(b) *$\{A,S\}$ and $\{B,T\}$ are limit coincidentally commuting,*
(c) *any one A, B, S, or T continuous.*
Then, A, B, S, and T have a unique common fixed point.

Proof. Let $x \in X$ be arbitrary, and define a sequence $\{y_n\} \subset X$ by (2.4), which is possible in view of condition (2.4). Now, taking $x = x_{2n}$ and $y = x_{2n+1}$

in (2.19),

$$\begin{aligned}\rho(y_{2n+1},y_{2n+2},z) &\le \lambda\max\{\rho(y_{2n},y_{2n+1},z),\rho(y_{2n},y_{2n+1},z),\rho(y_{2n+1},y_{2n+2},z)\}\\ &= \lambda\rho(y_{2n},y_{2n+1},z)\end{aligned} \tag{2.20}$$

for all $n \ge 0$ and $z \in \{y_n\}$. Similarly, taking $x = x_{2n}$ and $y = x_{2n-1}$ in (2.19),

$$\begin{aligned}\rho(y_{2n},y_{2n+1},z) &\le \lambda\max\{\rho(y_{2n-1},y_{2n},z),\rho(y_{2n},y_{2n+1},z),\rho(y_{2n-1},y_{2n},z)\}\\ &= \lambda\rho(y_{2n-1},y_{2n},z)\end{aligned} \tag{2.21}$$

for all $n \in \mathbb{N}$ and $z \in \{y_n\}$. Hence, in general,

$$\rho(y_n,y_{n+1},y_m) \le \lambda\rho(y_{n-1},y_n,y_m) \tag{2.22}$$

for all $m > n \in \mathbb{N}$ and $0 \le \lambda < 1$.

We prove the conclusion of our theorem in two cases.

Case 1. If $y_n = y_{n+1}$, then $y_n = y_{n+k}$ for all $k \ge 0$. If $y_{n+1} \ne y_{n+2}$, then, replacing n in (2.22) by $n+1$,

$$0 < \rho(y_{n+1},y_{n+2},y_{n+1}) \le \lambda\rho(y_n,y_{n+1},y_{n+1}) = 0, \tag{2.23}$$

which is a contradiction and $y_{n+1} = y_{n+2}$, and, by induction, $y_n = y_{n+k}$ for all $k \ge 0$. Therefore, by Lemma 2.2, there are points u and v in X such that $w_1 = Au = Su$ and $w_2 = Bv = Tv$.

We will show that $w_1 = w_2$. By (2.19),

$$\begin{aligned}\rho(w_1,w_2,w_1) &= \rho(Au,Bv,w_1)\\ &\le \lambda\max\{\rho(Su,Tv,w_1),\rho(Su,Au,w_1),\rho(Tv,Bv,w_1)\}\\ &= \lambda\max\{\rho(w_1,w_2,w_1),\rho(w_1,w_1,w_1),\rho(w_2,w_2,w_1)\}\\ &= \lambda\rho(w_1,w_2,w_2).\end{aligned} \tag{2.24}$$

Again,

$$\begin{aligned}\rho(w_1,w_2,w_2) &= \rho(Au,Bv,w_2)\\ &\le \lambda\max\{\rho(Su,Tv,w_2),\rho(Su,Au,w_2),\rho(Tv,Bv,w_2)\}\\ &= \lambda\max\{\rho(w_1,w_2,w_2),\rho(w_1,w_1,w_2),\rho(w_2,w_2,w_2)\}\\ &= \lambda\rho(w_1,w_2,w_1).\end{aligned} \tag{2.25}$$

Substituting (2.25) into (2.24),

$$\rho(w_1,w_2,w_1) \le \lambda^2\rho(w_1,w_2,w_1), \tag{2.26}$$

which is possible only when $w_1 = w_2$ since $\lambda < 1$. Hence, $Au = Bv = Su = Tv = w$. Next, we show that w is a coincidence of A, B, S, and T. Since $\{A,S\}$ and $\{B,T\}$ are limit coincidentally commuting, they commute at coincidence point. Therefore, $Sw = SAu = ASu = Aw$ and $Tw = TBv = BTv = Bw$. Now,

$$\begin{aligned}\rho(Aw,Bw,Aw) &\le \lambda\max\{\rho(Sw,Tw,Aw),\rho(Sw,Aw,Aw),\rho(Tw,Bw,Aw)\}\\ &= \lambda\max\{\rho(Aw,Bw,Aw),\rho(Bw,Bw,Aw)\}\\ &= \lambda\rho(Aw,Bw,Bw).\end{aligned} \tag{2.27}$$

Similarly,

$$\rho(Aw,Bw,Bw) \le \lambda\rho(Aw,Bw,Aw). \tag{2.28}$$

Substituting (2.28) into (2.27),

$$\rho(Aw,Bw,Aw) \le \lambda^2\rho(Aw,Bw,Aw), \tag{2.29}$$

which is possible only when $Aw = Bw$. Hence, $Aw = Sw = Tw = Bw$ is a coincidence point of the four maps A, B, S, and T. Finally, we prove that w is a common fixed point of A, B, S, and T. If $w \ne Aw$, then, by (2.19),

$$\begin{aligned}\rho(Aw,w,w) &= \rho(Aw,Bv,w)\\ &\le \lambda\max\{\rho(Sw,Tv,w),\rho(Sw,Aw,w),\rho(Tv,Bv,w)\}\\ &= \lambda\rho(Aw,w,Aw).\end{aligned} \tag{2.30}$$

Similarly,

$$\rho(Aw,w,Aw) \le \lambda\rho(Aw,w,w). \tag{2.31}$$

From (2.30) and (2.31),

$$\rho(Aw,w,w) \le \lambda^2\rho(Aw,w,w), \tag{2.32}$$

which is a contradiction to $Aw = w$ and hence, $w = Aw = Sw = Tw = Bw$.

Case 2. Suppose that $y_n \ne y_{n+1}$ for each n. Then, by Lemma 2.2, there exists a point $w \in X$ such that $\lim_n y_n = w$. By definition of $\{y_n\}$,

$$\begin{aligned}\lim_n y_{2n} = \lim_n Sx_{2n} = \lim_n Ax_{2n} = \lim_n Tx_{2n+1} = w,\\ \lim_n y_{2n+1} = \lim_n Tx_{2n+1} = \lim_n Bx_{2n+1} = \lim_n y_{2n+2} = w.\end{aligned} \tag{2.33}$$

Since $\{A,S\}$ and $\{B,T\}$ are limit coincidentally commuting, then

$$\begin{aligned}\lim_n ASx_{2n} = \lim_n SAx_{2n},\\ \lim_n BTx_{2n+1} = \lim_n TBx_{2n+1}.\end{aligned} \tag{2.34}$$

Suppose first that S is continuous on X. Then,

$$\lim_n SSx_{2n} = \lim_n SAx_{2n} = \lim_n ASx_{2n} = Sw. \tag{2.35}$$

First, we show that w is a fixed point of S. If $w \neq Sw$, then, by (2.19),

$$\begin{aligned}
&\rho(Sw,w,w)\\
&= \lim_n \rho(ASx_{2n},Bx_{2n+1},w)\\
&\leq \lambda \lim_n \max\{\rho(SSx_{2n},Tx_{2n+1},w),\rho(SBx_{2n},ASx_{2n},w),\rho(Tx_{2n+1},Bx_{2n+1},w)\}\\
&= \lambda \max\{\rho(Sw,w,w),\rho(Sw,Sw,w)\}\\
&= \lambda\rho(Sw,Sw,w).
\end{aligned} \tag{2.36}$$

Similarly,

$$\rho(Sw,w,Sw) \leq \lambda\rho(Sw,w,w). \tag{2.37}$$

Substituting (2.37) into (2.36),

$$\rho(Sw,w,w) \leq \lambda^2\rho(Sw,w,w), \tag{2.38}$$

which is a contradiction and hence, $Sw = w$. Similarly,

$$\begin{aligned}
&\rho(Aw,w,w)\\
&\quad\leq \lim_n \rho(Aw,Bx_{2n+1},w)\\
&\quad\leq \lambda \lim_n \max\{\rho(Sw,Tx_{2n+1},w),\rho(Tx_{2n+1},Bx_{2n+1},w),\rho(Sw,Aw,w)\}\\
&\quad= \lambda \max\{0,0,\rho(Aw,w,w)\}\\
&\quad= \lambda\rho(Aw,w,w),
\end{aligned} \tag{2.39}$$

which implies that $Aw = w$, since $\lambda < 1$. From the condition $A(X) \subseteq T(X)$, it follows that there is a point $p \in X$ such that $w = Aw = Tp$. We show that $Bp = Tp$. If not, then

$$\begin{aligned}
\rho(Tp,Bp,w) &= \rho(Aw,Bp,w)\\
&\leq \lambda \max\{\rho(Sw,Tp,w),\rho(Sw,Aw,w),\rho(Tp,Bp,w)\}\\
&= \lambda\rho(Tp,Bp,w),
\end{aligned} \tag{2.40}$$

which is a contradiction. Hence, $Bp = Tp$. Since $\{B,T\}$ are limit coincidentally commuting, we obtain $Bw = BTp = TBp = Bw$. Now,

$$\begin{aligned}
\rho(Aw,Bw,w) &\leq \lambda \max\{\rho(Sw,Tw,w),\rho(Sw,Aw,w),\rho(Tw,Bw,w)\}\\
&= \lambda\rho(Aw,Bw,Bw).
\end{aligned} \tag{2.41}$$

Similarly,

$$\rho(Aw,Bw,Bw) \leq \lambda\rho(Aw,Bw,w). \tag{2.42}$$

Substituting (2.42) into (2.41),

$$\rho(Aw,Bw,w) \leq \lambda^2\rho(Aw,Bw,w), \tag{2.43}$$

which is possible only when $Aw = Bw$. Thus, w is a common fixed point of A, B, S, and T.

Similarly, if T is continuous, then it is proved in an analogous way that A, B, S, and T have a common fixed point.

Next, suppose that A is continuous. Then, we have

$$\lim_n AAx_{2n} = \lim_n ASx_{2n} = \lim_n SAx_{2n} = Aw. \tag{2.44}$$

First, we show that $Aw = w$. If $Aw \neq w$, then

$$\begin{aligned}
&\rho(Aw,w,w) \\
&= \lim_n \rho(AAx_{2n},Bx_{2n+1},w) \\
&\leq \lambda \lim_n \max\{\rho(SAx_{2n},Tx_{2n+1},w),\rho(SAx_{2n},AAx_{2n},w),\rho(Tx_{2n+1},Bx_{2n+1},w)\} \\
&= \lambda \max\{\rho(Aw,w,w),\rho(Aw,Aw,w)\} \\
&= \lambda\rho(Aw,Aw,w).
\end{aligned} \tag{2.45}$$

Similarly,

$$\rho(Aw,w,Aw) \leq \lambda\rho(Aw,w,w). \tag{2.46}$$

Substituting (2.46) into (2.45),

$$\rho(Aw,w,w) \leq \lambda^2\rho(Aw,w,w), \tag{2.47}$$

which is a contradiction. Hence, $Aw = w$. Using condition (2.2), there exists a point $p \in X$ such that $Tp = Aw = w$. We show that $Bp = Tp$. Now,

$$\begin{aligned}
\rho(Aw,Bp,w) &\\
&= \lim_n \rho(AAx_{2n},Bp,w) \\
&\leq \lambda \lim_n \max\{\rho(SAx_{2n},Tp,w),\rho(SAx_{2n},AAx_{2n},w),\rho(Tp,Bp,w)\} \\
&= \lambda\rho(w,Bp,w),
\end{aligned} \tag{2.48}$$

which gives that $Bp = Tp$. Since $\{B,T\}$ are limit coincidentally commuting, they commute at coincidence point. Hence, $Tw = TBp = BTp = Bw$. Now,

$$\rho(Ax_{2n},Bw,w) \le \lambda \max\{\rho(Sx_{2n},Tw,w),\rho(Sx_{2n},Ax_{2n},w),\rho(Tw,Bw,w)\}. \tag{2.49}$$

Taking the limit as $n \to \infty$,

$$\rho(w,Bw,w) \le \lambda\rho(w,Bw,Bw). \tag{2.50}$$

Similarly,

$$\rho(w,Bw,Bw) \le \lambda\rho(w,Bw,w). \tag{2.51}$$

Substituting (2.51) into (2.50),

$$\rho(w,Bw,w) \le \lambda^2\rho(w,Bw,w), \tag{2.52}$$

which implies that $w = Bw = Tw = Aw$. Since $B(X) \subseteq S(X)$, there is a point $q \in X$ such that $w = Bw = Sq$. We show that $Aq = Sq$. Now,

$$\begin{aligned}\rho(Aq,Sq,w) &= \rho(Aq,Bw,w)\\ &\le \lambda \max\{\rho(Sq,Tw,w),\rho(Sq,Aq,w),\rho(Tw,Bw,w)\}\\ &= \lambda\rho(Sq,Aq,w),\end{aligned} \tag{2.53}$$

which implies that $Aq = Sq$. Since $\{A,S\}$ are limit coincidentally commuting $Sw = SAq = ASq = Aw = w$. Thus, $Aw = Sw = Tw = Bw$, that is, w is a common fixed point of A, B, S, and T. Similarly, if B is continuous, it is proved that A, B, S, and T have a common fixed point.

To prove the uniqueness, let $w^*(\ne w)$ be common fixed point of A, B, S, and T. Then,

$$\begin{aligned}\rho(w,w^*,w^*) &= \rho(Aw,Bw^*,w^*)\\ &\le \lambda \max\{\rho(Sw,Tw^*,w^*),\rho(Sw,Aw,w^*),\rho(Tw^*,Bw^*,w^*)\}\\ &= \lambda\rho(w,w,w^*).\end{aligned} \tag{2.54}$$

Similarly,

$$\rho(w,w,w^*) \le \lambda\rho(w,w^*,w^*). \tag{2.55}$$

Substituting (2.54) into (2.55),

$$\rho(w,w,w^*) \le \lambda^2\rho(w,w,w^*), \tag{2.56}$$

which is a contradiction. Hence, $w = w^*$. This completes the proof. □

Letting $S = T$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *Let A, B, and S be three selfmappings of a D-metric space satisfying (2.11) and*

$$\rho(Ax,By,z) \le \lambda \max\{\rho(Sx,Sy,z),\rho(Sx,Ax,z),\rho(Sy,By,z)\} \tag{2.57}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$.

Further assume that

(a) $\overline{O_{A,B}(Sx)}$ *is complete for each $x \in X$,*

(b) $\{A,S\}$ *and* $\{B,S\}$ *are limit coincidentally commuting,*

(c) *any one of A, B, and S is continuous.*

Then A, B, and S have a unique common fixed point.

Acknowledgments. We would like to express our thanks to the referee for his helpful suggestions. This work was supported by grant no. 2001-1-10100-005-2 from the Basic Research Program of the Korea Science and Engineering Foundation.

References

[1] B. C. Dhage, *On two general existence principles for proving the common fixed point theorems in D-metric spaces*, to appear in Indian J. Pure Appl. Math.

[2] ———, *A study of some fixed point theorems*, Ph.D. thesis, Marathwada University, Aurangabad, India, 1984.

[3] ———, *Generalised metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), no. 4, 329–336.

[4] ———, *On common fixed points of pairs of coincidentally commuting mappings in D-metric spaces*, Indian J. Pure Appl. Math. **30** (1999), no. 4, 395–406.

[5] ———, *Generalized metric spaces and topological structure. I*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **46** (2000), no. 1, 3–24.

[6] B. C. Dhage and B. E. Rhoades, *Proving fixed point theorems in D-metric spaces via general existence principle*, to appear in Indian J. Pure Appl. Math.

B. C. Dhage: Gurukul Colony, Ahmedpur-413 515, Latur District (Maharashtra), India
E-mail address: bcd20012001@yahoo.co.in

Smrati Arya: Chhatrapati Shivaji Ward, Kothari Gali, Harda-461331, India

Jeong Sheok Ume: Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea
E-mail address: jsume@changwon.ac.kr