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ASYMPTOTIC TRACTS OF HARMONIC FUNCTIONS III

KARL F. BARTH

Department of Mathematics
Syracuse University
Syracuse, NY 13244, USA
e-mail: kfbarth@mailbox.syr.edu

DAVID A. BRANNAN

The Open University
Department of Pure Mathematics
Milton Keynes MK7 6AA, United Kingdom
e-mail: D.A.Brannan@open.ac.uk

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ABSTRACT. A *tract* (or *asymptotic tract*) of a real function u harmonic and nonconstant in the complex plane $\mathcal{C}$ is one of the n_c components of the set $\{z : u(z) \neq c\}$, and the *order* of a tract is the number of non-homotopic curves from any given point to ∞ in the tract. The authors prove that if $u(z)$ is an entire harmonic polynomial of degree n, if the critical points of any of its analytic completions f lie on the level sets $\tau_j = \{z : u(z) = c_j\}$, where $1 \leq j \leq p$ and $p \leq n-1$, and if the total order of all the critical points of f on τ_j is denoted by σ_j, then

$$\{n_c : c \in \Re\} = \{n+1\} \cup \{n+1+\sigma_j : 1 \leq j \leq p\}.$$

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1. INTRODUCTION

This paper continues a study, begun in [1] and [2] , of the asymptotic tracts of functions harmonic in $\mathcal{C}$ (entire harmonic functions).

Definition 1. *An asymptotic tract (or tract) of a real function $u(z)$ harmonic and nonconstant in $\mathcal{C}$ is a component of the set $\{z : u(z) \neq c\}$ for some real number c.*

It was shown in [1] that each tract T is necessarily simply-connected and unbounded, and that u is necessarily unbounded in each tract T; in addition, ∞ is an accessible boundary point (in $\mathcal{C}$) of each tract T. The local mapping properties of analytic functions show that the set $\{z : u(z) \neq c\}$ consists of a finite or countable number of curves which are locally analytic, except at the zeros of $f'(z)$ (where f' is any analytic completion of u)—where the set $\{z : u(z) = c\}$ branches. Observe that the angle between the 'branches' must be equal to $2\pi/n$ for some $n \geq 1$.

We continue the study of harmonic polynomials in the plane initiated in [3], where it was shown that, if $u(z)$ is a harmonic polynomial in $\mathcal{C}$ of degree n, then the number, k, of tracts of u satisfies the sharp inequality

$$n+1 \leq k \leq 2n \tag{1}$$

A special case of our results, putting Example 2 together with Theorem 1, shows that, given any pair of positive integers n and k that satisfy the inequality (1) there is a harmonic polynomial $u(z)$ of degree n with k tracts. This is stronger than [3, Theorem 3] where it was shown that there exists a harmonic polynomial of degree n that has $2n$ tracts for the case $c = 0$.We also discover a restriction, for each given harmonic polynomial $u(z)$ in $\mathcal{C}$, on the number of tracts of $u(z) - c$, as the constant c varies over $\Re$.

Definition 2. *An unbounded simply-connected domain T in C is said to be* branched of order *n_T (possibly $n_T = +\infty$) if it has the following property: There exists a family $\mathcal{T}_T$ of n_T non-homotopic (in T) and disjoint (except for the end-point z_T) Jordan curves in T connecting some fixed point in T, z_T say, to ∞; in addition, any Jordan curve in T joining z_T to ∞ is homotopic* *(in T) to one of the elements of $\mathcal{T}_T$. If $n_T = 1$, we say that T is* unbranched*; if $n_T < +\infty$, we say that T is* finitely branched*; if $n_T = +\infty$, we say that T is* infinitely-branched.

2. Numbers of tracts

Let $u(z)$ be an entire harmonic polynomial of degree n. Then, if $z = re^{i\theta}$, we have that

$$u(z) = a_n r^n cos(n\theta + \theta_n) + O(r^{n-1}), \text{ where } a_n \neq 0. \tag{2}$$

It follows that near ∞ there must be on $\{z : |z| = r\}$ at least n arcs (each of angular length about π/n) on which $u(z) > 0$, and at least n arcs (each of angular length about π/n) on which $u(z) < 0$. Since u is a polynomial of degree n and so can have at most $2n$ zeros on $\{z : |z| = r\}$, it follows that for sufficiently large r there are precisely n arcs of each type. Also, it is easy to prove that the boundaries separating the $2n$ regions comprising $\{z : |z| = r, u(z) \neq 0\}$ tend to radial lines of angular separation π/n as $r \to +\infty$.

We will denote by n_c the number of components of the set $\{z : u(z) - c \neq 0\}$. It will be useful to examine how n_c varies with c. For sufficiently large r, the set $\{z : |z| > R\} \cap \{z : u(z) \neq 0\}$ consists of precisely $2n$ unbounded disjoint domains. Then, for such an r, we define

$$M = 1 + \max\{u(z) : |z| \leq r\}. \tag{3}$$

It follows that the set $\{z : u(z) - M \neq 0\}$ has exactly n components in which $\{z : u(z) - M > 0\}$ and exactly one component in which $\{z : u(z) - M < 0\}$. Thus $n_M = n + 1$. Also, it follows from the Phragmen-Lindelörf Principle that $n_c = n + 1$ when $c > M$. We now look at how n_c varies as c decreases from the value M. The components (tracts) of $\{z : u(z) - c \neq 0\}$ vary continuously with c, in terms of kernel convergence. Hence, as c decreases, n_c is an integer and varies continuously with c (hence remains constant)— except at those values of c for which a critical point of the analytic completion of u lies on the set $\{z : u(z) = c\}$.

Now two tracts of $u(z) - c$ in which $u(z) - c$ has opposite signs can never lie in a single tract of $u(z) - c_1$, for $c_1 \neq c$, since u is unbounded in any tract; however their boundaries may meet in a point or in an arc. No two tracts of $u(z) - c$ can have the property that their boundaries meet in a set with more than one component: for, if they did, then there would be a bounded (non-empty) domain on whose boundary $u(z) = c$, and so we would have $u(z) \equiv c$ in $\mathcal{C}$.

Suppose that T_1 and T_2 are two tracts of $u(z) - c$ in which $u(z) - c > 0$; we will call such tracts upper tracts (for the value c). (Lower tracts are defined similarly.) It may be that $\partial T_1 \cap \partial T_2 = \emptyset$. However we cannot have a situation where $\partial T_1 \cap \partial T_2$ contains an arc in $\mathcal{C}$, by the Maximum Principle. It follows, then, that, if ∂T_1 meets ∂T_2, the set $\partial T_1 \cap \partial T_2$ must be a singleton.

If T_1 and T_2 are both upper tracts or both lower tracts for which $\partial T_1 \cap \partial T_2 = \{z_0\}$, then there must exist an equal number of upper and lower tracts whose boundaries contain z_0. Since z_0 must thus be a critical point of any analytic completion of u, there can be at most $(n - 1)$ such points z_0 (since u is a polynomial of degree n). Note also that, as c decreases, the upper tracts individually increase in size. Hence their total number must decrease as c decreases.

Our main result in this Section is the following.

Theorem 1. *Let $u(z)$ be an entire harmonic polynomial of degree n. Let the critical points of any of its analytic completions f lie on the level sets $\tau_j = \{z : u(z) = c_j\}$, where $1 \leq j \leq p$ and $p \leq n - 1$, and let the total order of all the critical points of f on τ_j be denoted by σ_j. (In particular, $\sum_{j=1}^{p} \sigma_j = n - 1$.) Then $\{n_c : c \in \Re\} = \{n + 1\} \cup \{n + 1 + \sigma_j : 1 \leq j \leq p\}$.*

Proof. Let f be any analytic completion of u.

Case 1. All the critical points of f lie on different level sets for u.

Assume first that all the critical points of f are simple; then we may choose our notation so that they lie on the level sets $\tau_j = \{z : u(z) = c_j\}, 1 \leq j \leq n - 1$, where $c_1 > c_2 > ... > c_{n-1}$. Then, by the previous comments, for $c > c_1$ (for example, when $c = M$ (see (3)), we have $n_c = n + 1$ and there are n upper tracts of u and one lower tract. Next, $n_{c_1} = n + 2$ and there are, for the value $c = c_1$, n upper tracts and two lower tracts (the lower tract has 'split' in two). Finally, for $c_1 > c > c_2$, we have $n_c = n + 1$, and there are $(n - 1)$ upper tracts (two upper tracts have 'combined') and 2 lower tracts.

As c decreases further, a similar argument holds for each c_j in turn, $2 \leq j \leq n - 1$. For $c_{j-1} > c > c_j$, we have $n_c = n + 1$ and there are $(n + 1 - j)$ upper tracts and j lower tracts; when $c = c_j$, we have $n_{c_j} = n + 2$ and there are $(n + 1 - j)$ upper tracts and $(j + 1)$ lower tracts; and, for $c_j > c > c_{j+1}$ (with the convention that $c_n = -\infty$), we have $n_c = n + 1$ and there are $(n - j)$ upper tracts and $(j + 1)$ lower tracts.

Assume next that the critical points of f are not necessarily simple. First, suppose that the level set $\{z : u(z) = c_j\}$, for some particular value of j, contains a critical point of f (at z_j where f' has a zero of order b_j).Let I be an open interval of $\Re$ that contains c_j but contains no other c's corresponding to critical points of f Then, for a sufficiently small neighborhood $\mathcal{U}$ of z_j there are $(2b_j + 2)$ tracts of $u(z) - c_j$ that meet $\mathcal{U}$, namely $(b_j + 1)$ upper tracts and $(b_j + 1)$ lower tracts. However, when $c > c_j, c \in I$ and $c - c_j$ is sufficiently small, there are only $(b_j + 2)$ tracts of $u(z) - c$ that meet $\mathcal{U}$, namely $(b_j + 1)$ upper tracts and 1 lower tract; similarly, when $c < c_j, c \in I$ and $c_j - c$ is sufficiently small, there are $(b_j + 2)$ tracts of $u(z) - c$ that meet $\mathcal{U}$, namely $(b_j + 1)$ lower tracts and 1 upper tract.

Now consider the level set $\{z : u(z) = c\}$ for an arbitrary c. Since, except for values of c corresponding to critical points of f (and even then locally only in small neighborhoods of the critical points themselves) the tracts vary continuously with c (in the sense of kernel convergence), it follows from the above argument that there is some number N such that, for $|c - c_j|$ sufficiently small and non-zero, we have $n_c = N + 1$ whereas $n_{c_j} = N + 1 + b_j$. But $n_M = n + 1$, so that we must have $N = n$. This completes the proof of Case 1 of the theorem.

Case 2. More than one critical point of f lies on a given level set for u.

Assume first that, for some c_j, the level set $\{z : u(z) = c_j\}$ contains just two branch points, z_1 and z_2, of orders b_1 and b_2 respectively , and that z_1 and z_2 lie on different components, C_1 and C_2 respectively, of $\{z : u(z) = c_j\}$; thus $C_1 \cap C_2 = \emptyset$. It follows that there exists some Jordan curve from ∞ to ∞ that separates C_1 from C_2; this curve can be chosen to lie either in a single component of $\{z : u(z) > c_j\}$ or in a single component of $\{z : u(z) < c_j\}$. By considering the local behavior of u near z_1 and z_2, and by using the fact that components of $\{z : u(z) - d \neq 0\}$ vary continuously with d (except when their boundaries coalesce), it follows that, when $|d - c_j|$ is sufficiently small, we have $n_d = n + 1$ and $n_{c_j} = (n + 1) + b_1 + b_2$. A similar argument works in the case of more than two branch points on a single level set of u, so long as each such branch point lies on a different component of that level set.

Assume next that, for some c_j, the level set $\{z : u(z) = c_j\}$ contains just two branch points, z_1 and z_2, of orders b_1 and b_2 respectively (corresponding to zeros of f' of these orders), and that z_1 and z_2 lie on the same component, C, of $\{z : u(z) = c_j\}$. Then there is a Jordan subarc Γ of C joining z_1 to z_2; let z' be any interior point of this subarc. Since C cannot contain any closed Jordan curves, it follows that there are precisely two tracts, T_1 and T_2, say, of $u(z) - c_j$ that have $\Gamma - \{z_1, z_2\}$ as part of their boundaries; we may assume that $u(z) > c_j$ in T_1 and so that $u(z) < c_j$ in T_2. Similar considerations also show that there is a Jordan curve J_1 in $T_1 \cup \{z'\}$ that joins z' to ∞ inside T_1, and a Jordan curve J_2 in $T_2 \cup \{z'\}$ that joins z' to ∞ inside T_2.

We define $J' = J_1 \cup J_2$. Then J' plays the same role as J did earlier (when it separated C_1 from C_2), and a similar argument to the previous one shows that

$$n_d = \begin{cases} n + 1, & \text{if } d \neq c_j, \text{ and } |d - c_j| \text{ sufficiently small,} \\ n + 1 + (b_1 + b_2), & \text{if } d = c_j. \end{cases} \tag{4}$$

Again a similar argument can be used even when there are more than two branch points on the same component of the level set.

The result of this theorem is stronger than [3, Theorem 1] , where it was shown that $\{n_c : c \in \Re\}$ is a subset of $\{n + 1, n + 2, ..., 2n\}$.

Notice that for the function $u_1(z) = \text{Re}(z^n)$ we have $n_0 = 2n$ and $n_1 = n + 1$, and that in fact $\{n_c : c \in \Re\} = \{n + 1, 2n\}$. The next two examples show that, while this particular function u_1 is extremal in a certain sense, the conclusion of Theorem 1 concerning the range of possible values of n_c (as c varies) is best-possible.

Example 1. *There exists a harmonic polynomial u of degree n for which $\{n_c : c \in \Re\} = \{n + 1, n + 2\}$, and all the critical points of any analytic completion f of u are simple and lie on different level sets of u.*

Let $u(z) = \text{Re}(z^n - Az)$, for a complex number A yet to be specified. The analytic completion $f(z) \equiv z^n - Az$ of u has critical points where $nz^{n-1} - A = 0$; that is, where

$$z = z_k = \left(\frac{A}{n}\right)^{\frac{1}{n-1}} \exp\left(\frac{2\pi i k}{n - 1}\right), \quad 1 \le k \le n - 1. \tag{5}$$

Now $u(z_k) = \mathrm{Re}(\frac{z_k A(1-n)}{n})$;and it follows that,if A is chosen with $|A| = 1$ and with $\arg A$ not a rational multiple of 2π, then all the values of $\{u(z_k)\}_1^{n-1}$ are distinct. Thus u has the desired properties.

Example 2. *Let p be an integer, such that $1 \leq p \leq n-1$, and let $b_1, b_2, .., b_p$ be any integers in $[1, n-2]$ for which $\sum_{j=1}^{p} b_j = n-1$. There exists a harmonic polynomial $u(z)$ of degree n with the properties that any analytic completion f of u has critical points of orders $b_1, b_2, \ldots, b_p$, and that all these critical points lie on different level sets of u. Hence (from Theorem 1)*

$$\{n_c : c \in \Re\} = \{n+1\} \cup \{n+1+b_j : 1 \leq j \leq p\}.$$

First, let f_1 be the polynomial given by $f_1(0) = 0$ and

$$f_1'(z) = (z-1)^{b_1}(z-a_2)^{b_2} z^{(n-1)-(b_1+b_2)}, \tag{6}$$

where a_2 is chosen to be either $\frac{1}{2}$ or to be very close to $\frac{1}{2}$; in particular, we make our choice of a_2 to ensure that $u_1(1) \neq 0$, where $u_1(z) \equiv \mathrm{Re} f_1(z)$. It follows from Rolle's Theorem that the points 1 and a_2 lie on different level sets of u_1.

Next, let f_2 be the polynomial given by $f_2(0) = 0$ and

$$f_2'(z) = (z-1)^{b_1}(z-a_2)^{b_2}(z-a_3)^{b_3} z^{(n-1)-(b_1+b_2+b_3)}, \tag{7}$$

where a_3 is positive but small. By continuity arguments (on f) we see that we may choose a_3 sufficiently near to 0 that 1 and a_2 lie on different level sets of $u_2(z) \equiv \mathrm{Re} f_2(z)$.

We have to check that a_3 can be chosen so that a_3 lies on a different level set of u_2 from those that contain either 1 or a_2. But, if a_3 is sufficiently small, we have that

$$u_2(1) \approx \int_0^1 (t-1)^{b_1}(t-a_2)^{b_2} t^{(n-1)-(b_1+b_2)}\, dt, \tag{8}$$

$$u_2(a_2) \approx \int_0^{a_2} (t-1)^{b_1}(t-a_2)^{b_2} t^{(n-1)-(b_1+b_2)}\, dt, \tag{9}$$

and

$$u_2(a_3) \approx (-1)^{b_1+b_2+b_3}(a_2)^{b_2}(a_3)^{(n-(b_1+b_2))} \int_0^1 (1-x)^{b_3} x^{(n-1)-(b_1+b_2+b_3)}\, dx;$$

hence, for all sufficiently small a_3, the values of $u_2(1)$, $u_2(a_2)$ and $u_2(a_3)$ are all distinct.

A similar argument shows, after a further $(p-3)$ steps, that the polynomial $u(z) \equiv \mathrm{Re} f(z)$, where $f(0) = 0$ and

$$f'(z) = (z-1)^{b_1}(z-a_2)^{b_2}(z-a_3)^{b_3}(z-a_4)^{b_4} \ldots (z-a_p)^{b_p}, \tag{10}$$

and the sequence $\{a_j\}_{j=3}^{p}$ decreases to 0 sufficiently quickly, has the desired properties.

Finally, as was mentioned in the Introduction, suppose n and k are positive integers such that $n+1 \leq k \leq 2n$. If $k = n+1$, we see from Example 1 that there exists a harmonic polynomial $u(z)$ such that $n_c = \{n+1, n+2\}$. If $n+1 < k \leq 2n$ and we set $b_1 = k-(n+1)$ and $b_2 = 2n-k$, Example 2 shows that there exists a harmonic polynomial $u(z)$ such that

$$\{n_c\} = \{n+1, n+1+b_1, n+1+b_2\} = \{n+1, k, 3n+1-k\}.$$

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