Internat. J. Math. & Math. Sci.
VOL. 19 NO. 3 (1996) 581-586

EIGENVALUES OF THE NEGATIVE LAPLACIAN FOR ARBITRARY MULTIPLY CONNECTED DOMAINS

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(Received October 6, 1994 and in revised form January 17, 1995)

ABSTRACT. The purpose of this paper is to derive some interesting asymptotic formulae for spectra of arbitrary multiply connected bounded domains in two or three dimensions, linked with variation of positive distinct functions entering the boundary conditions, using the spectral function $\sum_{k=1}^{\infty}\{\mu_k(\sigma_1,...,\sigma_n)+P\}^{-2}$ as $P\to\infty$. Further results may be obtained.

KEY WORDS AND PHRASES. Inverse problem, negative Laplacian, eigenvalues, spectral functions.
1991 AMS SUBJECT CLASSIFICATION CODES. 35K, 35P.

1. INTRODUCTION.

The underlying inverse eigenvalue problem (1.1)-(1.2) has been discussed recently by Zayed [1] and has shown that some geometric quantities associated with a bounded domain can be found from a complete knowledge of the eigenvalues $\{\mu_k(\sigma)\}_{k=1}^{\infty}$ for the negative Laplacian $-\Delta_n = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i}\right)^2$ in $R^n (n=2 \text{ or } 3)$.

Let Ω be a simply connected bounded domain in R^n with a smooth boundary $\partial\Omega$ in the case $n=2$ (or a smooth bounding surface S in the case $n=3$). Consider the impedance problem

$$-\Delta_n u=\lambda u \quad in \quad \Omega\,, \tag{1.1}$$

$$\left(\frac{\partial}{\partial n}+\sigma\right)u=0 \quad \text{on} \quad \partial\Omega \quad (\text{or } S)\,, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial\Omega$ or S, and σ is a positive function.

Denote its eigenvalues, counted according to multiplicity, by

$$0<\mu_1(\sigma)\le\mu_2(\sigma)\le\ldots\le\mu_k(\sigma)\le\ldots\to\infty \quad \text{as} \quad k\to\infty\,. \tag{1.3}$$

It is well known [2] that in the case $n=2$

$$\mu_k(\sigma)\sim\left(\frac{4\pi}{|\Omega|}\right)k \quad \text{as} \quad k\to\infty\,, \tag{1.4}$$

while in the case $n=3$

$$\mu_k(\sigma) \sim \left(\frac{6\pi^2}{V}k\right)^{2/3} \quad \text{as} \quad k \to \infty , \tag{1.5}$$

where $|\Omega|$ and V are respectively the area and the volume of Ω.

The purpose of this paper is to discuss the following more general inverse problem: Let Ω be an arbitrary multiply connected bounded domain in $R^n (n = 2 \text{ or } 3)$ which is surrounded internally by simply connected bounded domains Ω_i with smooth boundaries $\partial\Omega_i$ in the case $n = 2$ (or smooth bounding surfaces S_i in the case $n = 3$) where $i = 1, 2, ..., m-1$, and externally by a simply connected bounded domain Ω_m with a smooth boundary $\partial\Omega_m$ in the case $n = 2$ (or a smooth bounding surface S_m in the case $n = 3$). Suppose that the eigenvalues

$$0 < \mu_1(\sigma_1, ..., \sigma_m) \leq \mu_2(\sigma_1, ..., \sigma_m) \leq ... \leq \mu_k(\sigma_1, ..., \sigma_m) \leq ... \longrightarrow \infty \quad \text{as} \quad k \to \infty \tag{1.6}$$

are known exactly for the impedance problem

$$-\Delta_n u = \lambda u \quad \text{in} \quad \Omega , \tag{1.7}$$

$$\left(\frac{\partial}{\partial n_i} + \sigma_i\right) u = 0 \quad \text{on} \quad \partial\Omega_i \quad (\text{or } S_i) , \tag{1.8}$$

where $\frac{\partial}{\partial n_i}$ denote differentiations along the inward pointing normals to $\partial\Omega_i$ or S_i respectively and σ_i are positive functions $(i = 1, ..., m)$.

In Theorem 2.1, we determine some geometric quantities associated with the multiply connected domain Ω from the complete knowledge of the eigenvalues (1.6) for the problem (1.7)-(1.8) using the asymptotic expansion of the spectral function

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma_1, ..., \sigma_m) + P]^2} \quad \text{as} \quad P \to \infty , \tag{1.9}$$

where P is a positive constant, while σ_i are positive functions defined on $\partial\Omega_i$ or S_i $(i = 1, ..., m)$ and satisfying the Lipschitz condition.

In Theorem 2.2, we show that the asymptotic expansion of (1.9) as $P \to \infty$ plays an important role in establishing a method to study the asymptotic behavior of the difference

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} [\mu_k(\alpha_1, ..., \alpha_m) - \mu_k(\beta_1, ..., \beta_m)] \quad \text{as} \quad \lambda \to \infty , \tag{1.10}$$

where $\sigma_i(Q)$, $\alpha_i(Q)$, $\beta_i(Q)$ $Q \in \partial\Omega_i$ (or $Q \in S_i$), $(i = 1, ..., m)$ are, generally speaking, distinct functions and satisfying the Lipschitz condition and the summation is taken over all values of k for which $\mu_k(\sigma_1, ..., \sigma_m) \leq \lambda$.

Note that theorems and corollaries of this paper contain further results similar to those obtained recently by Zayed and Younis [2].

2. STATEMENT OF RESULTS

Using methods similar to those obtained in [1], [2], we can easily prove the following theorems:

THEOREM 2.1. If the functions $\sigma_i(Q)$, $Q \in \partial\Omega_i$ (or $Q \in S_i$), $(i = 1, ..., m)$ satisfy the Lipschitz condition, then in the case $n = 2$

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma_1,\ldots,\sigma_m)+P]^2} = \frac{|\Omega|}{4\pi P} + \left(\sum_{i=1}^{m} L_i\right) \Big/ 16P^{3/2}$$
$$+ \frac{1}{6P^2}\left\{(2-m) + \frac{3}{\pi}\left[\sum_{i=1}^{m} \int_{\partial\Omega_i} \sigma_i(Q)dQ - 2\sum_{i=2}^{m} \int_{\partial\Omega_i} \sigma_i(Q)dQ\right]\right\}$$
$$+ \frac{21}{1024P^{5/2}} \sum_{i=1}^{m} \int_{\partial\Omega_i} \left\{K_i^2(Q) - \frac{32}{7}[\sigma_i(Q)K_i(Q)\right.$$
$$\left. - 2\sigma_i^2(Q)]\right\}dQ + 0\left(\frac{1}{P^3}\right) \quad \text{as} \quad P \to \infty , \tag{2 1}$$

where L_i and $K_i(Q)$ $(i = 1,\ldots,m)$ are respectively the total lengths and the curvatures of $\partial\Omega_i$ at the point Q, while in the case $n = 3$ we deduce that

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma_1,\ldots,\sigma_m)+P]^2} = \frac{V}{8\pi P^{1/2}} + \left(\sum_{i=1}^{m} |S_i|\right) \Big/ 16\pi P$$
$$+ \frac{1}{24\pi P^{3/2}} \sum_{i=1}^{m} \int_{S_i} [H_i(Q) - 3\sigma_i(Q)]dQ$$
$$+ \frac{7}{128\pi P^2} \sum_{i=1}^{m} \int_{S_i} \left\{[H_i(Q) - 3\sigma_i(Q)]^2\right.$$
$$\left. - \left[N_i(Q) - \frac{26}{7}\sigma_i(Q)H_i(Q) + \frac{47}{7}\sigma_i^2(Q)\right]\right\}dQ$$
$$+ \frac{39}{5760\pi P^{5/2}} \sum_{i=1}^{m} \int_{S_i} [H_i(Q) - 3\sigma_i(Q)]^3 dQ$$
$$+ 0\left(\frac{1}{P^3}\right) \quad \text{as} \quad P \to \infty , \tag{2 2}$$

where $|S_i|$, $H_i(Q)$ and $N_i(Q)$ are respectively the surface areas, mean curvatures and Gaussian curvatures of the bounding surfaces $S_i (i = 1,..,m)$

Formulae (2 1) and (2 2) can be considered as a generalization of the formula (2 3) obtained by Zayed [3] and the formula (2 3) obtained by Zayed [4] respectively

THEOREM 2.2. If the functions $\sigma_i(Q)$, $\alpha_i(Q)$, $\beta_i(Q)$, $Q \in \partial\Omega_i$ (or $Q \in S_i$) $(i = 1,\ldots,m)$ are distinct and satisfy the Lipschitz condition, then we deduce for $\lambda \to \infty$ that

$$\sum_{\mu_k(\sigma_1,\ldots,\sigma_m)\le\lambda} [\mu_k(\alpha_1,\ldots,\alpha_m) - \mu_k(\beta_1,\ldots,\beta_m)] = \begin{cases} \left[\frac{a_1}{2\pi} + o(1)\right]\lambda & \text{in the case } n = 2, \quad (2\ 3) \\ \left[\frac{b_1}{3\pi^2} + o(1)\right]\lambda^{3/2} & \text{in the case } n = 3, \quad (2\ 4) \end{cases}$$

where

$$a_1 = 2\sum_{i=2}^{m} \int_{\partial\Omega_i} [\alpha_i(Q) - \beta_i(Q)]dQ - \sum_{i=1}^{m} \int_{\partial\Omega_i} [\alpha_i(Q) - \beta_i(Q)]dQ ,$$

and

$$b_1 = \sum_{i=1}^{m} \int_{S_i} [\alpha_i(Q) - \beta_i(Q)]dQ .$$

3. FURTHER RESULTS

COROLLARY 3.1. Using formulae (1 4) and (1.5), we deduce as $M \to \infty$ that

$$\sum_{k=1}^{M} [\mu_k(\alpha_1, \dots, \alpha_m) - \mu_k(\beta_1, \dots, \beta_m)] = \begin{cases} \left[\frac{2a_1}{|\Omega|} + o(1)\right] M & \text{in the case } n = 2, \quad (3.1) \\ \left[\frac{2b_1}{V} + o(1)\right] M & \text{in the case } n = 3, \quad (3.2) \end{cases}$$

Using Theorem 2.2, we easily prove the following theorems:

THEOREM 3.1. Let the functions $\sigma_i(Q)$, $\alpha_i(Q)$, $\beta_i(Q)$, $Q \in \partial\Omega_i$ $(i = 1, \dots, m)$ and the quantity $a_1 \neq 0$ be the same as in (2.3) Furthermore, on the half-axis $[c, +\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[c, d]$, $d < \infty$; further we assume that the expression $\lambda f'(\lambda)/f(\lambda)$ is bounded almost everywhere. Then

(i) If $\int_c^{+\infty} f(\lambda) d\lambda = \infty$, we deduce for $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\dots,\sigma_m)\le\lambda} f[\mu_k(\sigma_1, \dots, \sigma_m)]\{\mu_k(\alpha_1, \dots, \alpha_m) - \mu_k(\beta_1, \dots, \beta_m)\} = \left[\frac{a_1}{2\pi} + o(1)\right] \int_c^{\lambda} f(t) dt . \quad (3.3)$$

(ii) If $\lim_{\lambda\to\infty} f(\lambda) = \infty$, we deduce for $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\dots,\sigma_m)\le\lambda} \{f[\mu_k(\alpha_1 \dots, \alpha_m)] - f[\mu_k(\beta_1, \dots, \beta_m)]\} = \left[\frac{a_1}{2\pi} + o(1)\right] f(\lambda) . \quad (3.4)$$

THEOREM 3.2. Let the functions $\sigma_i(Q)$, $\alpha_i(Q)$, $\beta_i(Q)$, $Q \in S_i$ $(i = 1, \dots, m)$ and the quantity $b_1 \neq 0$ be the same as in (2.4). Furthermore, on the half-axis $[c, +\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[c, d]$, $d < \infty$; further we assume that the expression $\lambda f'(\lambda)/f(\lambda)$ is bounded almost everywhere. Then

(i) If $\int_c^{+\infty} \lambda^{1/2} f(\lambda) d\lambda = \infty$, we deduce for $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\dots,\sigma_m)\le\lambda} f[\mu_k(\sigma_1, \dots, \sigma_m)]\{\mu_k(\alpha_1, \dots, \alpha_m) - \mu_k(\beta_1, \dots, \beta_m)\} = \left[\frac{b_1}{2\pi^2} + o(1)\right] \int_c^{\lambda} |t|^{1/2} f(t) dt . \quad (3.5)$$

(ii) If $\int_c^{+\infty} \lambda^{1/2} f'(\lambda) d\lambda = \infty$, we deduce for $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\dots,\sigma_m)\le\lambda} \{f[\mu_k(\alpha_1, \dots, \alpha_m)] - f[\mu_k(\beta_1, \dots, \beta_m)]\} = \left[\frac{b_1}{2\pi^2} + o(1)\right] \int_c^{\lambda} |t|^{1/2} f'(t) dt . \quad (3.6)$$

COROLLARY 3.2. Assuming that $f(\lambda)$ of Theorem 3.1(i) has the form $f(\lambda) = \lambda^j$, $(j \geq -1)$ then we deduce as $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,\dots,\sigma_m)\le\lambda} \mu_k^j(\sigma_1, \dots, \sigma_m)\{\mu_k(\alpha_1, \dots, \alpha_m) - \mu_k(\beta_1, \dots, \beta_m)\}$$

$$= \begin{cases} \left[\frac{a_1}{2\pi(j+1)} + o(1)\right] \lambda^{j+1} & \text{if } j > -1, \quad (3.7) \\ \left[\frac{a_1}{2\pi} + o(1)\right] \ln \lambda & \text{if } j = -1. \quad (3.8) \end{cases}$$

COROLLARY 3.3. Assuming that $f(\lambda)$ of Theorem 3 2(i) has the form $f(\lambda) = \lambda^j$, $(j \geq -\frac{3}{2})$ then we deduce as $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} \mu_k^j(\sigma_1,...,\sigma_m)\{\mu_k(\alpha_1,...,\alpha_m) - \mu_k(\beta_1,...,\beta_m)\}$$

$$= \begin{cases} \left[\frac{b_1}{\pi^2(2j+3)} + o(1)\right]\lambda^{j+\frac{3}{2}} & \text{if } j > -\frac{3}{2}, \quad (3\ 9) \\ \left[\frac{b_1}{2\pi^2} + o(1)\right]\ln\lambda & \text{if } j = -\frac{3}{2}. \quad (3\ 10) \end{cases}$$

COROLLARY 3.4. Assuming that $f(\lambda)$ of Theorem 3 1(ii) has the form $f(\lambda) = \lambda^j$, $(j > 0)$ then we deduce as $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} [\mu_k^j(\alpha_1,...,\alpha_m) - \mu_k^j(\beta_1,...,\beta_m)] = \left[\frac{a_1}{2\pi} + o(1)\right]\lambda^j . \quad (3\ 11)$$

COROLLARY 3.5. Assuming that $f(\lambda)$ of Theorem 3 2(ii) has the form $f(\lambda) = \lambda^j$, $(j > -\frac{1}{2})$ then we deduce as $\lambda \to \infty$ that

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} [\mu_k^j(\alpha_1,...,\alpha_m) - \mu_k^j(\beta_1,...,\beta_m)]$$

$$= \begin{cases} \left[\frac{jb_1}{\pi^2(2j+1)} + o(1)\right]\lambda^{j+\frac{1}{2}} & \text{if } j > -\frac{1}{2}, \quad (3\ 12) \\ \left[\frac{-b_1}{4\pi^2} + o(1)\right]\ln\lambda & \text{if } j = -\frac{1}{2}. \quad (3.13) \end{cases}$$

COROLLARY 3.6. If $\mu_k(\beta_1,...,\beta_m) \neq 0$ we deduce from Corollaries 3 1 and 3 3 that as $M \to \infty$

$$\sum_{k=1}^{M} \frac{\mu_k(\alpha_1,...,\alpha_m)}{\mu_k(\beta_1,...,\beta_m)} = \begin{cases} M + \left[\frac{a_1}{2\pi} + o(1)\right]\ln\left(\frac{4\pi}{|\Omega|}M\right) & \text{if } n = 2, \quad (3\ 14) \\ M + \left[b_1\left(\frac{6}{\pi^4 V}\right)^{1/3} + o(1)\right]M^{1/3} & \text{if } n = 3. \quad (3\ 15) \end{cases}$$

THEOREM 3.3. Let the functions $\sigma_i(Q)$, $\alpha_i(Q)$, $\beta_i(Q)$ $Q \in \partial\Omega_i$, $(i = 1,...,m)$ and the quantity $a_1 \neq 0$ be the same as in (3 11). Furthermore, on the half-axis $[c + \infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[c, d]$, $d < \infty$, further we assume that the expression $\lambda f'(\lambda)/f(\lambda)$ is bounded almost everywhere and $\int_c^{+\infty} \lambda^{j-1} f(\lambda)d\lambda = \infty (j > 0)$ Then as $\lambda \to \infty$

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} f[\mu_k(\sigma_1,...,\sigma_m)]\left\{\mu_k^j(\alpha_1,...,\alpha_m) - \mu_k^j(\beta_1,...,\beta_m)\right\}$$

$$= \left[\frac{a_1}{2\pi}j + o(1)\right]\int_c^\lambda |t|^{j-1} f(t)dt . \quad (3\ 16)$$

THEOREM 3.4. Let the functions $\sigma_i(Q)$, $\alpha_i(Q)$, $\beta_i(Q)$ $Q \in S_i$, $(i = 1,...,m)$ and the quantity $b_1 \neq 0$ be the same as in (3.12) Furthermore, on the half-axis $[c, +\infty)$ let a function $f(\lambda)$ of constant sign be given which is absolutely continuous on each interval $[c, d]$, $d < \infty$, further we assume that the expression $\lambda f'(\lambda)/f(\lambda)$ is bounded almost everywhere and $\int_c^{+\infty} \lambda^{j-\frac{1}{2}} f(\lambda) d\lambda = \infty \, (j \geq -\frac{1}{2})$. Then as $\lambda \to \infty$

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} f[\mu_k(\sigma_1,...,\sigma_m)]\left\{\mu_k^j(\alpha_1,...,\alpha_m) - \mu_k^j(\beta_1,...,\beta_m)\right\}$$
$$= \left[\frac{b_1}{2\pi^2} j + o(1)\right] \int_c^{\lambda} |t|^{j-\frac{1}{2}} f(t) dt \,. \tag{3.17}$$

COROLLARY 3.7. Assuming that $f(\lambda)$ of Theorem 3.3 has the form $f(\lambda) = \lambda^r$, where r is a real number. Then as $\lambda \to \infty$ we get

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} \mu_k^r(\sigma_1,...,\sigma_m)\left\{\mu_k^j(\alpha_1,...,\alpha_m) - \mu_k^j(\beta_1,...,\beta_m)\right\}$$

$$= \begin{cases} \left[\dfrac{ja_1}{2\pi(r+j)} + o(1)\right]\lambda^{r+j} & \text{if } \; r+j>0, \quad (3.18) \\ \left[\dfrac{a_1}{2\pi} j + o(1)\right] \ln\lambda & \text{if } \; r+j=0. \quad (3.19) \end{cases}$$

COROLLARY 3.8. Assuming that $f(\lambda)$ of Theorem 3.4 has the form $f(\lambda) = \lambda^r$, where r is a real number. Then as $\lambda \to \infty$ we get

$$\sum_{0<\mu_k(\sigma_1,...,\sigma_m)\leq\lambda} \mu_k^r(\sigma_1,...,\sigma_m)\left\{\mu_k^j(\alpha_1,...,\alpha_m) - \mu_k^j(\beta_1,...,\beta_m)\right\}$$

$$= \begin{cases} \left[\dfrac{jb_1}{\pi^2(1+2\gamma+2j)} + o(1)\right]\lambda^{r+j+1/2} & \text{if } \; r+j>-\dfrac{1}{2}, \quad (3.20) \\ \left[\dfrac{b_1}{2\pi^2} j + o(1)\right] \ln\lambda & \text{if } \; r+j=-\dfrac{1}{2}. \quad (3.21) \end{cases}$$

COROLLARY 3.9. If $\mu_k(\beta_1,...,\beta_m) \neq 0$ we deduce from Corollaries 3.7 and 3.8 that as $M \to \infty$

$$\sum_{k=1}^{M}\left[\frac{\mu_k(\alpha_1,...,\alpha_m)}{\mu_k(\beta_1,...,\beta_m)}\right]^j = \begin{cases} M + \left[\dfrac{a_1}{2\pi} j + o(1)\right] \ln\left(\dfrac{4\pi}{|\Omega|}M\right) & \text{if } \; n=2, \quad (3.22) \\ M + \left[b_1\left(\dfrac{6}{\pi^4 V}\right)^{1/3} j + o(1)\right] M^{1/3} & \text{if } \; n=3. \quad (3.23) \end{cases}$$

ACKNOWLEDGMENT. The author expresses his grateful thanks to the referee for some interesting suggestions and comments.

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