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NEW CHARACTERIZATIONS FOR HANKEL TRANSFORMABLE SPACES OF ZEMANIAN

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ABSTRACT. In this paper we obtain new characterizations of the Zemanian spaces H_μ and H'_μ

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A H Zemanian [7, Ch 5] introduced the space H_μ ($\mu \in \mathbb{R}$) of functions as follows a complex valued smooth function $\phi(x)$, $x \in I = (0, \infty)$, is in H_μ if, and only if, the quantity

$$\gamma^\mu_{n,k}(\phi) = \sup_{x \in I} \left| x^n (x^{-1}D)^k (x^{-\mu-1/2}\phi(x)) \right| < \infty$$

is finite, for every $n, k \in \mathbb{N}$. This space endowed the topology generated by $\{\gamma^\mu_{n,k}\}_{n,k\in\mathbb{N}}$ is a Fréchet space In the sequel we will refer to the above topology as the usual topology of H_μ Zemanian introduced the space H_μ to extend the Hankel integral transformation defined by

$$(h_\mu \phi)(x) = \int_0^\infty (xt)^{1/2} J_\mu(xt)\, \phi(t)\, dt \ ,$$

where J_μ denotes the Bessel function of the first kind and order μ, to generalized functions. He proved that h_μ is an automorphism of H_μ provided that $\mu \geq -\frac{1}{2}$. The generalized Hankel transform $h'_\mu f$ of $f \in H'_\mu$, the dual space of H_μ, is defined as the transposed of H_μ through

$$\langle h'_\mu f, \phi \rangle = \langle f, h_\mu \phi \rangle \qquad \text{for} \qquad \phi \in H_\mu \ .$$

Thus if $\mu \geq -\frac{1}{2}$ h'_μ is an automorphism of H'_μ when this space is equipped with the weak* topology or with the strong topology.

In [2] J. J. Betancor and I. Marrero have studied the main topological properties of the spaces H_μ and H'_μ. Amongst other results, it is established (Theorem 3.3) that the space H_μ, $\mu \geq -\frac{1}{2}$, is constituted by all those complex valued smooth functions $\phi(x)$, $x \in I$, such that

$$\tau^\mu_{n,k}(\phi) = \sup_{x\in I} \left| x^n N_{\mu+k-1} \ldots N_\mu \phi(x) \right| < \infty$$

for every $n, k \in \mathbb{N}$. Moreover, the system of seminorms $\{\tau^\mu_{n,k}\}_{n,k\in\mathbb{N}}$ generates of H_μ its usual topology Moreover in [4] they gave new descriptions for the usual topology of H_μ through L_2-norms

A. H. Zemanian [7, p. 134] defined the space O formed by all those complex valued smooth functions $v(x)$, $x \in I$, satisfying that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $(1+x^2)^{n_k}(x^{-1}D)^k v(x)$ is a bounded function on I. He proved that O is a space of multiplier of H_μ Recently J. J. Betancor and I. Marrero [2, Theorems 2.3 and 4.9] have characterized O as the space of multipliers of H_μ and H'_μ.

In this paper we characterize the smooth complex valued functions in H_μ, $\mu \geq -\frac{1}{2}$, as the ones satisfying

$$Z_n(\phi) = \sup_{x\in I} |x^n \phi(x)| < \infty \tag{1}$$

and

$$y_n^\mu(\phi) = \sup_{x\in I} |N_{\mu+n-1}...N_\mu\phi(x)| < \infty \tag{2}$$

for every $n \in N$. Moreover we prove that the usual topology of H_μ can be defined by the family of seminorms $\{Z_n, y_n^\mu\}_{n\in\mathbb{N}}$ and a new characterization for the elements of H'_μ is obtained. In the sequel we will assume that $\mu \geq -\frac{1}{2}$.

PROPOSITION 1. A complex valued smooth function $\phi(x)$, $x \in I$, is in H_μ if, and only if, ϕ satisfies (1) and (2) for every $n \in \mathbb{N}$.

PROOF. It is clear that if $\phi \in H_\mu$ then ϕ satisfies (1) and (2) for every $n \in \mathbb{N}$.

Let now ϕ be a complex valued smooth function defined on I. To see that (1) and (2) ($n \in \mathbb{N}$) are sufficient conditions for ϕ belongs to H_μ we proceed by induction. Suppose, as induction hypothesis, that

$$\sup_{x\in I} |x^m N_{\mu+n-1}...N_\mu\phi(x)| < \infty, \qquad m \in \mathbb{N} \qquad \text{and} \qquad n \in \mathbb{N},\ 0 \leq n < \ell$$

for certain $\ell \in \mathbb{N}$, $\ell \geq 1$.

By using partial integration we can obtain

$$\begin{aligned}\|x^m N_{\mu+\ell-1}...N_\mu\phi(x)\|_2^2 &= \int_0^\infty |x^m N_{\mu+\ell-1}...N_\mu\phi(x)|^2\,dx \\ &= \int_0^\infty x^{2m} N_{\mu+\ell-1}...N_\mu(\phi(x))N_{\mu+\ell-1}...N_\mu(\overline{\phi}(x))\,dx \\ &= \int_0^\infty (Dx^{-1})^\ell(x^{2m+\mu+\ell+1/2}N_{\mu+\ell-1}...N_\mu(\phi(x)))x^{-\mu-1/2}\overline{\phi}(x)\,dx\end{aligned}$$

for every $m \in \mathbb{N}$, $\ell < 2m+2$, because

$$\Big[(Dx^{-1})^i(x^{2m+\mu+\ell+1/2}N_{\mu+\ell-1}...N_\mu(\phi(x)))(x^{-1}D)^{\ell-i-1}(x^{-\mu-1/2}\overline{\phi}(x))\Big]_0^\infty = 0 \tag{3}$$

for each $i, m \in \mathbb{N}$, $0 \leq i < \ell < 2m+2$. In effect, if $m, i \in \mathbb{N}$, $0 \leq i < \ell < 2m+2$ then Leibniz's rule leads to

$$\begin{aligned}&(Dx^{-1})^i(x^{2m+\mu+\ell+1/2}N_{\mu+\ell-1}...N_\mu(\phi(x)))(x^{-1}D)^{\ell-i-1}(x^{-\mu-1/2}\overline{\phi}(x)) \\ &\quad= \sum_{j=0}^{i} a_j x^{2m+2\ell+2\mu+1-2j}(x^{-1}D)^{\ell+i-j}(x^{-\mu-1/2}\phi(x))(x^{-1}D)^{\ell-i-1}(x^{-\mu-1/2}\overline{\phi}(x)) \\ &\quad= \sum_{j=0}^{i} a_j x^{2m+1-j} N_{\mu+\ell+i-j-1}...N_\mu(\phi(x))N_{\mu+\ell-i-2}...N_\mu(\phi(x))\end{aligned}$$

where a_j, $j \in \mathbb{N}$, $0 \leq j \leq i$, are suitable real numbers, and by virtue of induction hypothesis (3) follows.

Most straightforward manipulations allow us to write

$$(Dx^{-1})^\ell(x^{2m+\mu+\ell+1/2}N_{\mu+\ell-1}...N_\mu(\phi(x)))x^{-\mu-1/2}\overline{\phi}(x) = \sum_{j=0}^{\ell} a_j x^{2m-j}\overline{\phi}(x)N_{\mu+2\ell-j-1}...N_\mu\phi(x)$$

with $m \in \mathbb{N}$ and $a_j \in \mathbb{R}$, $j \in \mathbb{N}$, $0 \leq j \leq \ell$.

Hence we can establish

$$\begin{aligned}\|x^m N_{\mu+\ell-1}...N_\mu\phi(x)\|_2^2 &\leq C_1 \sum_{j=0}^{\ell}\int_0^\infty |x^{2m-j}\overline{\phi}(x)|\,|N_{\mu+2\ell-j-1}...N_\mu\phi(x)|\,dx \\ &\leq C_2 \sum_{j=0}^{\ell} \sup_{x\in I} |(1+x^2)x^{2m-j}\phi(x)| \sup_{x\in I} |N_{\mu+2\ell-j-1}...N_\mu\phi(x)| < \infty,\end{aligned} \tag{4}$$

provided that $m \in \mathbb{N},\ 2m \geq \ell$. Here $C_i,\ i = 1, 2$, denotes suitable positive constants.

Assume now that $m \in \mathbb{N},\ 2m < \ell$. We have

$$\|x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)\|_2^2 = \left(\int_0^1 + \int_1^\infty\right) |x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)|^2 \, dx$$
$$\leq \int_0^1 |N_{\mu+\ell-1} \ldots N_\mu \phi(x)|^2 \, dx + \int_0^\infty \left|x^\ell N_{\mu+\ell-1} \ldots N_\mu \phi(x)\right|^2 dx \, .$$

Therefore, by invoking (4) and the induction hypothesis we infer that

$$\|x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)\|_2 < \infty \, , \qquad \text{when} \qquad m \in \mathbb{N} \, , \ 2m \leq \ell \, .$$

Thus it is concluded that $\|x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)\|_2 < \infty,\ \ m \in \mathbb{N}$.

Also, for every $m \in \mathbb{N},\ \ m \geq 1$, and $x \in I$,

$$(x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x))^2 = \int_0^x D_t (t^m N_{\mu+\ell-1} \ldots N_\mu \phi(t))^2 \, dt$$
$$= \int_0^x 2t^m N_{\mu+\ell-1} \ldots N_\mu (\phi(t)) ([m + \mu + \frac{1}{2} + \ell] t^{m-1} N_{\mu+\ell-1} \ldots N_\mu (\phi(t)) + t^m N_{\mu+\ell} \ldots N_\mu (\phi(t))) \, dt \, .$$

Hence if $m \in \mathbb{N},\ \ m \geq 1$, and $x \in I$ by using Holder's inequality we can find $C \geq 0$ such that

$$|x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)|^2 \leq C \Big(\|x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)\|_2 \left\|x^{m-1} N_{\mu+\ell-1} \ldots N_\mu \phi(x)\right\|_2$$
$$+ \sup_{x \in I} |N_{\mu+\ell} \ldots N_\mu \phi(x)| \left[\|x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)\|_2 + \left\|x^{m+1} N_{\mu+\ell-1} \ldots N_\mu \phi(x)\right\|_2\right] \Big)$$

and then $\sup_{x \in I} \ |x^m N_{\mu+\ell-1} \ldots N_\mu \phi(x)| < \infty,\ \ m \in \mathbb{N}$.

Thus the proof is finished.

The last proposition allows us to define the usual topology of H_μ through a family of seminorms simpler than $\{\gamma^\mu_{m,k}\}_{m,k \in \mathbb{N}}$.

PROPOSITION 2. The usual topology of H_μ is defined by the system of seminorms $\{Z_n, y^\mu_n\}_{n \in \mathbb{N}}$.

PROOF. It is clear that the topology generated by $\{\gamma^\mu_{m,k}\}_{m,k \in \mathbb{N}}$ is finer than the one defined by $\{Z_n, y^\mu_n\}_{n \in \mathbb{N}}$ on H_μ. Moreover by proceeding in a way similar to A. H. Zemanian [7, Lemma 5.2-2] we can prove that H_μ endowed with the topology generated by $\{Z_n, y^\mu_n\}_{n \in \mathbb{N}}$ is a Fréchet space. Hence the desired result is an immediate consequence of the Open Mapping Theorem [6, Corollary 2.12].

We now prove a new characterization for the elements of H'_μ the dual space of H_μ. The procedure employed is analogous to the one used by the author [1] and by J. J. Betancor and I. Marrero [2].

PROPOSITION 3. Let f be a linear functional defined on H_μ. Then f is in H'_μ if, and only if, there exist $r \in \mathbb{N}$ and $f_k,\ g_k \in L_\infty(0, \infty)$ (the space of essentially bounded functions on $(0, \infty)$), $k \in \mathbb{N},\ 0 \leq k \leq r$, such that

$$f = \sum_{k=0}^{r} h'_\mu (x^k f_k + x^{-\mu+1/2} (x^{-1} D)^k x^{k+\mu-1/2} g_k) \, . \tag{5}$$

PROOF. Let $f \in H'_\mu$. By virtue of a well-known result ([7, Theorem 1.8-1]) there exist $r \in \mathbb{N}$ and $C > 0$ such that

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \ \{Z_k(\phi), y^\mu_k(\phi)\} \ , \qquad \phi \in H_\mu \ . \tag{6}$$

According to [7, Lemma 5.4-1(2), (3) and Theorem 5.4-1] and since $z^{1/2} J_\mu(z)$ is a bounded function on I for every $k \in \mathbb{N}$ one has

$$\sup_{x\in I} |x^k\phi(x)| = \sup_{x\in I} |x^k h_\mu(h_\mu\phi)(x)| \le C\int_0^\infty |N_{\mu+k-1}...N_\mu(h_\mu\phi)(t)|\,dt \tag{7}$$

and

$$\sup_{x\in I} |N_{\mu+k-1}...N_\mu\phi(x)| = \sup_{x\in I} |N_{\mu+k-1}...N_\mu h_\mu(h_\mu\phi)(x)| \le C\int_0^\infty |t^k(h_\mu\phi)(t)|\,dt \tag{8}$$

for a suitable $C > 0$.

The linear mapping

$$j : H_\mu \to JH_\mu \subset L_1(0,\infty)^{2r+2}$$
$$\phi \to \left(x^k h_\mu\phi, N_{\mu+k-1}...N_\mu h_\mu\phi\right)_{k=0}^r$$

is one to one because h_μ is an automorphism of H_μ ([7, Theorem 5.4-1]). Here $L_1(0,\infty)$ denotes the usual Lebesgue space of order 1.

On the other hand, the inequalities (6), (7) and (8) imply that the linear mapping

$$L : JH_\mu \subset L_1(0,\infty)^{2r+2} \to \mathbb{C}$$
$$(x^k h_\mu\phi, N_{\mu+k-1}...N_\mu h_\mu\phi)_{k=0}^r \to \langle f,\phi\rangle$$

is continuous when JH_μ is endowed with the topology induced by $L_1(0,\infty)^{2r+2}$. Hence, by invoking the Hahn-Banach Theorem L can be extended to $L_1(0,\infty)^{2r+2}$ as a member of $(L_1(0,\infty)^{2r+2})'$, the dual space of $L_1(0,\infty)^{2r+2}$. Since, as it is well known, $L_1(0,\infty)' = L_\infty(0,\infty)$ there exist f_k, $g_k \in L_\infty(0,\infty)$, $k \in \mathbb{N}$, $0 \le k \le r$, such that

$$\langle f,\phi\rangle = \sum_{k=0}^r \left(\langle f_k, x^k h_\mu\phi\rangle + \langle g_k, x^{k+\mu+1/2}(x^{-1}D)^k(x^{-\mu-1/2}\phi)\rangle\right), \qquad \phi \in H_\mu .$$

Therefore

$$f = \sum_{k=0}^r h'_\mu\left(x^k f_k + (-1)^k x^{-\mu+1/2}(x^{-1}D)^k x^{k+\mu-1/2} g_k\right) .$$

Thus the proof of necessity if finished.

Conversely, if f is a linear functional defined on H_μ by (5) for certain $r \in \mathbb{N}$ and f_k, $g_k \in L_\infty(0,\infty)$, $k \in \mathbb{N}$, $0 \le k \le r$, then

$$|\langle f,\phi\rangle| \le C\sum_{k=0}^r \left(\|f_k\|_\infty \sup_{x\in I} |(1+x^2)x^k(h_\mu\phi)(x)| + \|g_k\|_\infty \sup_{x\in I} |(1+x^2)N_{\mu+k-1}...N_\mu(h_\mu\phi)(x)|\right)$$

for $\phi \in H_\mu$, where $\|\bullet\|_\infty$ denotes the usual norm in $L_\infty(0,\infty)$. Hence, according to [7, Theorem 5.4-1] and [2, Theorem 3.3], f is in H'_μ.

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