Internat.J. Math. & Math. Sci.
VOL. 18 NO. 2 (1995) 721-724

ON CERTAIN SEQUENCE SPACES II

HÜSNÜ KIZMAZ
Karadeniz Technical University
Department of Mathematics
61080 Trabzon, Turkey

(Received February 2, 1994 and in revised form July 1, 1994)

ABSTRACT In this paper we define the space $c_0(\Delta)=\{x=(x_k)/x_k-k_{k-1}\to 0\ (k\to\infty),\ x_0=0,\ x_k\in\mathbb{C}\}$ and compute its duals. (Continuous dual, β-dual and N-dual) The aim of this paper is to give same results about matrix mapping of $c_0(\Delta)$ into other sequence spaces including the convergent sequences, null sequences and bounded sequences

KEY WORDS AND PHRASES: Sequence spaces, matrix maps, Δ-norm, β-dual, Null-dual

1991 AMS SUBJECT CLASSIFICATION CODES: 40C05

1. Introduction

Let l_∞, c and c_0 be the linear spaces of complex bounded, convergent and null sequences $x=(x_k)$ respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k\in\mathbb{N}=\{1,2,\ldots\}$ the positive integers. On the other hand we defined $l_\infty(\Delta)=\{x=(x_k)/\Delta x\in l_\infty\}$, $c(\Delta)=\{x=(x_k)/\Delta x\in c\}$ and $c_0(\Delta)=\{x=(x_k)/\Delta x\in c_0\}$ where $\Delta x=(x_k-x_{k-1})$, $x_0=0$ [2]. (Throughout this paper it is assumed that $x_0=0$)

$c_0(\Delta)$, $c(\Delta)$ and $l_\infty(\Delta)$ are Banach Spaces with the norm

$$\|x\|_\Delta = \sup_k |x_k-x_{k-1}| \qquad [2].$$

c_0, c, l_∞ and $M_0=l_\infty\cap c_0(\Delta)$ are Banach with the norm $\|\cdot\|_\infty$ but they aren't Banach with the norm $\|\cdot\|_\Delta$. If we say $sx=(\sum_{k=1}^{n} x_k)$ then we have $m_s=\{x=(x_k)/sx\in l_\infty\}$, $c_s=\{x=(x_k)/sx\in c\}$ and $(c_0)_s=\{x=(x_k)/\ sx\in c_0\}$ [4]. l_∞, c and c_0 are isometrically isomorphic to m_s, c_s and $(c_0)_s$, respectively with their natural norms. For instance $f: l_\infty\to m_s$, $f(x)=\Delta x$ and $f^{-1}: m_s\to l_\infty$ $f^{-1}(x)=sx$ are isometric isomorphisms. Similary $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are isometrically isomorphic to l_∞, c and c_0 respectively. Obviously

$$f: (c_0(\Delta), \|\cdot\|_\Delta) \to (c_0, \|\cdot\|_\infty),\ f(x)=\Delta x$$

and

$$f^{-1}: (c_0, \|\cdot\|_\infty) \to (c_0(\Delta), \|\cdot\|_\Delta), \quad f(x)= sx$$

are isometric isomorphisms. (1.1)

We have investigated matrix maps and related questions connected with $l_\infty(\Delta)$ and $c(\Delta)$ in [2]. We know that c_0 and c have Schauder basis but l_∞ has no basis with the norm $\|\cdot\|_\infty$. Write $e_k=(0,0,\ldots,0,\overset{k}{1},0,\ldots)$. Then (e_k) is a basis for c_0 and (e_{k-1}) $(e_0=(1,1,1,\ldots))$ is a basis for c, with $\|\cdot\|_\infty$ and $\|\cdot\|_\Delta$. On the other hand $(E_k)=(\overbrace{0,0,\ldots,0}^{k-1},1,1,1,\ldots))$ is a basis for M_0 and $c_0(\Delta)$ with the norm $\|\ \|_\Delta$. So $c_0(\Delta)$ is a separable Banach Space.

We know that the continuous dual of c_0 and c is $l_1=\{x=(x_k)/\ \sum_{k=1}^{\infty}|x_k|<\infty,\ x_k\in\mathbb{C}\}$ [3] (Page 110) ($\mathbb{C}$ the set of complex numbers) Thus l_1, is continuous dual of $c_0(\Delta)$ by (1 1) Moreover, we can prove that

$$\overline{c}=\overline{M}_o=c_o(\Delta)$$

with the norm $\|\cdot\|_\Delta$, where the bar denotes closure. For this, let $x\in c_o(\Delta)$ and $\varepsilon>o$ be any number Then there exists one and only one $y=(y_k)\in c_o$ such that $x_k=\sum_{i=1}^{k} y_i$ (1.1) and a corresponding index $M=M(\varepsilon)\in\mathbb{N}$ such that $|y_k|<\varepsilon/2$ for all $k\geq M$. Now we take

$$z_k=\begin{cases} x_k, & 1\leq k\leq M \\ x_M, & k>M \end{cases},$$

thus $z=(z_k)\in c\subset c_o(\Delta)$ belongs to the open ball $B(x,\varepsilon)$ which is in $(c_o(\Delta), \|\cdot\|_\Delta)$

2. β-dual, N-dual and Matrix Maps

If X is a sequence space, we define

$X^\beta=\{a=(a_k)/\ \sum_{k=1}^{\infty} a_k x_k$ is Convergent for each $x\in X\}$

$X^N=\{a=(a_k)/\lim_k a_k x_k=0$, for each $x\in X\}$. X^β is called the β-(or generalized Köthe-Toeplitz) dual [1] and we will say that X^N is N-(or null) dual space of X. We have that if $X\subset Y$. then $Y^\beta\subset X^\beta$. The N-dual has similar properties with the β-dual. For instance if $X\subset Y$ than $Y^N\subset X^N$ and $X^\beta\subset X^N$.

Obviously $c_o^N=l_\infty$, $l_\infty^N=M_o^N=c^N=c_o$,

$c^N(\Delta)=l_\infty^N(\Delta)=\{a=(a_k)/(ka_k)\in c_o\}$. Let (X,Y) denote the set of all infinite matrices $A=(a_{nk})$ which map X into Y.

LEMMA 1. Let $(a_k)\in l_1$ and if $\lim_k|a_k x_k|=L$ exists for an $x\in c_o(\Delta)$, than L=0.

Proof. It is trivial if $x=(x_k)$ is bounded. Suppose that $x\in c_o(\Delta)$ is unbounded and $\lim_k|a_k x_k|=L>0$. Then x can't have a bounded subbequence. If (x_{k_n}) is bounded then $\lim_n|a_{k_n}x_{k_n}|=0$ implies L=0. So we can take $x_k\neq 0$ for all $n\in\mathbb{N}$.

Now let $\varepsilon=\frac{L}{2}>0$, than there exists an $M_1=M_1(\varepsilon)\in\mathbb{N}$ such that $\frac{L}{2}<|a_k x_k|<\frac{3L}{2}$ for all $k\geq M_1$. Thus we get $|a_k|>\frac{L}{2}\frac{1}{|x_k|}$ for all $k\geq M_1$ and

$$\sum_{k=1}^{\infty}\frac{1}{|x_k|}<\infty \tag{2.1}$$

We have that $\frac{x_k}{k}\to 0$ $(k\to\infty)$ [2]. Let $\varepsilon=1$, then we have $\frac{|x_k|}{k}<1$ and $\frac{1}{|x_k|}>\frac{1}{k}$ for all $k\geq M_2(1)\in\mathbb{N}$. If we take $\max\{M_1,M_2\}=M$ then $\sum_{k=1}^{\infty}\frac{1}{|x_k|}\geq\sum_{k=M}^{\infty}\frac{1}{|x_k|}=\infty$. This contradicts with (2.1). So L must be zero.

LEMMA 2. $c_o^N(\Delta)=\{a=(a_k)/(ka_k)\in l_\infty\}=E$.

Proof. Suppose that $a=(a_k)\in E$. Since $\lim_k\frac{x_k}{k}=0$ for all $x=(x_k)\in c_o(\Delta)$ [2], then we get $\lim_k a_k x_k=\lim_k ka_k\frac{x_k}{k}=0$. This implies that $a\in c_o^N(\Delta)$.

Now let $a\in c_o^N(\Delta)$. Then $\lim_k a_k x_k=0$, for all $x\in c_o(\Delta)$, then there exists one and only one $y=(y_k)\in c_o$, such that $x_n = \sum_{k=1}^{n} y_k$ (1 1)

$\lim_n a_n x_n = \lim_n \sum_{k=1}^{n} a_n y_k = 0$ for all $y=(y_k)\in c_o$ If we take

$$a_{nk} = \begin{cases} a_n, & 1\le k\le n \\ 0, & k>n \end{cases}$$

we get $\lim_n \sum_{k=1}^{\infty} a_{nk} y_k = 0$, for all $x\in c_o$. Then $A=(a_{nk})\in(c_o, c_o)$ and we have

$\text{Sup}_n \sum_{k=1}^{\infty} |a_{nk}| = \sup_n \sum_{k=1}^{n} |a_n| = \text{Sup}_n\, n|a_n| < \infty$ [4] This completes the proof

For the next results we introduce the sequence (R_k) [resp matrix R] given by $R_k = \sum_{i=k}^{\infty} a_i$ [resp matrix $R=(R_{nk}) = (\sum_{i=k}^{\infty} a_{ni})$].

LEMMA 3. $c_o^{\beta}(\Delta)=\{a=(a_k)\in l_1/(R_k)\in l_1 \cap c_o^N(\Delta)\}=D$

Proof. Suppose that $a\in D$ If $x\in c_o(\Delta)$ then we use Abel's summation formula to get

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (\sum_{i=1}^{k} a_i)(x_k - x_{k+1}) + (\sum_{k=1}^{n} a_k) x_{n+1}$$

$$= \sum_{k=1}^{n} (R_1 - R_{k+1})(x_k - x_{k+1}) + (R_1 - R_{n+1}) x_{n+1}$$

$$= \sum_{k=1}^{n+1} R_k (x_k - x_{k-1}) - R_{n+1} x_{n+1} \qquad (2\ 2)$$

This implies that $\sum_{k=1}^{\infty} a_k x_k$ is convergent, then $a\in c_o^{\beta}(\Delta)$.

If $a\in c_o^{\beta}(\Delta)$ then $\sum_{k=1}^{\infty} a_k x_k$ is convergent for all $x\in c_o(\Delta)$ Obviously $a\in l_1$ If $x\in c_o(\Delta)$, then there exists $y=(y_k)\in c_o$ such that $x_k = \sum_{i=1}^{k} y_i$ (1.1)

Then

$\sum_{k=1}^{n} R_k y_k = \sum_{k=1}^{n} (\sum_{i=1}^{k} y_i) a_k + R_{n+1} \sum_{k=1}^{n} y_k$ with Abel summation formula Thus we have

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (R_k - R_{n+1}) y_k = \sum_{k=1}^{n} (\sum_{i=k}^{n} a_i) y_k \qquad (2.3)$$

If we take

$$a_{nk} = \begin{cases} \sum_{i=k}^{n} a_i, & 1\le k\le n \\ 0, & k>n \end{cases}$$

then $A=(a_{nk})\in (c_o, c)$ since $\lim_n \sum_{k=1}^{\infty} a_{nk} y_k = \lim_n \sum_{k=1}^{n} a_{nk} y_k$ exists for all $y\in c_o$ (2 3). This implies that

$\mathrm{Sup}_n \sum_{k=1}^{\infty} |a_{nk}| = \mathrm{Sup}_n \sum_{k=1}^{n} |\sum_{i=k}^{n} a_i| < \infty$ [4]. Thus we get $\sum_{k=1}^{\infty} |R_k| < \infty$. Furthermore (2.2) implies that $\lim_n R_{n+1} x_{n+1}$ exists for each $x \in c_o(\Delta)$ then we get $(R_n) \in c_o^N(\Delta)$ by lemma 1. This completes the proof.

THEOREM 1. $A=(a_{nk}) \in (c_o(\Delta), c)$ iff

T_1. $(R_{nk}) \in c_o^N(\Delta)$, for each $n \in \mathbb{N}$

T_2. $R=(R_{nk}) \in (c_o, c)$

Proof. If $a \in (c_o(\Delta), c)$ then the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ are convergent for each $n \in \mathbb{N}$ and for all $x \in c_o(\Delta)$, this implies that $\mathrm{Sup}_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and $\lim_n \sum_{k=p}^{\infty} a_{nk} = a_p$ exists for each $p \in \mathbb{N}$ [3] (page 166). From lemma 3 we have $\sum_{k=1}^{\infty} |R_{nk}| < \infty$, $\lim_k R_{nk} x_k = 0$ for each $n \in \mathbb{N}$ and for all $x \in (c_o(\Delta)$. This proves T_1. If we write again (2.2) we get

$$\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m+1} R_{nk} (x_k - x_{k-1}) - R_{n\,m+1} x_{m+1} \qquad (2.4)$$

and

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} R_{nk} (x_k - x_{k-1}) \qquad (2.5)$$

This shows that $R \in (c_o, c)$. If we use again lemma 3 and (2.5) we get the sufficiency of T_1 and T_2.

Similarly we can prove that

i) $A \in (c_o(\Delta), c_o)$ iff T_1 and $R \in (c_o, c_o)$

ii) $A \in (c_o(\Delta), l_\infty)$ iff T_1 and $R \in (l_\infty, l_\infty)$

iii) $A \in (c_o(\Delta), M_o)$ iff T_1, $R \in (l_\infty, l_\infty)$ and

$$B=(b_{nk})=(a_{nk} - a_{n\,k+1}) \in (c_o(\Delta), c_o)$$

iv) $A \in (c_o(\Delta), c_o(\Delta))$ iff $(a_{nk}) \in c_o^\beta(\Delta)$, for each $n \in \mathbb{N}$ and $C=(c_{nk})=(a_{nk} - a_{n-1,k}) \in (c_o(\Delta), c_o)$ $(a_{ok}=o)$

Open questions

1) Matrix maps for M_o.

2) M_o has a Schauder basis with $\|\cdot\|_\Delta$. It is (E_k). (we can write $x = \sum_{k=1}^{\infty} (x_k - x_{k-1}) E_k$, each $x \in M_o$)

Then $(M_o, \|\cdot\|_\Delta)$ is separable.

Is M_o separable or have a Schauder basis with $\|\cdot\|_\infty$?

3) It is obvious that $c_o \subset c \subset M_o \subset l_\infty$ and inclusions are strict. In this order, is there a separable space E which is $c \subset E \subset l_\infty$ with the norm $\|\cdot\|_\infty$? If not, is c an upper bound according to separability?

REFERENCES

[1] Garling D.J.H., The α-and β-duality of sequence spaces, Proc. Comb. Phil soç. 1967 63(963-981)

[2] Kızmaz H., On Certain Sequence Spaces, Canad. Math. Bull. 25(2) 1981 (169-176)

[3] Maddox I.J., Elements of Functional Analysis Cambridge 1970

[4] Stieglitz M. - Tietz H., Matrixtransformationen Von Fongenraumen Eine Ergebnisübersicht Math. Z. 154 (1977) (1-16)