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APPROXIMATING COMMON FIXED POINTS OF FAMILIES OF QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT. This paper deals with a family of quasi-nonexpansive mappings in a uniformly convex Banach space, and the convergence of iterates generated by this family. A fixed point theorem for two quasi-nonexpansive mappings is then proved. This theorem is then extended for a finite family of quasi-nonexpansive mappings. It is shown that Ishikawa's [1] result follows as special cases of results proved in this paper.

KEY WORDS AND PHRASES: Uniformly convex Banach spaces, fixed point theorems, families of quasi-nonexpansive mappings, and convergence of iterates.
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1. INTRODUCTION.

Ishikawa [1] proved that if T is a Lipschitzian pseudo-contractive map of a compact convex subset E of a Hilbert space into itself and x_1 is any point in E, then the sequence

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] \tag{1.1}$$

converges to a fixed point of T where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers that satisfy certain conditions. De Marr [2] proved a theorem for existence of common fixed points for a commutative family of continuous (not necessarily linear) transformations. Diaz and Metcalf [3] considered the structure of the set of subsequential limit points of the sequence of iterates and investigated when this sequence converges. Dotson [4] applied Mann's iterative process to the approximation of fixed points of quasi-nonexpansive mappings in Hilbert space and in uniformly convex and strictly convex Banach spaces. He also generalized Mann's results to a locally convex Hausdorff linear topological space.

The main purpose of this paper is to consider a family of quasi-nonexpansive mappings and the convergence of iterates generated by this family. A fixed point theorem for two quasi-nonexpansive mappings is proved and is then extended for a finite family of mappings It is shown that results of this paper reduces to those of Ishikawa as special cases.

2. ITERATES GENERATED BY A FAMILY OF QUASI-NONEXPANSIVE MAPPINGS.

Before we discuss our results, we state the following definition from Diaz and Metcalf [3] and a lemma due to Dotson [4].

DEFINITION. Suppose B is a Banach space and D is a convex subset of B A mapping T is said to be *quasi-nonexpansive* on a subset D of B provided T maps D into itself, and if $p \in D$ and $Tp = p$ then $\| Tx - p \| \leq \| x - p \|$ hold for all $x \in D$.

LEMMA 2.1 If $\{v_n\}$ and $\{w_n\}$ are sequences in the closed unit ball of a uniformly convex Banach space and if $\{z_n\} = \{(1-\alpha_n)v_n + \alpha_n w_n\}$ satisfies $\lim_{n\to\infty} \| z_n \| = 1$ where $0 < a \leq \alpha_n \leq b < 1$, then

$$\lim_{n\to\infty} \| v_n - w_n \| = 0 . \tag{2.1}$$

Suppose D is a convex subset of a Banach space B, and T and S are two self mappings of D. For a $x_1 \in D$, we define a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$x_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n , \tag{2 2}$$

where

$$y_n = (1-\beta_n)Sx_n + \beta_n Tx_n , \tag{2.3}$$

α_n and β_n satisfy (i) $0 < a \leq \alpha_n \leq b < 1$, and (ii) $0 \leq \beta_n \leq \beta < 1$. If $S = I$, the identity mapping, the iterates (2.2) are identical with those of Ishikawa

We now state the following condition:

Condition C: The mappings $T, S : D \to D$ are said to satisfy condition C if there exists a nondecreasing function $f : [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$ such that

$$\| Ty - Sx \| \geq f(d(x, \mathrm{F})) \tag{2.4}$$

for all $x, y \in D$ with $y = (1-\mu)Sx + \mu Tx$, where $0 \leq \mu \leq \beta < 1$ and F is the common fixed point set of T and S.

If we set $S = I$ in (2.4), then condition C becomes identical with condition A as stated by Maiti and Ghosh [5].

We now state our main result for two quasi-nonexpansive mappings:

THEOREM 2.1 Suppose D is a closed convex subset of a uniformly convex Banach space B and T, S are two quasi-nonexpansive mappings of D into itself. If T, S satisfy condition C, then, for any $x_1 \in D$, sequence (2.2) converges to a member of the common fixed point set F of T and S.

PROOF. If $x_1 \in F$, then the result follows trivially. We assume $x_1 \in D - F$. Then, for an arbitrary $p \in F$, we have, from (2.2),

$$\begin{aligned}
\| x_{n+1} - p \| &= \| (1-\alpha_n)Sx_n + \alpha_n Ty_n - p \| \\
&= \| (1-\alpha_n)(Sx_n - p) + \alpha_n(Ty_n - \mathrm{p}) \| \\
&\leq (1-\alpha_n) \| Sx_n - p \| + \alpha_n \| Ty_n - p \| \\
&\leq (1-\alpha_n) \| x_n - p \| + \alpha_n \| y_n - p \| \\
&= (1-\alpha_n) \| x_n - p \| + \alpha_n \| (1-\beta_n)Sx_n + \beta_n Tx_n - p \| \\
&= (1-\alpha_n) \| x_n - p \| + \alpha_n \| (1-\beta_n)(Sx_n - p) + \beta_n(Tx_n - p) \| \\
&\leq (1-\alpha_n) \| x_n - p \| + \alpha_n(1-\beta_n) \| Sx_n - p \| + \alpha_n\beta_n \| Tx_n - p \| \\
&\leq (1-\alpha_n) \| x_n - p \| + \alpha_n(1-\beta_n) \| x_n - p \| + \alpha_n\beta_n \| x_n - p \| \\
&= \| x_n - p \| .
\end{aligned}$$

This implies that $d(x_{n+1}, F) \leq d(x_n, F)$ and hence that the sequence $\{d(x_n, F)\}$ is nonincreasing. Thus $\lim_{n\to\infty} d(x_n, F)$ exists. We next show that this limit is zero.

Suppose, if possible, $\lim_{n\to\infty} d(x_n, F) = l > 0$. Then, for $p \in F$, $\lim_{n\to\infty} \| x_n - p \| l' \geq l > 0$ We choose a positive integer N such that $\| x_n - p \| \leq 2l'$ for $n \geq N$. Now, we set

$$v_n = \frac{Sx_n - p}{\| x_n - p \|}\ , \quad w_n \frac{Ty_n - p}{\| x_n - p \|}\ . \tag{2.5}$$

Then $\| v_n \| \leq 1$, $\| w_n \| \leq 1$ for all n. This implies that the sequences $\{v_n\}$ and $\{w_n\}$ are in the closed unit ball of B. Further,

$$\begin{aligned}\lim_{n\to\infty} \| (1-\alpha_n)v_n + \alpha_n w_n \| &= \lim_{n\to\infty} \frac{\| (1-\alpha_n)(Sx_n - p) + \alpha_n(Ty_n - p) \|}{\| x_n - p \|} \\ &= \lim_{n\to\infty} \frac{\| (1-\alpha_n)Sx_n + \alpha_n Ty_n - p \|}{\| x_n - p \|} \\ &= \lim_{n\to\infty} \frac{\| x_{n+1} - p \|}{\| x_n - p \|} = \frac{l'}{l'} = 1\ . \end{aligned} \tag{2.6}$$

Thus the sequences $\{v_n\}$, $\{w_n\}$ and $\{(1-\alpha_n)v_n + \alpha_n w_n\}$ satisfy all the conditions of Lemma 2.1 and hence we must have

$$\lim_{n\to\infty} \| v_n - w_n \| = 0\ . \tag{2.7}$$

But for $n \geq N$, we have

$$\| v_n - w_n \| = \frac{\| Ty_n - Sx_n \|}{\| x_n - p \|} \geq \frac{f(d(x_n, F))}{\| x_n - p \|} \geq \frac{f(l)}{2l'} > 0\ . \tag{2.8}$$

Thus we conclude that $\lim_{n\to\infty} \| v_n - w_n \| \neq 0$, which is a contradiction. Thus $\lim_{n\to\infty} d(x_n, F) = 0$. We now show that this implies $\{x_n\}$ converges to a member of F. Since $\lim_{n\to\infty} d(x_n, F) = 0$. For a given $\epsilon > 0$ there exists $N_\epsilon > 0$ and $y_\epsilon \in F$ such that $\| x_{N_\epsilon} - y_\epsilon \| < \epsilon$, which implies $\| x_n - y_\epsilon \| < \epsilon$ for all $n \geq N_\epsilon$. Thus, if $\epsilon_k = 2^{-k}$ for $k \in P$, where P is the set of positive integers, then corresponding to each ϵ_k, there is an $N_k > 0$ and a $y_k \in F$ such that $\| x_n - y_k \| \leq \frac{\epsilon_k}{4}$ for all $n \geq N_k$. We require $N_{k+1} \geq N_k$ for all $k \in P$. We have, for all $k \in P$,

$$\| y_k - y_{k+1} \| = \| y_k - x_{N_{k+1}} + x_{N_{k+1}} - y_{k+1} \| < \frac{\epsilon_k}{4} + \frac{\epsilon_{k+1}}{4} = \frac{3\epsilon_{k+1}}{4}\ .$$

We now suppose that $S(y, \epsilon) = \{x \in B : \| x - y \| \leq \epsilon\}$ represents the closed sphere with center at y of radius ϵ. For $x \in S(y_{k+1}, \epsilon_{k+1})$, we have

$$\| y_k - x \| = \| y_k - y_{k+1} + y_{k+1} - x \| < \left(\frac{3\epsilon_{k+1}}{4} + \epsilon_{k+1}\right) < 2\epsilon_{k+1} = \epsilon_k\ .$$

Thus

$$S(y_{k+1}, \epsilon_{k+1}) \subseteq S(y_k, \epsilon_k) \qquad \text{for } k \in P\ .$$

Thus, $\{S(y_k, \epsilon_k)\}$ is a nested sequence of nonvoid closed spheres with radii ϵ_k tending to zero. By the Cantor intersection theorem, $\bigcap_{k\in P} S(y_k, \epsilon_k)$ contains exactly one point p. Since T and S are quasi-nonexpansive, the fixed point set F is closed by a result due to Dotson [6], and the sequence $\{y_k\}$ from F converges to p, and hence $p \in F$. In view of the result

$$\| x_n - y_k \| < \frac{1}{4}\,\epsilon_k \qquad \text{for } n \geq N_k\ , \tag{2.9}$$

the sequence $\{x_n\}_{n=1}^{\infty}$ converges to p. This completes the proof.

3. AN EXTENSION OF THEOREM 2.1.

We extend Theorem 2.1 for the case of more than two mappings. For the sake of clarity, we consider three quasi-nonexpansive mappings T, S, R of D into itself. For any $x_1 \in$ D, we define a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$x_{n+1} = \alpha_n^1 Rx_n + \alpha_n^2 Sx_n + \alpha_n^3 Ty_n \tag{3.1}$$

with

$$y_n = \beta_n^1 Rx_n + \beta_n^2 Sx_n + \beta_n^3 Tx_n \ , \tag{3.2}$$

where,

$$0 < a \leq \alpha_n^i \leq b < 1 \ , \ \sum_{i=1}^{3} \alpha_n^i = 1 \ , \tag{3.3}$$

$$0 \leq \beta_n^i \leq \beta < 1 \ , \ \sum_{i=1}^{3} \beta_n^i = 1 \ , \tag{3.4}$$

In the present situation, it is necessary to modify Condition A as stated by Maiti and Ghosh [5] in the following way.

Condition D. The mappings T, S, $R : D \to D$ are said to satisfy condition D, if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$ such that

$$\| Ty - \lambda Sx - (1 - \lambda) Rx \| \geq f(d(x, F)) \ , \ \lambda \in (0, 1) \tag{3.5}$$

for all $x, y \in D$ with $y_1 = a_1 Rx + a_2 Sx + a_3 Tx$, where $0 \leq a_i \leq \beta < 1$ with $\sum_{i=1}^{3} a_i = 1$ and F is the common fixed point set of T, S, R in D.

If we set $S = R = I$ in (3.5), this condition reduces to condition A as stated in a paper by Maiti and Ghosh [5].

THEOREM 3.1 Suppose D is a closed convex subset of a uniformly convex Banach space B. If T, S, R are three quasi-nonexpansive and self-mappings of D satisfying condition D, then, for any $x_1 \in D$, the sequence (3.1) with (3.3) and (3.4) converges to an element of the common fixed point set F.

PROOF. The proof of this theorem is similar to that of Theorem 2.1. It suffices to show that

$$\lim_{n \to \infty} d(x_n, F) = 0 \tag{3.6}$$

If possible, we let $\lim_{n \to \infty} d(x_n, F) = l > 0$. Then for $p \in F$, $\lim_{n \to \infty} \| x_n - p \| = l' \geq l > 0$. We now choose a positive integer N such that $\| x_n - p \| \leq 2l'$ for $n \geq N$.

We now write

$$v_n = \frac{\alpha_n^1}{1 - \alpha_n^3} \cdot \frac{Rx_n - p}{\| x_n - p \|} + \frac{\alpha_n^2}{1 - \alpha_n^3} \cdot \frac{Sx_n - p}{\| x_n - p \|} \ , \tag{3.7}$$

and

$$w_n = \frac{Ty_n - p}{\| x_n - p \|} \ . \tag{3.8}$$

Then

$$\| v_n \| \leq \frac{\alpha_n^1}{1-\alpha_n^3}\left(\frac{\| Rx_n - p \|}{\| x_n - p \|}\right) + \frac{\alpha_n^2}{1-\alpha_n^3}\left(\frac{\| Sx_n - p \|}{\| x_n - p \|}\right)$$
$$\leq \frac{\alpha_n^1 + \alpha_n^2}{1-\alpha_n^3} = 1$$

and

$$\| w_n \| \leq \frac{\| Ty_n - p \|}{\| x_n - p \|} \leq \frac{\| x_n - p \|}{\| x_n - p \|} = 1 \ .$$

Thus the sequence $\{v_n\}$ and $\{w_n\}$ are in the closed unit ball of B Now,

$$\lim_{n\to\infty} \| (1-\alpha_n^3)v_n + \alpha_n^3 w_n \| = \lim_{n\to\infty} \frac{\| \alpha_n^1(Rx_n - p) + \alpha_n^2(Sx_n - p) + \alpha_n^3(Ty_n - p) \|}{\| x_n - p \|}$$
$$= \lim_{n\to\infty} \frac{\| \alpha_n^1 Rx_n + \alpha_n^2 Sx_n + \alpha_n^3 Ty_n - p \|}{\| x_n - p \|}$$
$$= \lim_{n\to\infty} \frac{\| x_{n+1} - p \|}{\| x_n - p \|} = \frac{l'}{l'} = 1 \ . \tag{3 9}$$

Then, by Lemma 2.1, we must have $\lim_{n\to\infty} \| w_n - v_n \| = 0$ But, for $n \geq N$, we get

$$\| w_n - v_n \| = \frac{1}{\| x_n - p \|} \| (Ty_n - p) - \frac{\alpha_n^1}{1-\alpha_n^3}(Rx_n - p) - \frac{\alpha_n^2}{1-\alpha_n^3}(Sx_n - p) \|$$
$$= \frac{1}{\| x_n - p \|} \| Ty_n - \frac{\alpha_n^1}{1-\alpha_n^3} Rx_n - \frac{\alpha_n^2}{1-\alpha_n^3} Sx_n \|$$
$$\geq \frac{f(d(x_n, F))}{\| x_n - p \|} \geq \frac{f(l)}{2l'} > 0 \ . \tag{3 10}$$

This implies that $\lim_{n\to\infty} \| w_n - v_n \| \neq 0$ which leads to a contradiction, and hence $\lim_{n\to\infty} d(x_n,F) = 0$ This completes the proof.

It is obvious that the above results may be extended readily to any finite family of quasi-nonexpansive mappings.

We now give a variant of Theorem 2 1.

THEOREM 3.2 Suppose D is a closed, bounded and convex subset of a uniformly convex Banach space B. If T and S are two commuting and non-expansive self-mappings of D satisfying condition C, then, for any $x_1 \in D$, sequence (2.2) converges to a member of the common fixed point set F of T and S.

PROOF. In Theorem 2.1, the existence of common fixed point set F is assumed, whereas the existence of F is ensured here by a Theorem of Bowder [7] Since, with the existence of fixed points, nonexpansive mappings are also quasi-nonexpansive, the present theorem may be treated as a special case of Theorem 2 1.

Similarly, a variant of Theorem 3 1 may be obtained.

REMARK. If, in Theorem 3 2, D is assumed to be compact and convex (which is a stronger condition), then the existence of common fixed point set F is guaranteed by the theorem of DeMarr [2]

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