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MAXIMUM PRINCIPLES FOR PARABOLIC SYSTEMS COUPLED IN BOTH FIRST-ORDER AND ZERO-ORDER TERMS

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ABSTRACT. Some generalized maximum principles are established for linear second-order parabolic systems in which both first-order and zero-order terms are coupled.

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1. INTRODUCTION.

Hile and Protter [2] proved that the Euclidean length of the solution vector $u \in C^2(D) \cap C(\overline{D})$ of the second-order elliptic system

$$\sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 u_s}{\partial x_i \partial x_k} + \sum_{i=1}^{n}\sum_{j=1}^{m} b_{sij}(x)\frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj}(x)u_j = 0, \qquad s = 1, \cdots, m,$$

can be bounded by a constant times the maximum of its boundary values under a "small" condition which requires that either the domain D or the coefficients b_{sij} and c_{sj} are sufficiently small. In this paper, we have established the same kind of maximum principle for the second-order parabolic system

$$\sum_{i,k=1}^{n} a_{ik}(x,t)\frac{\partial^2 u_s}{\partial x_i \partial x_k} - \frac{\partial u_s}{\partial t} + \sum_{i=1}^{n}\sum_{j=1}^{m} b_{sij}(x,t)\frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj}(x,t)u_j = 0, 1 \leq s \leq m.$$

Moreover, our parabolic version of the maximum principle holds without any "small" conditions.

When the coupling occurs only in the zero-order terms (i.e., in the case of $b_{sij} = 0$ for all i, j, s except when $j = s$), the above systems are called weakly coupled systems. For weakly coupled second-order parabolic systems, similar maximum principles have been obtained by Stys [4] and Zhou [6]. Under different assumptions, different maximum principles in which the components rather than the Euclidean length of the solution vector are bounded can be found in Protter and Weinberger [3] and Dow [1]. In Weinberger's paper [5], both kinds of maximum principles have been reformulated and studied in terms of invariant sets.

2. MAIN RESULTS.

Consider a second-order parabolic operator with real coefficients,

$$M \equiv \sum_{i,k=1}^{n} a_{ik}(x,t)\frac{\partial^2}{\partial x_i \partial x_k} - \frac{\partial}{\partial t}, \qquad a_{ij} = a_{ji}$$

in a general bounded domain Ω in $\mathbb{R}^n \times \mathbb{R}_t$ ($n \geq 1$) with the boundary $\partial\Omega := \partial_p\Omega \cup \partial_t\Omega$. Here $\partial_p\Omega$ is the parabolic boundary of Ω and $\partial_t\Omega := \partial\Omega \backslash \partial_p\Omega$. We suppose that $\Omega \subset D \times (0,T)$ where D is a bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. The operator M is assumed to be uniformly parabolic in Ω; i.e., there is a constant $\delta > 0$ such that for all $(x,t) \in \Omega$ and all $(y_1, \cdots, y_n)$ in $\mathbb{C}^n$ the inequality

$$\sum_{i,k=1}^{n} a_{ik}(x,t)\, y_i \overline{y}_k \geq \delta \sum_{i=1}^{n} |y_i|^2 \tag{2.1}$$

holds. The operator M is the principal part of each equation in the second-order parabolic system

$$Mu_s + \sum_{i=1}^{n} \sum_{j=1}^{m} b_{sij}(x,t) \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj}(x,t) u_j = 0, \qquad s = 1, 2, \cdots, m. \tag{2.2}$$

We suppose that the complex-valued coefficients b_{sij}, c_{sj} have the property that for all $\xi \in \mathbb{C}^m$ and all $(x,t) \in \Omega$.

$$\sum_{r,s=1}^{m} \left[c_{sr} + \overline{c}_{rs} + \frac{1}{2} \sum_{j=1}^{m} \sum_{k,i=1}^{n} A_{ki} b_{sij} \overline{b}_{rkj} \right] \xi_r \overline{\xi}_s \leq K |\xi|^2, \text{for some } K > 0. \tag{2.3}$$

Here $(A_{ki}) = (A_{ik})$ denotes the inverse matrix of (a_{ik}). A solution $u = (u_1, u_2, \cdots, u_m)$ is a complex-valued $C^{2,1}(\Omega \cup \partial_t \Omega) \cap C(\overline{\Omega})$ function which satisfies (2) in Ω. Here $C^{k,h}(\Omega)$ is defined as the set of functions $f(x,t)$ having all x (space) derivatives of order $\leq k$ and t (time) derivatives of order $\leq h$ continuous in Ω.

THEOREM 1. Assume conditions (1.1) and (1.3) hold. If u is a solution of (2.2) and α is a positive $C^{2,1}(\Omega \cup \partial_t \Omega)$ function, then the product $\alpha |u|^2 = \alpha \sum_{j=1}^{m} |u_j|^2$ cannot attain a positive maximum at any point in $\Omega \cup \partial_t \Omega$ where α satisfies

$$\alpha^{-1} M\alpha - 2\alpha^{-2} \sum_{i,k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} > K. \tag{2.4}$$

PROOF. We set $p = |u|^2 = \sum_{s=1}^{m} |u_s|^2$ and find

$$M(\alpha p) = pM\alpha + \alpha Mp + 2 \sum_{i,k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial p}{\partial x_k}. \tag{2.5}$$

At a point $(x,t) \in \Omega \cup \partial_t \Omega$ where αp attains a maximum, we have

$$0 \leq \frac{\partial(\alpha p)}{\partial t}, \qquad 0 = \frac{\partial(\alpha p)}{\partial x_k} = \alpha \frac{\partial p}{\partial x_k} + p \frac{\partial \alpha}{\partial x_k}, \qquad 1 \leq k \leq n,$$

and (2.5) becomes

$$M(\alpha p) = p \left[M\alpha - 2\alpha^{-1} \sum_{i,k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} \right] + \alpha Mp. \tag{2.6}$$

A direct computation yields

$$\begin{aligned}
Mp &= \sum_{s=1}^{m} \left[u_s M\overline{u}_s + \overline{u}_s Mu_s + 2 \sum_{i,k=1}^{n} a_{ik} \frac{\partial u_s}{\partial x_i} \frac{\partial \overline{u}_s}{\partial x_k} \right] \\
&= \sum_{s=1}^{m} \left\{ -2Re \left[\overline{u}_s \left(\sum_{i=1}^{n} \sum_{j=1}^{m} b_{sij} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj} u_j \right) \right] + 2 \sum_{i,k=1}^{n} a_{ik} \frac{\partial u_s}{\partial x_i} \frac{\partial \overline{u}_s}{\partial x_k} \right\} \\
&= 2 \left\{ \sum_{j=1}^{m} \sum_{i,k=1}^{n} a_{ik} \left[\frac{\partial u_j}{\partial x_i} - \frac{1}{2} \sum_{q=1}^{n} \sum_{r=1}^{m} A_{iq} \overline{b}_{rqj} u_r \right] \left[\frac{\partial \overline{u}_j}{\partial x_k} - \frac{1}{2} \sum_{q=1}^{n} \sum_{s=1}^{m} A_{kq} b_{sqj} \overline{u}_s \right] \right. \\
&\qquad \left. - \frac{1}{4} \sum_{r,s=1}^{m} \left[\sum_{k,q=1}^{n} \sum_{j=1}^{m} A_{kq} b_{sqj} \overline{b}_{rkj} \right] u_r \overline{u}_s \right\} - \sum_{r,s=1}^{m} (c_{sr} + \overline{c}_{sr}) u_r \overline{u}_s \\
&\geq -K \sum_{s=1}^{m} |u_s|^2 = -Kp.
\end{aligned}$$

Hence, from (2.6), we have

$$M(\alpha p) \geq \alpha p \left[\alpha^{-1} M\alpha - 2\alpha^{-2} \sum_{i,k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} - K \right]. \tag{2.7}$$

This inequality holds at any point in $\Omega \cup \partial_t \Omega$ where αp attains a maximum. Thus αp cannot achieve a positive maximum at any point in $\Omega \cup \partial_t \Omega$ where the quantity in brackets in (2.7) is positive. The theorem is established. □

REMARK. If for all $(x,t) \in \Omega$,

$$|c_{sj}| \leq K_0, \ |b_{sij}| \leq K_1, 1 \leq i \leq n, 1 \leq j, s \leq m, \text{ for some } K_0, K_1 \in \mathbb{R}, \tag{2.8}$$

then for any $\xi \in \mathbb{C}^m$,

$$\sum_{r,s=1}^{m} \left[c_{sr} + \overline{c}_{rs} + \tfrac{1}{2} \sum_{j=1}^{m} \sum_{k,i=1}^{n} A_{ki} b_{sij} \overline{b}_{rkj} \right] \xi_r \overline{\xi}_s$$

$$\leq \sum_{r,s=1}^{m} |c_{sr}| \left(|\xi_r|^2 + |\xi_s|^2 \right) + \tfrac{1}{2} \sum_{j=1}^{m} \sum_{k,i=1}^{n} A_{ki} \left(\sum_{s=1}^{m} b_{sij} \overline{\xi}_s \right) \left(\sum_{r=1}^{m} \overline{b}_{rkj} \xi_r \right)$$

$$\leq 2mK_0 \sum_{s=1}^{m} |\xi_s|^2 + \frac{1}{2\delta} \sum_{j=1}^{m} \sum_{i=1}^{n} \left| \sum_{s=1}^{m} b_{sij} \overline{\xi}_s \right|^2$$

$$\leq 2mK_0 |\xi|^2 + \frac{m}{2\delta} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{s=1}^{m} |b_{sij} \overline{\xi}_s|^2 \leq [2mK_0 + (2\delta)^{-1} nm^2 K_1^2] |\xi|^2,$$

which is the condition (2.3) with $K := 2mK_0 + (2\delta)^{-1} nm^2 K_1^2$. Hence, the single bound (2.3) in Theorem 1 can be replaced by the separate bounds (2.8) with $K := 2mK_0 + (2\delta)^{-1} nm^2 K_1^2$.

Under the conditions (2.1) and (2.3) (or (2.1) and (2.8)), by choosing $\alpha(x,t) = e^{-(K+\varepsilon)t}, \varepsilon > 0$, the condition (2.4) will be satisfied. Hence from Theorem 1, we get the following maximum principle:

COROLLARY 2 (Maximum Principle). For any solution u of the system (2.2), the function

$$|u(x,t)|^2 exp[-(K+\varepsilon)t], \varepsilon > 0,$$

does not attain a positive maximum in $\Omega \cup \partial_t \Omega$, and

$$\| u \|_{0,\Omega} \leq exp(KT/2) \| u \|_{0,\partial_p \Omega}. \tag{2.9}$$

Here $K = (2\delta)^{-1} nm^2 K_1^2 + 2mK_0$ and $\| u \|_{0,\Omega} := \sup_{(x,t) \in \Omega} |u(x,t)|$.

REMARK. Results similar to Theorem 1 and Corollary 2 for second-order elliptic systems were proven by Hile and Protter [2] (under a condition which is similar to (2.8)). But their maximum principle for elliptic systems only holds under the restriction that either the domain D is sufficiently small or the coefficients of the elliptic system are restricted sufficiently. Corollary 2 tells us that these restrictions can be lifted for parabolic systems.

COROLLARY 3 (Uniqueness). The system (2.2) with the initial-boundary condition

$$u|_{\partial_p \Omega} = \varphi(x,t)$$

has at most one solution $u \in C^{2,1}(\Omega \cup {}_t\Omega) \cap C(\overline{\Omega})$.

Theorem 1 can be used to obtain bounds on the gradient of the $C^{3,2}$ solution of the parabolic system (2.2), provided the coefficients are C^1 and

$$\| a_{ik} \|_{1,\Omega} \leq L_2, \ \| b_{sij} \|_{1,\Omega} \leq L_1, \ \| c_{sj} \|_{1,\Omega} \leq L_0, \text{ for some } L_2, L_1, L_0 \in \mathbb{R}. \tag{2.10}$$

Here $\| f \|_{1,\Omega} := \| f \|_{0,\Omega} + \sum_{i=1}^{n} \| \frac{\partial f}{\partial x_i} \|_{0,\Omega} + \| \frac{\partial f}{\partial t} \|_{0,\Omega}$.

We differentiate (2.2) with respect to x_h and t, and get $m(n+1)$ equations:

$$M\left(\frac{\partial u_s}{\partial x_h}\right) + \sum_{i,k=1}^{n} \frac{\partial a_{ik}}{\partial x_h} \frac{\partial}{\partial x_i}\left(\frac{\partial u_s}{\partial x_k}\right) + \sum_{i=1}^{n} \sum_{j=1}^{m} b_{sij} \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_h}\right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial b_{sij}}{\partial x_h} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj} \frac{\partial u_j}{\partial x_h} + \sum_{j=1}^{m} \frac{\partial c_{sj}}{\partial x_h} u_j = 0,$$

$$s = 1,2,\cdots,m \text{ and } h = 1,2,\cdots,n; \tag{2.11}$$

$$M\left(\frac{\partial u_s}{\partial t}\right) + \sum_{i,k=1}^{n} \frac{\partial a_{ik}}{\partial t} \frac{\partial}{\partial x_i}\left(\frac{\partial u_s}{\partial x_k}\right) + \sum_{i=1}^{n} \sum_{j=1}^{m} b_{sij} \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial t}\right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial b_{sij}}{\partial t} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj} \frac{\partial u_j}{\partial t} + \sum_{j=1}^{m} \frac{\partial c_{sj}}{\partial t} u_j = 0,$$

$$h = 1,2,\cdots,n. \tag{2.12}$$

By combining (2.2), (2.11) and (2.12) we get a system (of the form (2.2)) consisting of $m(n+2)$ equations in the $m(n+2)$ unknowns u_s, $\frac{\partial u_s}{\partial x_h}$, $\frac{\partial u_s}{\partial t}$, $s = 1,2,\cdots,m$, $h = 1,2,\cdots,n$.

THEOREM 4. Let $K: = (2\delta)^{-1}n(n+2)^2m^2(Max\{L_1, L_2\})^2 + 2m(n+2)Max\{L_0, L_1\}$ and suppose that u is a $C^{3,2}(\Omega \cup \partial_t\Omega) \cap C^1(\bar{\Omega})$ solution of (2.2) and α is a positive $C^{2,1}(\Omega \cup \partial_t\Omega)$ function. Then the product

$$\alpha(x,t)[\,|u(x,t)|^2 + |\nabla u(x,t)|^2] = \alpha(x,t) \sum_{s=1}^{m} \left[|u_s|^2 + \sum_{i=1}^{m} \left|\frac{\partial u_s}{\partial x_i}\right|^2 + \left|\frac{\partial u_s}{\partial t}\right|^2 \right]$$

cannot attain a positive maximum at any point in $\Omega \cup \partial_t\Omega$ where α satisfies (2.4).

COROLLARY 5. Let K be the same number of Theorem 4. Then, for any $C^{3,2}(\Omega \cup \partial_t\Omega) \cap C^1(\bar{\Omega})$ solution u of the system (2.2), we have

$$\| u \|_{0,\Omega}^2 + \| \nabla u \|_{0,\Omega}^2 \leq exp(KT)\left(\| u \|_{0,\partial_p\Omega}^2 + \| \nabla u \|_{0,\partial_p\Omega}^2 \right),$$

or equivalently,

$$\| u \|_{1,\Omega} \leq exp(KT/2) \cdot \| u \|_{1,\partial_p\Omega}.$$

REMARK. Under the condition that either $(c_{sj})_{m \times m}$ is a constant matrix or $(c_{sj})_{m \times m}$ is invertible for all $(x,t) \in \Omega$, the unknowns $u_s, s = 1,\cdots,m$, can be eliminated from the system (2.2), (2.11), (2.12), and then a system of $m(n+1)$ equations in the gradient of u yields a maximum principle for $\alpha\,|\nabla u|^2$.

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