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A CLASS OF BOUNDED STARLIKE FUNCTIONS

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ABSTRACT. We consider functions $f(z) = z + \cdots$ that are analytic in the unit disk and satisfy there the inequality $Re\ (f'(z) + zf''(z)) > \alpha,\ \alpha < 1$. We find extreme points and then determine sharp lower bounds on $Re\ f'(z)$ and $Re\ (f(z)/z)$. Sharp results for the sequence of partial sums are also found.

KEY WORDS AND PHRASES. Univalent, starlike.
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1. INTRODUCTION.

Denote by A the family of functions $f(z) = z + \cdots$ that are analytic in the unit disk $\Delta = \{z\colon |z| < 1\}$ and by S the subfamily of functions that are univalent in Δ. Let R be the functions f in A for which $Re(f'(z) + zf''(z)) > 0, z \in \Delta$. Chichra [1] showed that $R \subset S$. In fact, he proved that $Ref'(z) > 0, z \in \Delta$, and hence $R \subset C$, the class of close-to-convex functions. R. Singh and S. Singh [4] showed that $R \subset S^*$, the family of starlike functions. They later found in [5] for $f \in R$ and $z \in \Delta$ that $Re(f(z)/z) > 1/2$ and that the partial sums $S_n(z,f)$ satisfy $Re(S_n(z,f)/z) > 1/3$. Neither of these results is sharp.

In this note, we find the sharp bounds. Our results will be put into a slightly more general context. Denote by $R(\alpha)$, $\alpha < 1$, the subfamily of A consisting of functions f for which $Re(f'(z) + zf''(z)) > \alpha, z \in \Delta$. Denote by $P(\alpha), \alpha < 1$, the subfamily of A consisting of functions f for which $Ref'(z) > \alpha, z \in \Delta$. It was shown in [5] that $R(\alpha) \subset S^*$ for $\alpha \geq -1/4$. We improve this lower bound and also find the smallest α for which $R(\alpha) \subset S$. Our approach in this note will be to characterize the extreme points of $R(\alpha)$, which lead to sharp bounds for certain linear problems.

2. MAIN RESULTS.

THEOREM 1. (i) The extreme points of $R(\alpha)$ are

$$f_x(z) = \int_0^z \frac{(2\alpha - 1)t + (2\alpha - 2)\,\overline{x}\,log(1 - xt)}{t}\,dt,\ |x| = 1.$$

(ii) A function f is in $R(\alpha)$ if and only if f can be expressed as

$$F(z) = \int_X f_x(z)d\mu(x),$$

where μ varies over the probability measures defined on the unit circle X.

PROOF of (i). Hallenbeck [2] showed that the extreme points of $P(\alpha)$ are

$$\{(2\alpha - 1)z + (2\alpha - 2)\overline{x}\,log(1 - xz),\ |x| = 1\}. \tag{2.1}$$

Since $(zf')' = f' + zf''$, we have $f \in R(\alpha)$ if and only if $zf' \in P(\alpha)$. Hence the operator L defined by $L(f) = \int_0^z (f(t)/t)dt$ is a linear homeomorphism $L: P(\alpha) \to R(\alpha)$ and thus preserves extreme points.

PROOF of (ii). The family $R(\alpha)$ is convex and is therefore equal to its convex hull. This enables us to characterize $f \in R(\alpha)$ by $F(z) = \int_X f_x(z) d\mu(x)$.

COROLLARY 1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R(\alpha)$, then $|a_n| \leq 2(1-\alpha)/n^2$. The result is sharp.

PROOF. The coefficient bounds are maximized at an extreme point. Now $f_x(z)$ may be expressed as

$$f_x(z) = z + 2(1-\alpha) \sum_{n=2}^{\infty} \frac{x^{n-1} z^n}{n^2}, \; |x| = 1, \tag{2.2}$$

and the result follows.

COROLLARY 2. If $f \in R(\alpha)$, then $|f(z)| \leq (1-\alpha)\left(\frac{\pi^2}{3} - 1\right) + \alpha$.

PROOF. From (2.2), we see that $|f(z)| \leq r + 2(1-\alpha) \sum_{n=2}^{\infty} \frac{r^n}{n^2}$, $|z| = r$. Letting $r \to 1$, we get

$$|f(z)| \leq 1 + 2(1-\alpha)\left(\frac{\pi^2}{6} - 1\right) = (1-\alpha)\left(\frac{\pi^2}{3} - 1\right) + \alpha.$$

Corollary 2 shows that the family $R(\alpha)$ is bounded in Δ for all real α, $\alpha < 1$, even though its functions may not be univalent. Note from (2.1) that the extreme points of $P(\alpha)$ are unbounded in Δ for all $\alpha < 1$.

In the next two theorems, we will be looking at continuous linear operators $L(f) = Re f'$ and $L(f) = Re(f(z)/z)$ acting on $R(\alpha)$. It therefore suffices to investigate the extreme points in determining minima. Since $R(\alpha)$ is rotationally invariant, we may restrict our attention to the extreme point

$$g(z) = (2\alpha - 1)z - 2(1-\alpha)\int_0^z \frac{log(1-t)}{t} dt = z + 2(1-\alpha) \sum_{n=2}^{\infty} \frac{z^n}{n^2}. \tag{2.3}$$

THEOREM 2. If $f \in R(\alpha)$, then

$$Re\, f'(z) > (1-\alpha)(2\, log\, 2 - 1) + \alpha \qquad (z \in \Delta).$$

The result is sharp.

PROOF. We need only consider $g(z)$ defined by (2.3). We have

$$g'(z) = (2\alpha - 1) - 2(1-\alpha)\frac{log(1-z)}{z}. \tag{2.4}$$

In [2] it is shown that

$$Re - \frac{log(1-z)}{z} \geq \frac{log(1+r)}{r}, \; |z| = r, \tag{2.5}$$

so that $Re\, g'(z) \geq (2\alpha - 1) + 2(1-\alpha)\frac{log(1+r)}{r}$. Letting $r \to 1$, the result follows.

The case $\alpha = 0$ is found in [5].

COROLLARY 1. $R(\alpha) \subset S$ for $\alpha \geq -\frac{1}{2}\left(\frac{2log2 - 1}{1 - log2}\right) = \alpha_0 \approx -0.63$ and $R(\alpha) \not\subset S$ for $\alpha < \alpha_0$.

PROOF. We know that $P(0) \subset S$. Since $(1-\alpha)(2log2 - 1) + \alpha = 0$ for $\alpha = \alpha_0$, the first part is a consequence of Theorem 2. The result cannot be extended to $\alpha < \alpha_0$ because $g'(-1) = 0$ at $\alpha = \alpha_0$. Thus $g'(-r) = 0$ for some $r = r(\alpha) < 1$ when $\alpha < \alpha_0$.

COROLLARY 2. $\sum_{k=1}^{\infty} \frac{cos\, k\theta}{k+1} \geq \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} = log\, 2 - 1.$

PROOF. From (2.3) we have

$$Re\ g'(z) = 1 + 2(1-\alpha)\sum_{k=1}^{\infty} \frac{r^k \cos k\theta}{k+1},\ |z| = r,$$

which according to (2.4) and (2.5) is minimized when $\theta = \pi$. We then let $r \to 1$.

In [5] it is shown that $Re(f(z)/z) > 1/2$ for all f in R. The next theorem improves this lower bound to $\frac{\pi^2}{6} - 1 \approx 0.645$. But first we state

LEMMA 1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

PROOF. $\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$, so that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12}.$$

THEOREM 3. If $f \in R(\alpha)$, then

$$Re\,\frac{f(z)}{z} > (1-\alpha)\left(\frac{\pi^2}{6} - 1\right) + \alpha \qquad (z \in \Delta).$$

The result is sharp, with the extremal function g defined by (2.2).

PROOF. Again, we need only consider

$$\frac{g(z)}{z} = (2\alpha - 1) - 2(1-\alpha)\int_0^z \frac{log(1-t)}{tz}\,dt.$$

Setting $t = vz$, we may write

$$\frac{g(z)}{z} = (2\alpha - 1) - 2(1-\alpha)\int_0^1 \frac{log(1-vz)}{vz}\,dv. \tag{2.6}$$

Since $Re\left(-\frac{log(1-w)}{w}\right) \geq \frac{log(1+|w|)}{|w|}$, $|w| < 1$, we get from (2.6) that for $|z| = r$,

$$Re\,\frac{g(z)}{z} \geq (2\alpha - 1) + 2(1-\alpha)\int_0^1 \frac{log(1+vr)}{vr}\,dv = \frac{g(-r)}{-r}.$$

But from (2.3) we see that

$$\frac{g(-r)}{-r} = 1 + 2(1-\alpha)\sum_{n=2}^{\infty} \frac{(-r)^{n+1}}{n^2} > 1 + 2(1-\alpha)\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}.$$

An application of Lemma 1 yields

$$Re\,\frac{f(z)}{z} \geq \frac{g(-r)}{-r} > 1 + 2(1-\alpha)\left(\frac{\pi^2}{12} - 1\right) = (1-\alpha)\left(\frac{\pi^2}{6} - 1\right) + \alpha.$$

In [5], R. Singh and S. Singh showed that $R(\alpha) \subset S^*$ for

$$\alpha \geq -\frac{1}{2}\left[\inf_{f \in R(\alpha), z \in \Delta} Re\left(\frac{f(z)}{z}\right)\right].$$

This enabled them to conclude that $R(\alpha) \subset S^*$ for $\alpha \geq -1/4$. Our sharp bound in Theorem 3 gives the following improvement.

COROLLARY. $R(\alpha) \subset S^*$ for $\alpha \geq \frac{6-\pi^2}{24-\pi^2} \approx -0.2738$.

PROOF. The result follows from Theorem 3 upon solving the inequality

$$\alpha \geq -\frac{1}{2}\left((1-\alpha)\left(\frac{\pi^2}{6} - 1\right) + \alpha\right).$$

The next lemma, due to Rogosinski and Szegö, will be needed for our results on partial sums.

LEMMA 2 [3]. $\sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geq -\frac{1}{2}.$

THEOREM 4. Denote by $S_n(z,f)$ the nth partial sum of a function f in $R(\alpha)$. If $f \in R(\alpha)$, then

(i) $S_n(z,f) \in P(\alpha)$,

(ii) $Re\, \frac{S_n(z,f)}{z} > \frac{1+\alpha}{2}, \qquad z \in \Delta.$

The results are sharp, with extremal function $g(z)$ defined by (2.3) and $n = 2$.

PROOF of (i). As before, it suffices to prove our results when $f(z) = g(z)$. We have

$$S_n'(z,g) = 1 + 2(1-\alpha)\sum_{k=2}^{n} \frac{z^{k-1}}{k} = 1 + 2(1-\alpha)\sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{k+1}.$$

By Lemma 2 and the minimum principle for harmonic functions,

$$Re\, S_n'(z,g) > 1 + 2(1-\alpha)(-\tfrac{1}{2}) = \alpha$$

PROOF of (ii). We have

$$Re\, \frac{S_n(z,g)}{z} = 1 + 2(1-\alpha)\sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{(k+1)^2}. \tag{2.7}$$

Since $1/(k+1)$ is decreasing, we use Lemma 2 and summation by parts to obtain

$$\sum_{k=1}^{n-1}\left(\frac{1}{k+1}\right)\left(\frac{\cos k\theta}{k+1}\right) \geq \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}. \tag{2.8}$$

Substituting inequality (2.8) into (2.7) and applying the minimum principle, we get

$$Re\, \frac{S_n(z,g)}{z} > 1 + 2(1-\alpha)\left(-\frac{1}{4}\right) = \frac{1+\alpha}{2}.$$

In the special case $\alpha = 0$, (i) gives the result found in [5] and (ii) improves the estimate of 1/3 to the sharp bound of 1/2.

REMARK. This work was completed while the author was a Visiting Scholar at the University of Michigan.

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