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Θ-R-CONTINUOUS FUNCTIONS

C.W. BAKER

Department of Mathematics
Indiana University Southeast
New Albany, Indiana 47150

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ABSTRACT. A strong form of continuity, called θ-R-continuity, is introduced. It is shown that θ-R-continuity is stronger that R-continuity. Relationships between θ-R-continuity and various closed graph properties are investigated. Additional properties of θ-R-continuous functions are established.

KEW WORDS AND PHRASES. θ-R-continuity, R-continuity, closed graph property, θ-C-closed, almost-regular.
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1. INTRODUCTION.

In this paper a strong form of continuity, which we call θ-R-continuity, is introduced. This form of continuity is stronger than R-continuity developed by Konstadilaki-Savvopoulou and Janković in [1]. As is the case for R-continuity, θ-R-continuity is closely related to regularity and the closed graph property. For example, the range of a θ-R-continuous function is R_1 and a θ-R-continuous function into a T_1-space satisfies a strong form of the closed graph property. Many of the results in Konstadilaki-Savvopoulou and Janković [1] carry over, with some modification, to θ-R-continuous functions. However, one result in [1] is extended to θ-R-continuous functions.

2. PRELIMINARIES.

The symbols X, Y, and Z represent topological spaces with no separation axioms assumed unless explicitly stated. For a subset A of a space X the closure of A, interior of A, and boundary of A are denoted by Cl A, Int A, and Bd A, respectively. The θ-closure of A, denoted by Cl_θ A, is the set of all x in X such that every closed neighborhood of x intersects A nontrivially. A set A is θ-closed provided that A = Cl_θ A and a set is θ-open if its complement is

θ-closed. A set A is regular open (resp. regular closed) if Int Cl A = A (resp. Cl Int A = A). If $\mathcal{F}$ is a filter base on a space X, then by $Cl_\theta\ \mathcal{F}$ we mean the filter base $\{Cl_\theta\ F : F \in \mathcal{F}\}$.

DEFINITION 1. A function f: X → Y is R-continuous (Konstadilaki-Savvopoulou and Janković [1]) (resp. strongly θ-continuous (Long and Herrington [2]), weakly continuous (Levine [3])) if for every x ∈ X and every open subset V of Y containing f(x), there exists an open subset U of X containing x such that Cl f(U) ⊆ V (resp. f(Cl U) ⊆ V, f(U) ⊆ Cl V).

3. RELATIONSHIPS TO OTHER CONTINUITY CONDITIONS.

We define a function f: X → Y to be θ-R-continuous if for each x ∈ X and each open subset V of Y containing f(x), there exists an open subset U of X containing x for which $Cl_\theta\ f(U) \subseteq V$. The following theorem characterizes θ-R-continuity in terms of filter bases. The proof is straightforward and is omitted.

THEOREM 1. A function f: X → Y is θ-R-continuous if and only if for every filter base $\mathcal{F}$ on X, if $\mathcal{F}$ converges to x in X, then the filter base $Cl_\theta\ f(\mathcal{F})$ converges to f(x) in Y.

Since for any set A Cl A ⊆ Cl_θ A, θ-R-continuity implies R-continuity (and hence implies continuity). The following example shows that the converse implication fails.

EXAMPLE 1. Let X = {a, b, c, d}, $\mathfrak{T}$ = {X, ∅, {c}, {c, d}, {a, b, c}} and f: (X, $\mathfrak{T}$) → (X, $\mathfrak{T}$) be the constant map f(x) = a for each x ∈ X. It is easily checked that f is R-continuous but not θ-R-continuous.

Since in a regular space the closure and θ-closure operators agree, obviously if f: X → Y is R-continuous and Y is regular, then f is θ-R-continuous. Also because the closure and θ-closure operators agree on open sets, it follows that if f: X → Y is R-continuous and open, then f is θ-R-continuous. R-continuity (and hence θ-R-continuity) implies strong θ-continuity (Konstadilaki-Savvopoulou and Janković [1]). The following theorem establishes conditions under which this implication can be reversed.

DEFINITION 2. Janković and Rose [4]. A function f: X → Y is weakly θ-closed provided there exists an open basis $\mathcal{B}$ for the topology on X for which $Cl_\theta\ f(U) \subseteq f(Cl\ U)$ for every U ∈ $\mathcal{B}$.

THEOREM 2. If f: X → Y is strongly θ-continuous and weakly θ-closed, then f is θ-R-continuous.

PROOF. Let x ∈ X and let V be an open subset of Y containing f(x). Since f is strongly θ-continuous, there exists an open subset U of X containing x for which f(Cl U) ⊆ V. Let $\mathcal{B}$ be an open basis for the topology on X such that $Cl_\theta\ f(W) \subseteq f(Cl\ W)$ for every W ∈ $\mathcal{B}$. We may assume U ∈ $\mathcal{B}$, Thus $Cl_\theta\ f(U) \subseteq f(Cl\ U) \subseteq V$ and hence f is θ-R-continuous. □

In Konstadilaki-Savvopoulou and Janković [1] it is proved that a continuous function from a locally compact Hausdorff Space into a Hausdorff space is R-continuous. We shall prove that under these conditions the function is actually θ-R-continuous. The essential reason is that a compact subset of a Hausdorff space is θ-closed.

Except for the use of this fact, the proof is the same as in Konstadilaki-Savvopoulou and Jankovic [1].

THEOREM 3. If f: X → Y is continuous, X is locally compact Hausdorff, and Y is Hausdorff, then f is θ-R-continuous.

PROOF. Since X is locally compact Hausdorff, X is regular and therefore f is strongly θ-continuous. From the local compactness of X, we obtain an open basis $\mathcal{B}$ for the topology on X consisting of sets with compact closures. Let $U \in \mathcal{B}$. Since f is continuous, f(Cl U) is compact in Y and hence θ-closed because Y is Hausdorff. Thus $\mathrm{Cl}_\theta\, f(U) \subseteq \mathrm{Cl}_\theta\, f(\mathrm{Cl}\ U) = f(\mathrm{Cl}\ U)$. Therefore f is weakly θ-closed and hence by Theorem 2 f is θ-R-continuous. □

4. CLOSED GRAPH PROPERTIES.

From Konstadilaki-Savvopoulou and Jankovic [1] the graph of an R-continuous function into a T_1-space is θ-closed with respect to the domain. In this section an analogous result is proved for θ-R-continuous functions.

DEFINITION 3. Baker [5]. A subset U of a space X is θ-C-open provided there exists a subset A of X for which $U = X - \mathrm{Cl}_\theta\, A$.

DEFINITION 4. Let f: X → Y be a function. The graph of f, denoted by G(f), is θ-C-closed with respect to X if for each (x, y) G(f) there exist subsets U and V of X and Y, respectively, with $x \in U$, $y \in V$, U open, V θ-C-open, and $((\mathrm{Cl}\ U) \times V) \cap G(f) = \emptyset$ (or equivalently $f(\mathrm{Cl}\ U) \cap V = \emptyset$).

THEOREM 4. If f: X → Y is θ-R-continuous and Y is T_1, then G(f) is θ-C-closed with respect to X.

PROOF. Assume $(x, y) \notin G(f)$. Since $y \neq f(x)$ and Y is T_1, there exists an open subset V of Y such that $f(x) \in V$ and $y \notin V$. The θ-R-continuity of f implies the existence of an open subset U of X containing x such that $\mathrm{Cl}_\theta\, f(U) \subseteq V$. Therefore $(x, y) \in (\mathrm{Cl}\ U) \times (Y - \mathrm{Cl}_\theta\, f(U))$ which is disjoint from G(f) because if $a \in \mathrm{Cl}\ U$, then $f(a) \in f(\mathrm{Cl}\ U) \subseteq \mathrm{Cl}\ f(U) \subseteq \mathrm{Cl}_\theta\, f(U)$. Note that $Y - \mathrm{Cl}_\theta\, f(U)$ is θ-C-open. □

Next we establish conditions under which a weak form of continuity and some type of closed graph property imply θ-R-continuity.

DEFINITION 5. Jankovic and Rose [4]. The graph of a function f: X → Y is θ-closed with respect to Y if for each $(x, y) \notin G(f)$, there exist open subsets U and V of X and Y, respectively, with $x \in U$, $y \in V$, and $(U \times \mathrm{Cl}\ V) \cap G(f) = \emptyset$ (or equivalently $f(U) \cap \mathrm{Cl}\ V = \emptyset$).

DEFINITION 6. Singal and Arya [6]. A space X is almost-regular if each regular closed set C and each $x \in X - C$ can be separated by disjoint open subsets of X.

The assumption that the codomain is almost-regular is required in several of the following theorems. Note that under this assumption, θ-R-continuity is not equivalent to R-continuity. The space in Example 1 is almost-regular whereas the function is R-continuous but not θ-R-continuous.

From Jankovic and Rose [4] we have that a space X is almost-regular if and only if regular closed sets are θ-closed. This fact is used in the following proof.

DEFINITION 7. Konstadilaki-Savvopoulou and Jankovic [1]. A space X is rim-compact provided there exists an open basis for the topology on X consisting of sets with compact boundaries.

THEOREM 5. If $f: X \to Y$ is weakly continuous and has a θ-closed graph with respect to Y and Y is rim-compact and almost-regular, then f is θ-R-continuous.

PROOF. Let $x \in X$ and let V be an open subset of X containing f(x). Since Y is rim-compact, there exists an open subset W of Y for which $f(x) \in W \subseteq V$ and Bd W is compact. The weak continuity of f implies the existence of an open subset U of X containing x such that $f(U) \subseteq \text{Cl } W$.

Let $y \in \text{Bd } W$. Since $f(x) \in W$ and $W \cap \text{Bd } W = \emptyset$, $(x, y) \notin G(f)$. Because G(f) is θ-closed with respect to Y, there exist open subsets A_y and B_y of X and Y, respectively, for which $x \in A_y$, $y \in B_y$, and $f(A_y) \cap \text{Cl } B_y = \emptyset$. It follows that $(\text{Cl}_\theta f(A_y)) \cap B_y = \emptyset$.

The collection $\{B_y : y \in \text{Bd } W\}$ is an open cover of Bd W, which is compact. Hence there is a finite collection $\{B_{yi} : i = 1, 2, \ldots, n\}$ for which $\text{Bd } W \subseteq \cup \{B_{yi} : i = 1, 2, \ldots, n\}$. Let $U_o = U \cap (\cap \{A_{yi} : i = 1, 2, \ldots, n\})$. Then $\text{Cl}_\theta f(U_o) \subseteq \text{Cl}_\theta f(\cap \{A_{yi} : i = 1, 2, \ldots, n\}) \subseteq \text{Cl}_\theta \cap \{f(A_{yi}) : i = 1, 2, \ldots, n\} \subseteq \cap \{\text{Cl}_\theta f(A_{yi}) : i = 1, 2, \ldots, n\}$ which is disjoint from $\cup\{B_{yi} : i = 1, 2, \ldots, n\}$ and hence disjoint from Bd W. Therefore $(\text{Cl}_\theta f(U_o)) \cap \text{Bd } W = \emptyset$. However, $\text{Cl}_\theta f(U_o) \subseteq \text{Cl}_\theta f(U) \subseteq \text{Cl}_\theta \text{Cl } W = \text{Cl } W$. (This last equality follows from the fact that Cl W is regular closed and Y is almost-regular.) Thus we have that $\text{Cl}_\theta f(U_0) \subseteq (\text{Cl } W) - \text{Bd } W \subseteq W \subseteq V$ and therefore f is θ-R-continuous. □

5. ADDITIONAL PROPERTIES.

In Konstadilaki-Savvopoulou and Janković [1] it is proved that a function $f: X \to Y$ is R-continuous if and only if for every $x \in X$ and every closed subset F of Y with $f(x) \notin F$, there exist open subsets U and V of X and Y, respectively, such that $x \in U$, $F \subseteq V$, and $f(U) \cap V = \emptyset$. The following three theorems are analogous results for θ-R-continuous functions.

THEOREM 6. If $f: X \to Y$ is θ-R-continuous, then for every $x \in X$ and every closed subset F of Y such that $f(x) \notin F$, there exists an open subset U of X containing x and a θ-C-open subset V of Y with $F \subseteq V$ such that $f(\text{Cl } U) \cap V = \emptyset$.

PROOF. Let $x \in X$ and let F be a closed subset of Y with $f(x) \in X - F$. There exists an open subset U of X containing x such that $\text{Cl}_\theta f(U) \subseteq Y - F$. Let $V = Y - \text{Cl}_\theta f(U)$. Then V is θ-C-open and $F \subseteq V$. Since f is continuous, $f(\text{Cl } U) \subseteq \text{Cl } f(U) \subseteq \text{Cl}_\theta f(U)$. Therefore $f(\text{Cl } U) \cap V = \emptyset$. □

If the condition that V be θ-C-open is replaced with the stronger requirement that V be θ-open, then the implication in Theorem 6 can be reversed.

THEOREM 7. Let $f: X \to Y$ be a function. If for every $x \in X$ and every closed subset F of Y with $f(x) \notin F$ there exists an open subset U of X containing x and a θ-open subset V of Y with $F \subseteq V$ such that $f(U) \cap V = \emptyset$, then f is θ-R-continuous.

PROOF. Let $x \in X$ and let V be an open subset of Y with $f(x) \in V$. Let $F = Y - V$. Since $f(x) \notin F$, there exists an open subset U of X containing x and a θ-open subset W of Y with $F \subseteq W$ and $f(U) \cap W = \emptyset$. Then $f(U) \subseteq Y - W$. Thus $Cl_\theta\, f(U) \subseteq Cl_\theta\, (Y - W) = Y - W \subseteq Y - F = V$. Therefore f is θ-R-continuous. □

In an almost-regular space $Cl_\theta\, Cl_\theta\, A = Cl_\theta\, A$ for any set A (Janković and Rose [4]). It follows that in a almost-regular space θ-C-openness is equivalent to θ-openness. Therefore we have the following result.

THEOREM 8. Let X and Y be topological spaces with Y almost-regular. Then $f: X \to Y$ is θ-R-continuous if and only if for every $x \in X$ and every closed subset F of Y with $f(x) \notin F$, there exist an open subset U of X containing x and an θ-C-open subset V of Y with $F \subseteq V$ such that $f(U) \cap V = \emptyset$.

In the definition of R-continuity the condition "Cl f(U) ⊆ V" can be replaced with "Cl f(Cl U) ⊆ V" (Konstadilaki-Savvopoulou and Janković [1]). If we require the codomain to be almost-regular, the following similar result holds for θ-R-continuity.

THEOREM 9. Let X and Y be topological spaces with Y almost-regular. Then $f: X \to Y$ is θ-R-continuous if and only if for each $x \in X$ and each open subset V of Y containing f(x), there exists an open subset U of X containing x such that $Cl_\theta\, f(Cl\, U) \subseteq V$.

PROOF. Assume $f: X \to Y$ is θ-R-continuous. Let $x \in X$ and let V be an open subset of Y containing f(x). Then there exists an open subset U of X containing x such that $Cl_\theta\, f(U) \subseteq V$. Since f is continuous, $Cl_\theta\, f(Cl\, U) \subseteq Cl_\theta\, Cl\, f(U) \subseteq Cl_\theta\, Cl_\theta\, f(U)$ and since Y is almost-regular, $Cl_\theta\, Cl_\theta\, f(U) = Cl_\theta\, f(U) \subseteq V$. Thus $Cl_\theta\, f(Cl\, U) \subseteq V$.

The converse implication is immediate. □

In Konstadilaki-Savvopoulou and Janković [1] the range of an R-continuous function is shown to be R_0. Here we show that the range of a θ-R-continuous function satisfies the stronger R_1 condition. The following definition is implicit in Theorem 3.1 in Janković [7].

DEFINITION 8. Janković [7]. A space X is R_1 if for each open set U and each $x \in U$, $Cl_\theta\, \{x\} \subseteq U$.

THEOREM 10. If $f: X \to Y$ is a θ-R-continuous surjection, then Y is R_1.

PROOF. Let $y \in Y$ and let V be an open subset of Y containing y. Let $x \in X$ such that $y = f(x)$. Since f is θ-R-continuous, there exists an open subset U of X containing x for which $Cl_\theta\, f(U) \subseteq V$. Then $Cl_\theta\, \{y\} \subseteq Cl_\theta\, f(U) \subseteq V$. □

We close this section with a sample of the basic properties of θ-R-continuous functions concerning composition and restriction. Most of these are analogues of the corresponding properties for R-continuous or continuous functions. The proofs are straightforward and are omitted.

THEOREM 11. If $f: X \to Y$ is continuous and $g: Y \to Z$ is θ-R-continuous, then $g \circ f: X \to Z$ is θ-R-continuous.

COROLLARY 1. If both $f: X \to Y$ and $g: Y \to Z$ are θ-R-continuous, then $g \circ f: X \to Z$ is θ-R-continuous.

THEOREM 12. If $f: X \to Y$ and $g: Y \to Z$ are functions with $g \circ f: X \to Z$ θ-R-continuous, and f is an open surjection, then g is θ-R-continuous

THEOREM 13. If $f: X \to Y$ is θ-R-continuous, $A \subseteq X$, and $f(A) \subseteq B \subseteq Y$, then $f|_A: A \to B$ is θ-R-continuous.

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