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A CRITERION FOR P-VALENTLY STARLIKE FUNCTIONS

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ABSTRACT. The object of the present paper is to prove a criterion for p-valently starlike functions in the open unit disk.

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1. INTRODUCTION.

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ belonging to $A(p)$ is said to be p-valently starlike in U if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in U). \tag{1.2}$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions $f(z)$ which are p-valently starlike in U (cf. [1]).

Recently, Nunokawa [4] has shown that

THEOREM A. If $f(z) \in A(p)$ satisfies $f(z) \neq 0 (0 < |z| < 1)$ and

$$Re\left\{\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\} < 1 + \frac{1}{2p} \qquad (z \in U), \tag{1.3}$$

then $f(z) \in S(p)$.

In the present paper, we derive a new criterion for the class $S(p)$ involving the above result by Nunokawa [4].

2. A NEW CRITERION.

To derive our main result, we have to recall here the following lemma due to Jack [2] (also, due to Miller and Mocanu [3]).

LEMMA. Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then we can write

$$z_0 w'(z_0) = k w(z_0), \tag{2.1}$$

where k is a real number and $k \geqq 1$.

Now, we prove

THEOREM. If $f(z) \in A(p)$ satisfies $f(z) \neq 0 (0 < |z| < 1)$ and

$$\left| arg\left\{ \frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(1+\frac{1}{4p}\right)\right\}\right| > 0 \qquad (z \in U), \tag{2.2}$$

then $f(z) \in S(p)$ and

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p \qquad (z \in U). \tag{2.3}$$

PROOF. Define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = p(1+w(z)). \tag{2.4}$$

Then $w(z)$ is analytic in U and $w(0) = 0$. It follows from (2.4) that

$$1+\frac{zf''(z)}{f'(z)} = p(1+w(z))+\frac{zw'(z)}{1+w(z)}, \tag{2.5}$$

so that,

$$\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) = 1+\frac{zw'(z)}{p(1+w(z))^2}. \tag{2.6}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq -1).$$

Then, applying Lemma, we can write

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geqq 1)$$

and $w(z_0) = e^{i\theta} (\theta \neq \pi)$. Thus we have

$$\begin{aligned}\frac{f(z_0)}{z_0 f'(z_0)}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= 1+\frac{ke^{i\theta}}{p(1+e^{i\theta})^2}\\ &= 1+\frac{k}{2p(1+cos\theta)}\\ &\geqq 1+\frac{1}{4p}.\end{aligned} \tag{2.7}$$

Note that the condition (2.2) implies

$$\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \neq \alpha \qquad (z \in U), \tag{2.8}$$

where $\alpha \geqq 1 + 1/4p$. Therefore, (2.7) contradicts our condition (2.2). Consequently, we conclude that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p \qquad (z \in U), \tag{2.9}$$

that is, that $f(z) \in S(p)$.

Letting $p = 1$ in Theorem, we have

COROLLARY. If $f(z) \in A(1)$ satisfies $f(z) \neq 0 (0 < |z| < 1)$ and

$$\left| arg\left\{ \frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{5}{4}\right| > 0 \qquad (z \in U), \tag{2.10}$$

then $f(z) \in S(1)$ and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in U). \tag{2.11}$$

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