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EXISTENCE THEOREMS FOR THE IMPLICIT COMPLEMENTARITY PROBLEM

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ABSTRACT. Some existence theorems for the general implicit complementarity problem in an infinite dimensional space are considered.

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1. INTRODUCTION.

The study of Complementarity Problems is an interesting and important domain of applied mathematics [1], [5], [8], [10] etc. In this domain, a special chapter is the Implicit Complementarity Problem. It seems that the first Implicit Complementarity Problem was defined in 1973 by Bensoussan and Lions [2], as the mathematical model of some stochastic optimal control problems [2], [3], [4]. Now, it is well known that, the Implicit Complementarity Problem can be used to study the optimal stopping of Markov chains [6].

The first existence results for the Implicit Complementarity are the results obtained by Dolcetta and Mosco [7], [18], [19].

As numerical methods for solving the Implicit Complementarity Problem we remark the iterative methods proposed by Pang [20], [21] and Mosco [22].

In this paper, we study some existence theorems for the general Implicit Complementarity Problem in an infinite dimensional space. This paper can be considered as a complement of our paper [13].

2. DEFINITION OF PROBLEM AND PRELIMINARIES.

Let $<E,E^*>$ be a dual system of Banach spaces. Denote by $\mathbf{K}$ a pointed convex cone in E, that is, a subset of E satisfying the following properties:

1°) $\mathbf{K}+\mathbf{K}\subseteq\mathbf{K}$ 2°) $\lambda\mathbf{K}\subseteq\mathbf{K}$, for all $\lambda\in\mathbf{R}$ and 3°) $\mathbf{K}\cap(-\mathbf{K})=\{0\}$.

The closed convex cone $\mathbf{K}^*=\{y\in C^*\mid <x,y>\ \geq 0;$ for all $x\in\mathbf{K}\}$ is called the dual of $\mathbf{K}$.

Given a subset $D \subset E$ and the mappings $S: D \to \mathbf{K}$ and $T: D \to E^*$, the Implict Complementarity Problem associated to T, S and $\mathbf{K}$ is

$$ICP(T,S,\mathbf{K}): \left| \begin{array}{l} \text{find } x_o \in D \text{ such that} \\ T(x_o) \in \mathbf{K}^* \text{ and} \\ < S(x_o),\ T(x_o) > \ = 0. \end{array} \right.$$

We find applications and examples of this problem in [2], [3], [4], [6], [7], [18], [19], [20], [21].

When $D = \mathbf{K}$ and $S(x) = x$, for all $x \in \mathbf{K}$, the problem $ICP(T,S,\mathbf{K})$ is exactly the nonlinear complementarity problem, which has interesting applications in: Optimization, Game Theory, Economics, Mechanics, etc. [1], [5], [8-15].

If the problem $ICP(T,S,\mathbf{K})$ is defined, we consider the following special variational inequality:

$$SVI(T,S,\mathbf{K}): \left[\begin{array}{l} \text{find } x_o \in D \text{ such that} \\ < x - S(x_o),\ T(x_o) > \ \geq 0; \forall x \in \mathbf{K}. \end{array} \right.$$

PROPOSITION 1. The problem $SVI(T,S,\mathbf{K})$ is equivalent to the problem $ICP(T,S,\mathbf{K})$.

PROOF. Indeed, if x_o is a solution of the problem $SVI(T,S,\mathbf{K})$ then $S(x_o) \in \mathbf{K}$ and we have

$$(1): < x - S(x_o),\ T(x_o) > \ \geq 0;\ \forall x \in \mathbf{K}$$

Let $u \in \mathbf{K}$ be an arbitrary element. If we put $x = u + S(x_o)$ in (1) we obtain $< u, T(x_o) > \ \geq 0$, for every $u \in \mathbf{K}$, that is, we have $T(x_o) \in \mathbf{K}^*$.

If we put $x = 0$ in (1) we have $< S(x_o),\ T(x_o) > \ \leq 0$ and since $< S(x_o),\ T(x_o) > \ \geq 0$ we deduce $< S(x_o), T(x_o) > \ = 0$.

Conversely, let x_o be a solution of the problem $ICP(T,S,\mathbf{K})$. We have, $S(x_o) \in \mathbf{K}$, $T(x_o) \in \mathbf{K}^*$ and $< S(x_o),\ T(x_o) > \ = 0$ which imply $< x - S(x_o),\ T(x_o) > \ \geq 0$, for every $x \in \mathbf{K}$.

Given a nonempty subset $D \subset E$ and the mappings $T: D \to E^*$ and $S: D \to \mathbf{K}$ we consider the following problem

$$SVI(T,S,D): \left[\begin{array}{l} \text{find } x_o \in D \text{ such that} \\ < x - S(x_o),\ T(x_o) > \ \geq 0;\ \forall x \in D\ , \end{array} \right.$$

To solve the problem $SVI(T,S,D)$ we use the following classical result.

THEOREM 1. A mapping $T_o: D \to 2^D$, where $D \subset X$, have a fixed point if the following conditions are satisfied:

1°): X is a locally convex space and the set D is nonempty, compact, and convex,

2°): the set $T_o(X)$ is nonempty and convex for all $x \in D$ and the preimages $T_0^{-1}(y) = \{x \in D \mid y \in T_o(x)\}$ are relatively open with respect to D, for all $y \in D$.

PROOF. The proof is in [25][Proposition 9.9, p. 453].

THEOREM 2. Let $D \subset E$ be a nonempty compact convex set, $T: D \to E^*$ and $S: D \to \mathbf{K}$ two continuous mappings.

If for every $x \in D$ we have $< S(x),\ T(x) > \ \leq \ < x,\ T(x) >$, then the problem $SVI(T,S,D)$ has a solution.

PROOF. If the problem $SVI(T,S,D)$ does not have a solution then,

$$(2): (\forall x \in D)(\exists u \in D)(< u - S(x),\ T(x) > \ < 0)$$

Let $T_o: D \to D$ be the point-to-set mapping defined by, $T_o(x) = \{u \in D \mid \langle u - S(x), T(x) \rangle < 0\}$, for every $x \in D$.

We remark that $T_o(x)$ is nonempty and convex for every $x \in D$.

Since T and S are continuous, the mapping $v \to \langle x - S(v), T(v) \rangle$ is continuous too and we have that $T_o^{-1}(y) = \{x \in D \mid y \in T_o(x)\} = \{x \in D \mid \langle y - S(x), T(x) \rangle < 0\}$ is relatively open with respect to D.

Hence, by Theorem 1 there is an element $x_* \in D$ such that $x_* \in T_o(x_*)$, that is, $\langle x_* - S(x_*), T(x_*) \rangle < 0$, which is impossible since for every $x \in D$ we have (by assumption) $\langle S(x), T(x) \rangle \le \langle x, T(x) \rangle$.

Let $\mathbf{K}$ be a pointed convex cone in E. We say that a subset B of $\mathbf{K}$ is a base, if B is convex and for every $x \in \mathbf{K} \setminus \{0\}$ there is a unique $b_x \in B$ and a unique number $\lambda_x \in R_+ \setminus \{0\}$ such that $x = \lambda_x b_x$.

A closed pointed convex cone $\mathbf{K} \subset E$ is locally compact if and only if, it has a compact base [Klee's Theorem].

If $r \in \mathbf{R}_+ \setminus \{0\}$ we denote $\mathbf{K}_r^{\le} = \{x \in \mathbf{K} \mid \|x\| \le r\}$ and $\mathbf{K}_r^{<} = \{x \in \mathbf{K} \mid \|x\| < r\}$.

We say that a convex cone $\mathbf{K} \subset E$ is a **Galerkin cone** [10] if there exists a countable family of convex subcones $\{K_n\}_{n \in N}$ of $\mathbf{K}$ such that:

i) $\mathbf{K}_n$ is locally compact for every $n \in N$,

ii) if $n \le m$ then $\mathbf{K}_n \subseteq \mathbf{K}_m$,

iii) $\mathbf{K} = \overline{\bigcup_{n \in N} \mathbf{K}_n}$

A Galerkin cone will be denoted by $\mathbf{K}(\mathbf{K}_n)_{n \in N}$.

We recall that if $D \subset E$ is a closed convex set, we say that a continuous operator (not necessary linear) $P: E \to E$ is a projection onto D if $P(E) = D$ and $P(x) = x$ for every $x \in D$.

By the same proof as in our paper [12] we can prove that if $\mathbf{K}(\mathbf{K}_n)_{n \in N}$ is a Galerkin cone in a Banach space, then for every $n \in N$ there exists a projection P_n onto $\mathbf{K}_n$ such that for every $x \in \mathbf{K}$ we have $\lim_{n \to \infty} P_n(x) = x$.

Given two Banach spaces $(E, \|\ \|)$ and $(F, \|\ \|)$ we say that an operator (not necessary linear) $T: E \to F$ is **strongly continuous** if for every sequence $\{x_n\}_{n \in N} \subset E$, weakly convergent to x_* we have that $\{T(x_n)\}_{n \in N}$ is norm convergent to $T(x_*)$.

This class of operators is very important and was intensively studied by Vainberg [24] and Lipkin [17].

3. PRINCIPAL RESULTS.

The principal aim of this paper is to give some existence theorems for the problem $ICP(T, S, \mathbf{K})$.

In this sense, we suppose given a dual system $\langle E, E^* \rangle$ of Banach spaces. We consider on E^* the strong topology.

THEOREM 3. Let $\mathbf{K} \subset E$ be a pointed locally compact cone and $S: \mathbf{K} \to E$, $T: \mathbf{K} \to E^*$ continuous mappings. If the following assumptions are satisfied:

1°) there is a number $r > 0$ such that $S(\mathbf{K}_r^{\le}) \subseteq \mathbf{K}$,

2°) there is an element $u_o \in \mathbf{K}$ such that $S(u_o) \in \mathbf{K}$, $\|S(u_o)\| < r$ and $\langle x - S(u_o), T(x) \rangle \ge 0$, for all $x \in \mathbf{K}$ satisfying $r \le \|x\| \le max(r, r_o)$ where r_o is a number such that $sup\{\|S(u)\| \mid u \in \mathbf{K}_r^{\le}\} \le r_o$,

3°) $\langle S(x), T(x) \rangle \le \langle x, T(x) \rangle$; $\forall x \in \mathbf{K}_r^{\le}$,

then the problem $ICP(T, S, \mathbf{K})$ has a solution $x_* \in \mathbf{K}_r^{\le}$ such that $\|S(x_*)\| \le max(r, r_o)$.

PROOF. Since $\mathbf{K}$ is locally compact we have that $\mathbf{K}_r^{\leq}$ is a convex compact set. Applying Theorem 2 with $D = K_r^{\leq}$, we obtain an element $x_* \in \mathbf{K}_r^{\leq}$ such that

$$(3): < x - S(x_*),\ T(x_*) > \ \geq 0;\ \forall x \in \mathbf{K}_r^{\leq}$$

We have that $S(x_*) \in \mathbf{K}$. Two cases are possible:

I) $\| S(x_*) \| < r$. If $x \in \mathbf{K}$ is an arbitrary element then there is a sufficiently small $\lambda \in]0,1[$ such that $w = \lambda x + (1-\lambda)S(x_*) \in \mathbf{K}_r^{\leq}$. If in (3) we put $x = w$ we have, $\lambda < x - S(x_*), T(x_*) > \ \geq 0$, that is, $< x - S(x_*), T(x_*) > \ \geq 0$ for all $x \in \mathbf{K}$ and by Proposition 1 we obtain that x_* is a solution of the problem $ICP(T, S < \mathbf{K})$.

II) $\| S(x_*) \| \geq r$. In this case we have $r \leq \| S(x_*) \| \leq max(r, r_0)$ and by assumption 2°) we obtain,

$$(4): < S(x_*) - S(u_o),\ T(x_*) > \ \geq 0,$$

and since for every $x \in \mathbf{K}_r^{\leq}$ we have

$$(5): < x - S(x_*),\ T(x_*) > \ \geq 0$$

we deduce (using (4) and (5)), $< x - S(u_o),\ T(x_*) > \ = \ < x - S(x_*) + S(x_*) - S(u_o),\ T(x_*) >$ $= \ < x - S(x_*),\ T(x_*) > \ + \ < S(x_*) - S(u_o), T(x_*) > \ \geq 0$, that is, we have

$$(6): < x - S(u_o),\ T(x_*) > \ \geq 0;\ \forall x \in \mathbf{K}_r^{\leq}.$$

If $x \in \mathbf{K}$ is an arbitrary element then there is a sufficiently small $\lambda \in]0,1[$ such that $v = \lambda x + (1-\lambda)S(u_o) \in \mathbf{K}_r^{\leq}$. Now, if we put $x = v$ in (6) we obtain,

$$(7): < x - S(u_o),\ T(x_*) > \ \geq 0;\ \forall x \in \mathbf{K}.$$

Since $\| S(u_o) \| < r$ we can put $x = S(u_o)$ in (3) and we deduce,

$$(8): < S(u_o) - S(u_*),\ T(x_*) > \ \geq 0.$$

From (7) and (8) we obtain

$$(9): < x - S(x_*),\ T(x_*) > \ \geq 0;\ \forall x \in \mathbf{K}.$$

Since $S(x_*) \in \mathbf{K}$, from (9) and Proposition 1 we obtain that x_* is a solution of the problem $ICP(T, S, \mathbf{K})$ and the proof is finished.

Theorem 3 can be extended to Galerkin cones. To obtain this extension we need to introduce a new concept.

We say that $S: \mathbf{K} \to E$ is **subordinate** to the approximation $(\mathbf{K}_n)_{n \in N}$ if there exists $n_o \in N$ such that for every $n \geq n_o$ we have $S(\mathbf{K}_n) \subseteq \mathbf{K}_n$.

In [13] we indicated some examples of mappings with this property.

Independent of us in [16] is defined the concept of F-mapping which is similar to our concept.

In [16] we showed that every DC-mapping can be approximated by an F-mapping while the class of DC-mappings is very reach.

We say that $S: \mathbf{K} \to E$ is **r-subordinate** to the approximation $(\mathbf{K}_n)_{n \in N}$ if there exist $r > 0$ and $n_o \in N$ such that for every $n \geq n_o$ we have $S(\mathbf{K}_{rn}^{\leq}) \subset \mathbf{K}_n$, where $\mathbf{K}_{rn}^{\leq} = \{x \in \mathbf{K}_n \mid \| x \| \leq r\}$.

REMARK. If $S: \mathbf{K} \to E$ is continuous and r-subordinate to the approximation $(\mathbf{K}_n)_{n \in N}$ then $S(\mathbf{K}_r^{\leq}) \subset \mathbf{K}$.

Indeed, if $x \in \mathbf{K}_r^{\leq}$ then we have two cases:

a) $\|x\| < r$. Since $\mathbf{K}$ is a Galerkin cone there is a sequence $\{x_n\}_{n\in N}$ such that $x = \lim_{n\to\infty} x_n$ and for every $n \in N$, $x_n \in \mathbf{K}_n$.

There exists $n_1 \in N$ such that $\|x_n - x\| < r - \|x\|$, for every $n \geq n_1$, which implies, $\|x_n\| \leq \|x_n - x\| + \|x\| < r$.

Since, for every $n \geq max(n_o, n_1)$ we have $S(x_n) \in \mathbf{K}_n \subset \mathbf{K}$, we obtain by continuity that $S(x) \in \mathbf{K}$.

b) $\|x\| = r$. If for every $n \in N$, $x_n \in \mathbf{K}_n$ $\lim_{n\to\infty} x_n = x$ and $r < \|x_n\|$ then considering the sequence $y_n = \left(\frac{r}{\|x_n\|} - \varepsilon_n\right) x_n$, where $0 < \varepsilon_n < \frac{r}{\|x_n\|}$; $\forall n \in N$ and $\lim_{n\to\infty} \varepsilon_n = 0$ we have that $y_n \in \mathbf{K}_n$, $\|y_n\| < r$ and $\lim_{n\to\infty} y_n = x$, which imply that $S(x) = \lim_{n\to\infty} S(y_n) \in \mathbf{K}$.

THEOREM 4. Let $(E, \|\ \|)$ be a reflexive Banach space and $\mathbf{K}(\mathbf{K}n)n \in N$ a Galerkin cone in E. Let $S:\mathbf{K}\to E$ and $T:\mathbf{K}\to E^*$ be strongly continuous mappings.

If the following assumptions are satisfied:

1°) S is r-subordinate to the approximation $\mathbf{K}(\mathbf{K}n)n \in N$,

2°) there exist $m \in N$ and $u_o \in \mathbf{K}_m$ such that $\|S(u_o)\| < r$, $S(u_o) \in \mathbf{K}_m$ and $< x - S(u_o), T(x) > \geq 0$, for all $x \in \mathbf{K}_n$ satisfying $r \leq \|x_n\| \leq max(r, r_n)$, where r_n is a number such that $sup\{\|S(u)\| \mid u \in \mathbf{K}_n^{\leq}\} \leq r_n$ and for every $n \geq max(n_o, m)$,

3°) $< S(x), T(x) > \leq < x, T(x) >$; $\forall x \in \mathbf{K}_r^{\leq}$,

then the problem $ICP(T,S,\mathbf{K})$ has a solution x_* such that $\|x_*\| \leq r$.

PROOF. We remark that for every $n \geq max(n_o, m)$ the all assumptions of Theorem 3 are satisfied for every problem $ICP(T,S,\mathbf{K}_n)$ and hence we have a solutionx_n^* for each of these problems.

Since for ever x_n^* (with $n \geq max(n_o, m)$) we have $\|x_n^*\| \leq r$ we have that $\{x_n\}_{n\in N}$ is a bounded sequence.

Because E is reflexive $\{x_n^*\}_{n\in N}$ has a weakly convergent subsequence $\{x_{nk}^*\}_{k\in N}$. We denote again this subsequence by $\{x_n^*\}_{n\in N}$ and we put $x_* = (w) - \lim_{n\to\infty} x_n^*$. We have that $x_* \in \mathbf{K}$ and $\|x_*\| \leq r$, since $\mathbf{K}_r^{\leq}$ is closed and convex. Hence $S(x_*) \in \mathbf{K}$.

Let $x \in \mathbf{K}$ be an arbitrary element. For every $n \geq max(n_o, m)$ we have,

$$(10): < P_n(x) - S(x_n^*),\ T(x_n^*) > \geq 0,$$

where $\{P_n\}_{n\in N}$ is a sequence of projections. Since S and T are strongly continuous, computing the limit in (10) we obtain,

$$(11): < x - S(x_*),\ T(x_*) > \geq 0; \text{ for all } x \in \mathbf{K}.$$

The proof is finished since from (11) by Proposition 1 we have that x_* is a solution of the problem $ICP(T,S,\mathbf{K})$.

We consider now the case when $S(\mathbf{K}) \subseteq \mathbf{K}$.

THEOREM 5. Let $(E, \|\ \|)$ be a Banach space, $\mathbf{K} \subset E$ a pointed locally compact convex cone and $S:\mathbf{K}\to\mathbf{K}$, $T:\mathbf{K}\to E^*$ continuous mappings.

If the following assumptions are satisfied:

1°) $< S(x),\ T(x) > \leq < x,\ T(x) >$; $\forall x \in \mathbf{K}$,

2°) there is $r > 0$ such that for every $x \in \mathbf{K}$ with $r \leq \|x\|$ there is an element $v_x \in \mathbf{K}$ such that $\|v_x\| < r$ and $< S(x) - v_x,\ T(x) > > 0$, then the problem $ICP(T,S,\mathbf{K})$ has a solution x_* such that $\|x_*\| < r$.

PROOF. We denote $D_n = \{x \in \mathbf{K} \mid \|x\| \leq n\}$. Since $\mathbf{K}$ is locally compact we have that for every $n \in N$, D_n is a convex compact set.

We apply Theorem 2 with $D = D_n$ and we obtain a solution x_n^* for the problem $SVI(T,S,D_n)$.

So we have:

$$(12):\begin{cases} \text{for every } n\in N \text{ there is } x_n^*\in D \text{ such that} \\ < S(x_n^*)-v,\ T(x_n^*)> \ \leq 0;\ \forall v\in D_n \end{cases}$$

The sequence $\{x_n^*\}_{n\in N}$ is bounded. Indeed, supposing the contrary we have $(\forall k>0)(\exists n\in N)(\|x_n^*\|\geq k)$.

If $k\geq r$ then there is a natural number n such that, $n\geq \|x_n^*\|\geq k\geq r$. For this x_n^*, by assumption 2°) there is an element $v_{x^*_n}\in \mathbf{K}$ such that $\|v_{x^*_n}\|<r$ and,

$$(13):\ < S(x_n^*)-v_{x^*_n},\ T(x_n^*)> \ >0.$$

But, since $\|v_{x^*_n}\|<r<\|x_n^*\|\leq n$, from (12) we have $< S(x_n^*)-v_{x^*_n},\ T(x_n^*)> \ \leq 0$, which is a contradiction of (13).

Hence, $\{x_n^*\}_{n\in N}$ is bounded and because $\mathbf{K}$ is locally compact the sequence $\{x_n^*\}_{n\in N}$ has a norm convergent subsequence $\{x_{n_k}^*\}_{k\in N}$.

Let $x_*=\lim_{k\to\infty} x_{n_k}^*$. We show now that x_* is a solution of the problem $ICP(T,S,\mathbf{K})$.

Indeed, if $v\in\mathbf{K}$ is an arbitrary element, then there is $m\in N$ such that for every $n\geq m$ we have, $v\in D_n$ and for every $n_k\geq m$, $v\in D_{n_k}$ and $< S(x_{n_k}^*)-v,\ T(x_{n_k}^*)> \ \leq 0$.

Using the continuity of S and T we obtain,

$< S(x_*)-v,\ T(x_*)> \ \leq 0,\ \forall v\in\mathbf{K}$, that is x_* is a solution of the problem $SVI(T,S,\mathbf{K})$ which, by Proposition 1 is equivalent to the problem $ICP(T,S,\mathbf{K})$. Obviously, by assumption 2°) we must have $\|x_*\|<r$.

From Theorem 5 we deduce two important corollaries.

COROLLARY 1. Let $\mathbf{K}\subset E$ be a pointed locally compact cone and $S:\mathbf{K}\to\mathbf{K}$, $T:\mathbf{K}\to E^*$ continuous mappings. If the following assumptions are satisfied:

1°) $< S(x),T(x)> \ \geq \ < x,T(x)>;\ \forall x\in\mathbf{K}$,

2°) there is a number $r>0$ such that for every $x\in\mathbf{K}$ with $r\leq\|x\|$ we have $< S(x),T(x)> \ >0$,

then the problem $ICP(T,S,\mathbf{K})$ has a solution x_* such that $\|x_*\|\leq r$.

PROOF. We apply Theorem 5 with $v_*=0$ for every $x\in\mathbf{K}$ satisfying $\|x\|\geq r$.

COROLLARY 2. Let $\mathbf{K}\subset E$ be a pointed locally compact cone and $S:\mathbf{K}\to\mathbf{K}$, $T:\mathbf{K}\to E^*$ continuous mappings. If the following assumptions are satisfied:

1°) $< S(x),T(x)> \ \leq \ < x,\ T(x)>;\ \forall x\in\mathbf{K}$,

2°) there is a number $r_o\geq 0$ and $u_o\in\mathbf{K}$ such that for every $x\in\mathbf{K}$ with $r_o\leq\|x\|$ we have $< S(x)-u_o,\ T(x)> \ >0$.

then the problem $ICP(T,S,\mathbf{K})$ has a solution x_* such that $\|x_*\|<1+max(r_o,\|u_o\|)$.

PROOF. If we denote, $r=max(r_o,\|u_o\|)+1$, we have $r>r_o$ and $r>\|u_o\|$.

Now, we can apply Theorem 5 since assumption 2°) of this theorem is satisfied with $v_x=u_o$, for every $x\in\mathbf{K}$ with $\|x\|\geq r$.

REMARK. Condition 2°) of Corollary 2 is satisfied if T is semicoercive with respect to S in the following sense:

$$(\exists u_o\in\mathbf{K})\left(\lim_{\|x\|\to\infty}\frac{< S(x)-u_o,T(x)>}{\|x\|}=+\infty\right)$$

If $S(x) = x$, for every $x \in \mathbf{K}$, this notion is similar to the semicoercivity used by Stampacchia and Lions [22], [23].

Obviously, condition 2°) is satisfied if there is a number $\alpha > 0$ such that $< S(x), T(x) > \ \geq \alpha \| x \|^2$, for every $x \in \mathbf{K}$.

Finally, we give an extension of Theorem 5 to Galerkin cone.

THEOREM 6. Let $(E, \| \ \|)$ be a reflexive Banach space and $\mathbf{K}(\mathbf{K}_n)_{n \in N}$ a Galerkin cone in E. Let $S: \mathbf{K} \to \mathbf{K}$ and $T: \mathbf{K} \to E^*$ be strongly continuous mappings.

If the following assumptions are satisfied:

1°) S is subordinate to the approximation $(\mathbf{K})_{n \ n \in N}$,

2°) $< S(x), T(x) > \ \leq \ < x, T(x) >; \ \forall x \in \mathbf{K}$,

3°) there is a number $r > 0$ such that for every $n \geq n_o$ and every $x \in \mathbf{K}_n$ with $r \leq \| x \|$ there is an element $v_x \in \mathbf{K}_n$ such that $\| v_x \| < r$ and $< S(x) - v_x, T(x) > \ > 0$,

then the problem $ICP(T, S, \mathbf{K})$ has a solution x_* such that $\| x_* \| \leq r$.

PROOF. Since, for every $n \geq n_o$ we have $S(\mathbf{K}_n) \subseteq \mathbf{K}_n$ and the all assumptions of Theorem 5 are satisfied, we have that for every $n \geq n_o$ the problem $ICP(T, S, \mathbf{K}_n)$ has a solution x_n^*.

Because for every $n \geq n_o$ we have $\| x_n^* \| < r$ and E is reflexive the sequence $\{x_n^*\}_{n \in N}$ has a weakly convergent subsequence denoted again by $\{x_n^*\}_{n \in N}$. We put $x_* = (w) - \lim_{n \to \infty} x_n^*$.

Now, as in the proof of Theorem 4 we conclude that x_* is a solution of the problem $ICP(T, S, \mathbf{K})$. Obviously, $\| x_* \| \leq r$.

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