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AN INFINITE VERSION OF THE PÓLYA ENUMERATION THEOREM

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ABSTRACT. Using measure theory, the orbit counting form of Pólya's enumeration theorem is extended to countably infinite discrete groups.

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1. INTRODUCTION.

Let G be countable discrete group acting as permutations on a countable set D. Let S be a finite set with cardinality, $|S| = N$. Denote by S^D the set of functions from D to S. For $\gamma \in S^D$ define $g\gamma \in S^D$ by $g\gamma(d)=\gamma(g^{-1}d)$. For a subgroup K of G let Δ_K be a set of representatives for the orbits of K in S^D. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_\gamma\colon \gamma \in S^D\}$ and inner product $<\ ,\ >$. Define a unitary representation of G on $\mathcal{H}$ by $\pi(g)e_\gamma = e_{g\gamma}$.

The number of orbits of G in $S^D$is denoted by $|\Delta_G|$. For finite G and D this can be counted by the Pólya enumeration theorem. Specifically, for each $g \in G$, let $c_i(g)$ be the number of cycles of length i in the representation of g as a product of disjoint cycles in D and let $M(g)=y_1^{c_1(g)}\cdots y_n^{c_n(g)}$, where $n=|D|$. The cycle index of G on D is the polynomial $P_G=\frac{1}{|G|}\sum_{g\in G} M(g)$. Denote by σP_G the value P_G at $y_i=N$, $i=1$ to n. Pólya's enumeration theorem, see Pólya [1], says that $|\Delta_G| = \sigma P_G$.

Define the operator T_G on $\mathcal{H}$ by $T_G=\frac{1}{|G|}\sum_{g\in G}\pi(g)$. Then it can also be shown, see Williamson [2], that $|\Delta_G| = trace(T_G \text{ on } \mathcal{H})$. It is these two ways of measuring a set of representatives for orbits that we extend to infinite G and D.

2. THE MAIN RESULTS.

If we view S as a finite group with the discrete topology, then S^D is a compact group in the product topology. Let μ be normalized Haar measure on S^D.

For $g \in G$ and $\gamma \in S^D$ define $f(\gamma)=<\pi(g)e_\gamma, e_\gamma>$. Then $f(\gamma)=\begin{cases}1 & \text{if } g\gamma=\gamma\\ 0 & \text{otherwise.}\end{cases}$

LEMMA 1. f is measurable.

PROOF. Let $f_i(\gamma(d_i))=\begin{cases}1 & \text{if } \gamma(g^{-1}d_i)=\gamma(d_i)\\ 0 & \text{otherwise}\end{cases}$ and $h_n(\gamma)=\prod_{i=1}^{n} f_i(\gamma(d_i))$.

Then h_n is measurable for all n. Now $g\gamma=\gamma$ if and only if γ is constant on the orbits of g. But this happens if and only if $\gamma(g^{-1}d)=\gamma(d)$ for all $d \in D$. Therefore $f(\gamma)=1$ if and only if $f_i(\gamma(d_i))=1$

for all i. This shows that $f(\gamma)=\lim_{n\to\infty}h_n(\gamma)$ and therefore measurable by Hewett and Stromberg [3, 22.24b]. □

We write $D=\{d_1,d_2,d_3,\ldots\}$ and let $D_n=\{d_1,\ldots,d_n\}$. Let $<g>$ be the subgroup generated by g and $<g>d$ the orbit of d under $<g>$. For each n and each $k\leq n$ let $c_k^n(g)$ be the number of distinct cycles of g such that $|<g>d\cap D_n| = k$. Form the monomial $M^n(g) = \frac{1}{N^n}y_1^{c_1^n(g)}y_2^{c_2^n(g)}\ldots y_n^{c_n^n(g)}$.

LEMMA 2. $\int_{S^D} <\pi(g)e_\gamma,e_\gamma> d\mu(\gamma) = \lim_{n\to\infty}\sigma M^n(g)$.

PROOF. From the proof of Lemma 1 we saw that $<\pi(g)e_\gamma,\ e_\gamma> = \lim_{n\to\infty}h_n(\gamma)$. So by the dominated convergence theorem, $\int_{S^D} <\pi(g)e_\gamma,e_\gamma> d\mu(\gamma) = \lim_{n\to\infty}\int_{S^D} h_n(\gamma)d\mu(\gamma)$. But now $h_n(\gamma) = 1$ if and only if γ is constant on the intersection of the orbits of g with D_n otherwise $h_n(\gamma)=0$. Let $B_n=\{\gamma\colon \gamma \textit{ is constant on the intersection of the orbits of } g \textit{ with } D_n\}$. Then $\int_{S^D} h_n(\gamma)d\mu(\gamma) = \mu(B_n)$. Since there are N choices for the value of γ on each orbit meeting D_n and no restrictions on γ outside D_n, we get $\mu(B_n) = \frac{1}{N^n}N^{c_1^n(g)}\ldots N^{c_n^n(g)} = \sigma M^n(g)$. □

Let G_o be the subgroup of G consisting of all those $g\in G$ having only a finite number of cycles in D of length greater than 1.

LEMMA 3. $\int_{S^D} <\pi(g)e_\gamma,e_\gamma> d\mu(\gamma) = 0$ *for all* $g\notin G_o$.

PROOF. Suppose $g\notin G_o$. Then there either exists k_o such that $c_{k_o}^n(g)\to\infty$ as $n\to\infty$ or there exists an increasing sequence $\{k_n\}$ such that $c_{k_n}^n(g)\geq 1$. In the first case, for $n\geq k_o$, $n-\sum_{i=1}^{n}c_i^n(g) = \sum_{i=1}^{n}(i-1)c_i^n(g) \leq c_{k_o}^n(g)$. So with B_n as in the proof of Lemma 2, we get $0 \leq \int_{S^D} <\pi(g)e_\gamma,e_\gamma> d\mu(\gamma) = \lim_{n\to\infty}\mu(B_n) \leq \lim_{n\to\infty}N^{-c_{k_o}^n(g)} = 0$. In the second case we get $n-\sum_{i=1}^{n}c_i^n(g) \leq k_n-1$ and so $0 \leq \int_{S^D} <\pi(g)e_\gamma,e_\gamma> d\mu(\gamma) = \lim_{n\to\infty}\mu(B_n) \leq \lim_{n\to\infty}N^{-(k_n-1)} = 0$.

□

For each k let $F_k=\{g\in G\colon gd_i=d_i \textit{ for all } i>k\}$. Then $\{F_k\}$ is a nondecreasing sequence of subgroups with $\bigcup_{k=1}^{\infty}F_k=G_o$. Suppose $G = \{g_1,\ g_2,\ \ldots\}$ and let $G_m=\{g_1,\ \ldots,\ g_m\}$. Assume G is ordered in such a way that there exists a subsequence $\{m_k\}$ with $G_o\cap G_{m_k}=F_k$.

Let F be a finite subset of G. Define the n^{th}cycle index of F to be the polynomial $P_F^n = \frac{1}{|F|}\sum_{g\in F}M^n(g)$. Define the operator T_F on $\mathcal{H}$ by $T_F = \frac{1}{|F|}\sum_{g\in F}\pi(g)$. Write P_m^n for $P_{G_m}^n$ and T_m for T_{G_m}.

THEOREM 4. Δ_{G_o} *is closed and*

$$\mu(\Delta_{G_o}) = \lim_{k\to\infty}\left\{\frac{m_k}{|G_{m_k}\cap G_o|}\lim_{n\to\infty}\sigma P_{m_k}{}^n\right\} = \lim_{k\to\infty}\frac{m_k}{|G_{m_k}\cap G_o|}\int_{S^D} < T_{m_k}e_\gamma, e_\gamma > d\mu(\gamma) .$$

PROOF. Fix k and let $D_{k'}=\{d_{k+1}, d_{k+2},\dots\}$. If $\alpha_1,\dots,\alpha_s$ are representatives for the orbits of F_k in S^{D_k}, then $\Delta_{F_k} = \{\alpha_1,\dots,\alpha_s\}\times S^{D_{k'}}$. Therefore Δ_{F_k} is closed and $\mu(\Delta_{F_k}) = \frac{s}{N^k}$. Let $\mathcal{H}_k$ be a Hilbert space with orthonormal basis $\{e_\alpha: \alpha\in S^{D_k}\}$. By Williamson [2], $s = trace(T_{F_k} on\ \mathcal{H}_k) = \sigma P_{F_k}$, where P_{F_k} is the usual cycle index of F_k on D_k. Note that $\sigma P_{F_k} = N^k P_{F_k}{}^n$ for all $n\geq k$. By Lemma 3, $\frac{m_k}{|G_{m_k}\cap G_o|}\lim_{n\to\infty}\sigma P_{m_k}{}^n = \lim_{n\to\infty}\sigma P_{F_k}{}^n$. Therefore $\frac{m_k}{|G_{m_k}\cap G_o|}\lim_{n\to\infty}\sigma P_{m_k}{}^n = \frac{1}{N^k}\sigma P_{F_k}$. By Lemma 2, $\lim_{n\to\infty}\sigma P_{m_k}{}^n = \int_{S^D} < T_{m_k}e_\gamma, e_\gamma > d\mu(\gamma)$. So we get $\mu(\Delta_{F_k}) = \frac{m_k}{|G_{m_k}\cap G_o|}\lim_{n\to\infty}\sigma P_{m_k}{}^n = \frac{m_k}{|G_{m_k}\cap G_o|}\int_{S^D} < T_{m_k}e_\gamma, e_\gamma > d\mu(\gamma)$.

Since $F_k\subseteq G_o$ we can assume that $\Delta_{G_o}\subseteq\Delta_{F_k}$ for all k. Therefore $\Delta_{G_o}\subseteq\bigcap_{k=1}^{\infty}\Delta_{F_k}$. We claim that $\Delta_{G_o}=\bigcap_{k=1}^{\infty}\Delta_{F_k}$. To see this suppose that $\gamma\in\Delta_{F_k}$ for all k. Then there exists $\gamma'\in\Delta_{G_o}$ and $g\in G_o$ such that $\gamma=g\gamma'$. Since $G_o=\bigcup_{k=1}^{\infty}F_k$ there exists k_o such that $g\in F_{k_o}$. Therefore γ and γ' represent the same orbit of F_{k_o} in S^D. Since γ and $\gamma'\in\Delta_{F_{k_o}}$ we get $\gamma=\gamma'$. This proves the claim.

It follows that Δ_{G_o} is closed and hence measurable. Therefore $\mu(\Delta_{G_o}) = \lim_{k\to\infty}\mu(\Delta_{F_k})$. This completes the proof of the theorem. □

Suppose now that G is in no particular order. We show how to compute $\mu(\Delta_{G_o})$. Let $A_m = G_m\cap G_o$ and let $T_{A_m,n} = (T_{A_m})^n$.

THEOREM 5. $\mu(\Delta_{G_o}) = \lim_{m\to\infty}\lim_{n\to\infty}\int_{S^D} < T_{A_m,n}e_\gamma, e_\gamma > d\mu(\gamma)$.

PROOF. Exists m_o so that $1\in G_{m_o}$. Fix $m\geq m_o$ and let H_m be the subgroup of G_o generated by A_m. Define a probability measure ν on H_m by $\nu(g) = \frac{1}{|A_m|}$ if $g\in A_m$ and $\nu(g) = 0$ otherwise. Let ν^{*n} be the n-fold convolution of ν with itself and U the uniform probability measure on H_m. Then by Diaconis [4, pg23], $\|\nu^{*n}-U\|\to 0$ where $\|\cdot\|$ is the total variation norm. If we extend the representation π, in the usual way, to the set of measures on H_m we get $\pi(\nu^{*n}) = (T_{A_m})^n = T_{A_m,n}$ and $\pi(U) = T_{H_m}$. It follows, therefore, that $\lim_{n\to\infty} < T_{A_m,n}e_\gamma, e_\gamma > = < T_{H_m}e_\gamma, e_\gamma >$ for all $\gamma\in S^D$. By the dominated convergence theorem, $\lim_{n\to\infty}\int_{S^D} < T_{A_m,n}e_\gamma, e_\gamma > d\mu(\gamma) = \int_{S^D} < T_{H_m}e_\gamma, e_\gamma > d\mu(\gamma)$. Then as in the proof of Theorem 4, we get $\mu(\Delta_{H_m}) = \int_{S^D} < T_{H_m}e_\gamma, e_\gamma > d\mu(\gamma)$. The result follows since $G_o = \bigcup_{m=1}^{\infty}H_m$. □

3. EXAMPLE.

Suppose $D = \bigcup_{n=1}^{\infty} D_n$, where the D_n are disjoint and finite and that G sends D_n into itself. Then if G_n is G restricted to D_n, G is isomorphic to the product $\prod_{n=1}^{\infty} G_n$. In this case the product measure μ on S^D need no longer come from uniform measures on S.

Let $S = \{s_1,\cdots,s_k\}$ and let the measure ν on S be defined by $\nu(s_i) = a_i$. If $|D_n| = m_n$ define the measure μ_n on S^{D_n} by $\mu_n = \prod_{i=1}^{m_n} \nu$. Let Δ_n be representatives for the orbits of G_n in S^{D_n} and P_{G_n} the cycle index. Then using the pattern inventory from Pólya's enumeration theorem, see Pólya and Read [1], we get $\mu_n(\Delta_n) = P_{G_n}\left(\sum_{i=1}^{k} a_i, \sum_{i=1}^{k} a_i^2, \cdots, \sum_{i=1}^{k} a_i^n\right)$. Let $\mu = \prod_{n=1}^{\infty} \mu_n$ and let Δ be representatives for the orbits of G in R^D. Then, as in the proof of Theorem 4, we get that $\mu(\Delta) = \lim_{n\to\infty} \prod_{k=1}^{n} \mu_k(\Delta_k)$. Note that when $a_i = \frac{1}{k}$, $i = 1,\ldots,k$ and $|D_n| = n$ we get $\mu_n(\Delta_n) = \sigma P_{G_n}$, which is the situation in Theorem 4.

Now consider the plane tiled by one unit square tiles with sides parallel to the axis and center the coordinates (m,n), m and n integers. We color the tiles black or white and compute the measure the orbits of two groups of symmetries acting on the set of such tilings. For m a a positive integer let $D_m = \{\textit{tiles with centers } (\pm m,k) \textit{ or } (k,\pm m): k = -m,-m+1,\cdots,m-1,m\}$. Let $G_n = \prod_{k=1}^{2n^2+1} \mathbb{Z}_2$ act on D_{n^2} by interchanging tiles with central coordinates $(\pm n^2,k)$, $k=-n^2,\cdots,n^2$ and let $H_n = \prod_{k=1}^{2n+1} \mathbb{Z}_2$ act on D_{n^2} by interchanging tiles with central coordinates $(\pm n^2,k)$, $k=-n,\cdots,n$. Now let $G = \prod_{n=1}^{\infty} G_n$ and $H = \prod_{n=1}^{\infty} H_n$. With $S=\{\textit{black, white}\}$, we define probability measures μ_n on S^{D_n} by $\mu_n = \prod_{k=1}^{m_n} \gamma_n$, where $\nu_n(\textit{black}) = \sqrt{\exp\left\{-\frac{1}{n(2\sqrt{n}+1)}\right\} - \frac{3}{4}} + \frac{1}{2}$ and $\nu_n(\textit{white}) = 1-\nu_n(\textit{black})$. Let $\Delta(G_n)$ and $\Delta(H_n)$ be representatives for the orbits of G_n and H_n respectively on $S^{D_{n^2}}$ and let $\Delta(G)$ and $\Delta(H)$ be representatives for the orbits of G and H respectively on S^D. Then $\mu_n(\Delta(H_n)) = \exp(-1/n^2)$ and so $\mu(\Delta(H)) = \operatorname{limit}_{m\to\infty} \prod_{n=1}^{m} \mu_n(\Delta(H_n)) > 0$. But $\mu_n(\Delta(G_n)) = \exp\left\{-\frac{2n^2+1}{2n^3+n^2}\right\}$ and so $\mu(\Delta(G)) = \operatorname{limit}_{m\to\infty} \prod_{n=1}^{m} \mu_n(\Delta(G_n)) = 0$.

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