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SUBCLASSES OF UNIFORMLY STARLIKE FUNCTIONS

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ABSTRACT. We study subclasses of the class of uniformly starlike functions which were recently introduced by A.W. Goodman. One new subclass is defined and it is shown that it shares many properties of the parent class.

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1. INTRODUCTION.

Let ST be the class of analytic univalent functions f in the unit disk $D = \{\,|z| < 1\}$ that are normalized by $f(0) = 0, f'(0) = 1$, and that are starlike with respect to the origin. The subclass of uniformly starlike functions (UST) consists of normalized analytic function f in D such that for each ζ in D and any arc γ in D of a circle with center at ζ is mapped by f onto an arc $f(\gamma)$ that is starlike with respect to $f(\zeta)$. This class was recently introduced and studied by A.W. Goodman [1] who proved in particular the following analytic characterization.

THEOREM A. Let f be analytic in D, $f(0) = f'(0) - 1 = 0$. Then f is in UST if and only if

$$Q(z,\zeta) = \frac{f(z) - f'(\zeta)}{(z-\zeta)f'(z)} \quad (z \neq \zeta) \text{ and } Q(z,z) = 1 \tag{1.1}$$

has positive real part for all z and ζ in D.

Properties of the class UST are difficult to establish. One reason is that the usual transformations of univalent function theory generally do not preserve the UST class. The only known exceptions are rotations, $e^{-i\alpha}f(e^{i\alpha}z)$ for some real α, and the transformation $t^{-1}f(tz)$, $0 < t \leq 1$.

In order to obtain a coefficient bound $|a_n| \leq 2/n$ $(n = 2,3,...)$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in UST, Goodman proves that class UST is embedded in a larger subclass of starlike functions UST*. A function $f \in ST$ is in UST* if there is a real α such that $\text{Re}\{e^{i\alpha}f'(z)\} \geq 0$ for $z \in D$. (Goodman credits the result to Charles Horowitz.) This suggests that information about the UST class might be generated through the study of subclasses of UST as well. In this paper we study one such class that shares and extends some known properties of UST.

2. SOME SUBCLASSES OF UST.

If $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=0}^{\infty} b_j z^j$ are analytic in D, the Hadamard product $(f*g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$ is also analytic in D. In particular, if f is a normalized analytic function in D, then for all complex numbers $\alpha, \beta, 0 < |\alpha| \leq 1, \alpha \neq \beta$ we have

$$\frac{1}{\alpha}f(\alpha z) = f*\frac{z}{1-\alpha z}, \frac{f(\alpha z) - f(\beta z)}{\alpha - \beta} = f*\frac{z}{(1-\alpha z)(a - \beta z)}, zf'(\alpha z) = f*\frac{z}{(1-\alpha z)^2}. \tag{2.1}$$

These identities lead to an equivalent form for Theorem A.

THEOREM 1. Let f be a normalized analytic function in D. Then $f \in$ UST if and only if for all complex numbers α, β, $|\alpha| \leq 1$, $|\beta| \leq 1$, and for all $z \in D$ we have

$$\text{Re}\left\{\frac{f*\frac{z}{(1-\alpha z)(1-\beta z)}}{f*\frac{z}{(1-\alpha z)^2}}\right\} \geq 0. \tag{2.2}$$

In this form we can appeal to the extensive work on Hadamard products initiated by the proof of the Polya-Schoenberg conjecture by Ruscheweyh and Sheil-Small [4]. The fundamental result in this proof was the following theorem [4].

THEOREM B. If φ is a normalized convex univalent function in D and $g \in ST$, then for all $z \in D$

$$\text{Re}\left\{\frac{\varphi*(Fg)}{\varphi*g}\right\} \geq 0. \tag{2.3}$$

whenever F is an analytic function with positive real part in D.

These results, along with the following elementary observation about certain linear fractional transformations enable us to generate functions in UST from functions in the convex subclass K of ST.

LEMMA. Let $\rho > 0$ and let $\bar{D} = \{|z| \leq 1\}$. Then $\text{Re}\{(1-\alpha\rho z)/(1-\beta\rho z)\} > 0$ for $z \in D$ and for all α, β in $\bar{D}$ if any only if $\rho \leq 1/\sqrt{2}$.

The proof follows by considering the image of $|z| = 1$ under the transformation

$$T = \frac{(1-\alpha\rho z)}{(1-\beta\rho z)}.$$

THEOREM 2. If $f \in K$, then $\sqrt{2}f(z/\sqrt{2}) \in$ UST. The radius $\rho = 1/\sqrt{2}$ is best possible.

PROOF. Let $f \in K$. Since $z/(1-\alpha\rho z)^2$ is in ST when $|\alpha| \leq 1, \rho \in (0,1)$, we conclude from Theorem B and the Lemma that for $z \in D$

$$\text{Re}\left\{\frac{f*(\frac{1-\alpha\rho z}{1-\beta\rho z})\frac{z}{(1-\alpha\rho z)^2}}{f*\frac{z}{(1-\alpha\rho z)^2}}\right\} \geq 0$$

for all $\alpha, \beta \in \bar{D}, 0 < \rho \leq 1/\sqrt{2}$. But $f*g(\rho z)/\rho = g*f(\rho z)/\rho$. Hence, the expression above can be rewritten as

$$\operatorname{Re}\left\{\frac{\frac{1}{\rho}f(\rho z)*\frac{z}{(1-\beta z)(1-\alpha z)}}{\frac{1}{\rho}f(\rho z)*\frac{z}{(1-\alpha z)^2}}\right\}\geq 0 \quad z\in D,\ |\alpha|\leq 1,\ |\beta|\leq 1\ ,$$

where $0<\rho\leq 1/\sqrt{2}$. By Theorem 1 we have $\frac{1}{\rho}f(\rho z)\in \mathrm{UST}$ when $0<\rho\leq 1/\sqrt{2}$. For the convex function $f(z)=z/(1-z)$, Theorem 1 states $f(\rho z)/\rho = z/(1-\rho z)\in \mathrm{UST}$ if and only if Re $\{(1-\alpha\rho z)/(1-\beta\rho z)\}>0$ for $z\in D$, $|\alpha|\leq 1, |\beta|\leq 1, \rho>0$. This is the case by the Lemma only if $\rho\leq 1/\sqrt{2}$. The result is sharp. QED.

Goodman [1] proved $z/(1-Az)\in \mathrm{UST}$ whenever $|A|\leq 1/\sqrt{2}$ which in particular establishes the sharpness of Theorem 2. This function, however, is also in $\mathrm{UST}^*\supset \mathrm{UST}$.

The function $z/(1-\rho z)^2$ is starlike of order $\alpha=(1-\rho)/(1+\rho)$. Now a normalized analytic function f is said to be in the class R_α of prestarlike functions if $f*z/(1-z)^{2-2\alpha}$ is in the class ST_α of starlike functions of order α when $\alpha<1$ or $\operatorname{Re}\{f(z)/z\}>1/2$ when $\alpha=1$. Ruscheweyh [5, p. 54] proves a generalization of Theorem B to the case where $\varphi\in R_\alpha$ and $g\in ST_\alpha$. By an argument that is similar to the proof of Theorem 2, we obtain the following result.

THEOREM 3. If f is in the class R_α of prestarlike functions of order α, then $F(z)=f(\rho z)/\rho$ is in UST whenever $\rho=(1-\alpha)/(1+\alpha)$ and $(\sqrt{2}-1)/(\sqrt{2}+1)\leq\alpha<1$.

Except for the sharpness, Theorem 2 is a special case of Theorem 3 since $R_0=K\subset R_\alpha$ for $0<\alpha\leq 1$. The link between the convex case and the fundamental Theorem B is our justification of first proving the less general result.

It is interesting to note that for $\alpha>1/2$, the class R_α contains functions that are not univalent in D [6]. The function F of Theorem 3 is, of course, univalent and starlike in D.

To obtain another subset of UST, notice that

$$\frac{f(z)-f(\zeta)}{z-\zeta}\ \frac{1}{f'(z)}=\frac{1}{f'(z)}\int_0^1 f'(tz+(1-t)\zeta)\,dt \tag{2.4}$$

THEOREM 4. Let f be a normalized analytic function in D. Then $f\in \mathrm{UST}$ if for all w,z in D

$$\operatorname{Re}\left\{\frac{f'(w)}{f'(z)}\right\}\geq 0\ .$$

If $f\in \mathrm{UST}$, then for all w,z in D

$$\operatorname{Re}\left\{\frac{f'(w)}{f'(z)}\right\}^{1/2}\geq 0$$

and the 1/2 is best possible.

PROOF. The first part of the theorem follows from Theorem A and the real part of (2.4). For the second statement, we note that for $f\in \mathrm{UST}$

$$\left|\arg\left\{\frac{f(w)-f(z)}{w-z}\ \frac{1}{f'(z)}\right\}\right|\leq \pi/2\ .$$

Hence,

$$\left|\arg\left\{\frac{f'(w)}{f'(z)}\right\}^{1/2}\right|\leq\frac{1}{2}\left|\arg\frac{f(w)-f(z)}{w-z}\ \frac{1}{f'(z)}\right|+\frac{1}{2}\left|\arg\frac{w-z}{f(w)-f(z)}\ f'(w)\right|\leq\pi/2\ .$$

The function $f(z)=z/(1-\rho z)$ with $\rho=1/\sqrt{2}$ proves the exponent 1/2 is best possible.

The condition Re $\{f'(w)/f'(z)\} \geq 0$ for z, $w \in D$ implies $f'(D)$ lies in a quarter plane that contains 1. Indeed, $f'(0)=1$ and $f'(w) \neq itf'(z)$ for any real t. The condition Re $\{f'(w)/f'(z)\}^{1/2} \geq 0$ implies $f'(D)$ lies in a half plane containing 1. This is the result of Goodman and Horowitz that proves UST $\subset$ UST*.

COROLLARY 1. [1] If $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ as in UST, then there is an α, $-\pi/2 < \alpha < \pi/2$ such that $e^{i\alpha}f'(z)$ has positive real part in D. Furthermore, $|a_n| \leq 2/n$ $(n = 2,3,\ldots)$.

The sharp bounds for the coefficients of $f \in$ UST is an elusive open problem. Some information for this problem is contained in the next theorem.

THEOREM 5. For any integer $n \geq 2$, the function $f(z) = z + Az^n$ is in UST if $|A| \leq \sqrt{\frac{n+1}{2n^3}}$.

PROOF. Since $e^{-i\alpha}f(e^{i\alpha}z)$ is in UST whenever f is, we assume $A > 0$. For $w \neq \xi$ nonzero in D, we have

$$\frac{f(w)-f(\xi)}{w-\xi}\,\frac{1}{f'(w)} = \frac{1 + A(w^{n-1} + \xi w^{n-2} + \ldots + \xi^{n-1})}{1 + nAw^{n-1}}.$$

Replace w, and ξ respectively by $z^{1/(n-1)}$, $\alpha z^{1/(n-1)}$ where $|\alpha| \leq 1$, $\alpha \neq 1$. The above expression becomes

$$\frac{1 + A(1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1})z}{1 + nAz} = \frac{1 + A\zeta z + Az}{1 + nAz},$$

where $\zeta = \alpha + \alpha^2 + \ldots + \alpha^{n-1}$. The image of $|z| = 1$ by this linear fractional transformation is a circle with center c and radius R given by

$$c = \frac{1 - nA^2(\zeta+1)}{1 - n^2A^2}, R = \left|\frac{1 + A(\zeta+1)}{1+nA} - \frac{1 - nA^2(\zeta+1)}{1-n^2A^2}\right| = \frac{A}{|1-nA^2|}\,|(n-1) - \zeta|.$$

The image is in the right half plane if

$$\frac{|(n-1)-\zeta|}{|1-nA^2|}A \leq \left|\frac{1 - nA^2(\mathrm{Re}\,\zeta + 1)}{1 - n^2A^2}\right|.$$

Since $|\zeta| = |\alpha + \alpha^2 + \ldots + \alpha^{n-1}| \leq n-1$, the inequality holds when

$$A^2[2(n-1)^2 - 2(n-1)x] \leq (1-nA^2)^2 - 2nA^2(1-nA^2)x + n^2A^2x^2$$

where $x = \mathrm{Re}\,\zeta$, that is,

$$n^2A^4x^2 + 2(n^2A^2 - 1)A^2x + 1 - 2(n^2 - n + 1)A^2 + n^2A^4 \geq 0.$$

The minimum of the function of x on the left of this inequality occurs when

$$x = -(n^2A^2 - 1)/n^2A^2.$$

If we substitute this for x and simplify we obtain

$$(n-1)(n+1-2n^3A^2) \geq 0 \text{ or } \frac{n+1}{2n^3} \geq A^2. \qquad \text{QED.}$$

This improves the bound $|A| \leq 1/(\sqrt{2}n)$ of Goodman [1]. It does not appear to be the best possible except when $n = 2$. Godman [1] states that $z + Az^2 \in$ UST if and only if $|A| \leq \sqrt{3}/4$. We prove this result for a subclass of UST in the next section.

3. THE SUBCLASS UST*.

A rather natural way to construct a subclass of UST is to replace the derivative in (1.1) by a difference quotient. This generates a family of functions that shares most of the known sharp properties of the class UST. To be precise, we define UST* as the class of normalized analytic functions f in D such that

$$\operatorname{Re}\left\{\frac{f*z/(1-\alpha z)(1-\beta z)}{f*z/(1-\alpha z)(1-\gamma z)}\right\} \geq 0\,, \tag{3.1}$$

for all $z \in D$, and α, β, γ in $\bar{D}$. Since $\gamma = \alpha$ reduces (3.1) to (2.2), we have $\mathrm{UST}_* \subset \mathrm{UST}$.

The function $z/(1-\alpha\rho z)(1-\gamma\rho z)$ for $z \in D$, $\rho > 0$ and $\alpha, \gamma \in \bar{D}$ is starlike of order $\alpha = (1-\rho)/(1+\rho)$. We conclude from the generalized version of Theorem B that the functions F of Theorem 3 are in fact in the class UST_*.

THEOREM 6. If f is in the class R_α, then $F(z) = f(\rho z)/\rho$ is in UST_* whenever

$$\rho = (1-\alpha)/(1+\alpha) \text{ and } (\sqrt{2}-1)/(\sqrt{2}+1) \leq \alpha < 1\,.$$

COROLLARY 1. If $f \in K$, then $\sqrt{2}f(z/\sqrt{2}) \in \mathrm{UST}_*$. This result is sharp.

To prove the sharpness in the Corollary, we observe from (3.1) that $z/(1-\rho z) \in \mathrm{UST}_*$ if and only if $(1-\gamma\rho z)/(1-\beta\rho z)$ has positive real part for $z \in D$ and some $\rho > 0$ whenever $\gamma, \beta \in \bar{D}$. By the Lemma this requires $0 < \rho \leq 1/\sqrt{2}$.

There is another way to characterize UST_* (and UST) that can be useful.

THEOREM 7. Let f be a normalized analytic function in D. Then $f \in \mathrm{UST}_*$ if and only if for $z \in D$, $z \neq 0$,

$$f*\frac{z}{(1-\alpha z)}\;\frac{1-\frac{1}{2}[(1+x)\gamma+(1-x)\beta]z}{(1-\beta z)(1-\gamma z)} \neq 0 \tag{3.2}$$

for all $|x| = 1$ and γ, β, α in $\bar{D}$.

This result follows directly from (3.1). The expression in braces in (3.1) cannot equal $(x+1)/(x-1)$, $|x| = 1$, when its real part is positive. This yields (3.2) upon algebraic simplification.

COROLLARY 2. $z + Az^2 \in \mathrm{UST}_*$ if and only if $|A| \leq \sqrt{3}/4$.

PROOF. If a_2 is the coefficient of z^2 in the power series expansion of the second function in the Hadamard product (3.2), then with $f(z) = z + Az^2$ this product is not zero in D if $|Aa_2| \leq 1$. Now, with $\alpha = 1$, the second coefficient is

$$a_2 = 1 + \tfrac{1}{2}[(1-x)\gamma + (1+x)\beta]\,.$$

Now

$$|a_2| \leq |1 + \tfrac{1}{2}(\gamma+\beta)| + \tfrac{1}{2}|\gamma-\beta|$$

and we seek a maximum of the right hand side when $|\gamma| \leq 1$, $|\beta| \leq 1$. A computation shows a maximum occurs when $\gamma = 1, \beta = e^{it}, t = 2 \arcsin 1/\sqrt{3}$. This gives us $|a_2| \leq 4/\sqrt{3}$ and proves that $|A| \leq \sqrt{3}/4$. Since there is a choice of x, $|x| = 1$, such that $|a_2| = |1 + \frac{1}{2}(\gamma+\beta)| + \frac{1}{2}(\gamma - \beta|$, we conclude that the result is sharp.

4. ARC LENGTH FOR THE CLASS UST.

In the final section we state a result on the length of images of circles under UST mappings which extends a well-known result of Keogh for bounded starlike functions [3]. Let $C(\zeta, r)$ be a circle centered at ζ and radius r which is strictly inside the unit disk and let Γ be the image of this circle under the function f which is in the class UST. Furthermore, let

$$M_r = \operatorname*{Max}_{|z-\zeta|=r} |f(z)-f(\zeta)|,$$

and let L_r be the arc length of Γ.

THEOREM 8. Given the foregoing definitions we have

$$L_r \le \frac{M_r}{r}\frac{|\zeta|}{|f(\zeta)|}(2|\zeta|+r)\left\{2\pi + 4\log\frac{1+|\zeta|+r}{1-|\zeta|-r}\right\} \tag{3.3}$$

where $\zeta/f(\zeta) = 1$ if $\zeta = 0$. In particular, $L_r = O\left(\log\frac{1}{1-|\zeta|-r}\right)$. This result reduces to Keogh's when $\zeta = 0$.

The technique of the proof of (3.3) is similar to Keogh's except that we use the following result of R.M. Gabriel [2].

If $u(z)$ is subharmonic, positive, and continuous inside and on a circle Γ, and C is a circle inside Γ, then

$$\int_C u(z)\,|dz| < (1+\rho/R)\int_\Gamma u(z)\,|dz| ,$$

where ρ is the distance between centers of C and Γ and R is the radius of Γ.

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