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A PARABOLIC DIFFERENTIAL EQUATION WITH UNBOUNDED PIECEWISE CONSTANT DELAY

JOSEPH WIENER

Department of Mathematics
The University of Texas-Pan American
Edinburg, Texas 78539

LOKENATH DEBNATH

Department of Mathematics
University of Central Florida
Orlando, Florida 32816

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ABSTRACT. A partial differential equation with the argument $[\lambda t]$ is studied, where $[\cdot]$ denotes the greatest integer function. The infinite delay $t-[\lambda t]$ leads to difference equations of unbounded order.

KEY WORDS AND PHRASES. Partial differential equation, piecewise constant delay, boundary value problem, initial value problem.
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1. INTRODUCTION.

Functional differential equations (FDE) with delay provide a mathematical model for a physical or biological system in which the rate of change of the system depends upon its past history. The theory of FDE with continuous argument is well developed, and has numerous applications in natural and engineering sciences. This paper continues our earlier work [1-5] in an attempt to extend this theory to differential equations with discontinuous argument deviations. In these papers, ordinary differential equations having intervals of constancy have been studied. Such equations represent a hybrid of continuous and discrete dynamical systems and combine properties of both differential and difference equations. They include as particular cases loaded and impulse equations, hence their importance in control theory and in certain biomedical problems. Indeed, we consider the equation

$$x'(t)=ax(t)+bx([t]), \tag{1.1}$$

where $[t]$ denotes the greatest integer function, and write it as

$$x'(t)=ax(t)+\sum_{i=-\infty}^{\infty} bx(i)(H(t-i)-H(t-i-1)), \tag{1.2}$$

where $H(t)=1$ for $t>0$ and $H(t)=0$ for $t<0$. If we admit distributional derivatives, then differentiating the latter relation gives

$$x''(t)=ax'(t)+\sum_{i=-\infty}^{\infty} bx(i)(\delta(t-i)-\delta(t-i-1)), \tag{1.3}$$

where δ is the delta functional. This impulse equation contains the values of the unknown solution for the integral values of t. Within intervals of certain lengths, differential equations with piecewise constant argument (EPCA) describe continuous dynamical systems. Continuity of a solution at a point joining any two consecutive intervals implies recursion relations for the values of the solution at such points. Therefore, EPCA are intrinsically closer to difference equations rather than differential equations. The main feature of equations with piecewise constant argument is that it is natural to formulate initial and boundary value problems for them not on intervals but at a number of individual points.

In [6] boundary value problems for linear EPCA in partial derivatives were considered and the behavior of their solutions studied. The results were also extended to equations with positive definite operators in Hilbert spaces. In [7] initial value problems were studied for EPCA in partial derivatives. A class of loaded equations that arise in solving certain inverse problems was explored within the general framework of differential equations with piecewise constant delay. Integral transforms were successfully used to find the solutions of initial value problems for linear partial differential equations with piecewise constant delay. It has been shown in [6] and [7] that partial differential equations (PDE) with piecewise constant time naturally arise in the process of approximating PDE by simpler EPCA. Thus, if in the equation

$$u_t = a^2 u_{xx} - bu \, , \tag{1.4}$$

which describes heat flow in a rod with both diffusion $a^2 u_{xx}$ along the rod and heat loss (or gain) across the lateral sides of the rod, the lateral heat change is measured at discrete moments of time, then we get an equation with piecewise constant argument

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,nh) \, , \tag{1.5}$$

$$t \in [nh,(n+1)h] \, , \quad n = 0,1,\ldots$$

where $h > 0$ is some constant. This equation can be written in the form

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t/h]h) \, , \tag{1.6}$$

The purpose of the present paper is to investigate boundary value problems and initial value problems for linear PDE with the piecewise constant argument $[\lambda t/h]h$, where λ and $h > 0$ are constants and $0 < \lambda < 1$. Such equations are of both theoretical and applied interest. For instance, the equation

$$y'(t) = ay(t) + by(\lambda t) \, , \tag{1.7}$$

arises as a mathematical idealization of an industrial problem involving wave motion in the overhead supply line to an electrified railway system. The profound study [8] of Eq. (1.7) has led to numerous works in this direction, some of which were reviewed in [9]. In particular, in [10] and [11] distributional and entire solutions were explored for general classes of equations of type (1.7) with polynomial coefficients. While of considerable importance in their own right, solutions of EPCA with the argument $[\lambda t/h]h$ can be used to approximate solutions of equations of form (1.7) as $h \to 0$. Obviously, the lags $t - \lambda t$ and $t - [\lambda t/h]h$ become infinite as $t \to +\infty$.

2. MAIN RESULTS.

We consider the boundary value problem (BVP) consisting of the equation

$$\frac{\partial u(x,t)}{\partial t} + P\left(\frac{\partial}{\partial x}\right)u(x,t) = Q\left(\frac{\partial}{\partial x}\right)u(x,[\lambda t/h]h) \, , \tag{2.1}$$

where P and Q are polynomials of the highest degree m with coefficients that may depend only on x, the boundary conditions

$$L_j u = \sum_{k=1}^{m} (M_{jk} u^{(k-1)}(0) + N_{jk} u^{(k-1)}(1)) = 0 \, , \tag{2.2}$$

where M_{jk} and N_{jk} are constants, $j = 1, \dots, m$; and the initial condition

$$u(x,0) = u_0(x). \tag{2.3}$$

where $(x,t) \in [0,1] \times [0,\infty)$, and $h > 0$, $0 < \lambda < 1$ are constants. Equations (2.2) can be written as

$$Lu = 0. \tag{2.2'}$$

Following [6], we introduce the following definition.

DEFINITION 2.1. A function $u(x,t)$ is called a solution of the above BVP if it satisfies the conditions: (i) $u(x,t)$ is continuous in $G = [0,1] \times [0,\infty)$; (ii) $\partial u/\partial t$ and $\partial^k u/\partial x^k (k = 0,1,\dots,m)$ exist and are continuous in G, with the possible exception of the points $(x, nh/\lambda)$, where one-sided derivatives exist ($n = 0,1,2,\dots$); (iii) $u(x,t)$ satisfies equation (2.1) in G, with the possible exception of the points $(x, nh/\lambda)$, and conditions (2.2)-(2.3).

Let $u_n(x,t)$ be the solution of the given problem on the interval $nh/\lambda \le t < (n+1)h/\lambda$, then

$$\partial u_n(x,t)/\partial t + Pu_n(x,t) = Qc_n(x), \tag{2.4}$$

where

$$c_n(x) = u(x, nh).$$

We next write

$$u_n(x,t) = w_n(x,t) + v_n(x),$$

which gives the equation

$$\partial w_n/\partial t + Pw_n + Pv_n(x) = Qc_n(x),$$

and require that

$$\partial w_n/\partial t + Pw_n = 0, \tag{2.5}$$

$$Pv_n(x) = Qc_n(x). \tag{2.6}$$

Assuming both w_n and v_n satisfy (2.2′) leads to an ordinary BVP (2.6)-(2.2′), whose solution is denoted by

$$v_n(x) = P^{-1}Qc_n(x),$$

and to BVP (2.5)-(2.2′), whose solution is sought in the form

$$w_n(x,t) = X(x)T_n(t). \tag{2.7}$$

Separation of variables produces the ODE

$$T_n' + \mu T_n = 0$$

with a solution

$$T_n(t) = e^{-\mu(t - nh/\lambda)},$$

and the BVP

$$P(d/dx)X - \mu X = 0, \qquad LX = 0 \tag{2.8}$$

where L is defined in (2.2) and (2.2′). If BVP (2.8) has an infinite countable set of eigenvalues μ_j and corresponding eigenfunctions $X_j(x) \in C^m[0,1]$, then the series

$$w_n(x,t) = \sum_{j=1}^{\infty} C_{nj} e^{-\mu_j(t - nh/\lambda)} X_j(x), \qquad C_{nj} = const \tag{2.9}$$

represents a formal solution of problem (2.5)-(2.2′) and

$$u_n(x,t) = \sum_{j=1}^{\infty} C_{nj} e^{-\mu_j(t - nh/\lambda)} X_j(x) + P^{-1}Qc_n(x) \tag{2.10}$$

is a formal solution of (2.1)-(2.2). At $t = nh/\lambda$ we have

$$s_n(x) = \sum_{j=1}^{\infty} C_{nj} X_j(x) + P^{-1} Q c_n(x), \tag{2.11}$$

where

$$s_n(x) = u_n(x, nh/\lambda).$$

Therefore, assuming the sequence $\{X_j(x)\}$ is complete and orthonormal in $C^m[0,1]$ yields for the coefficients C_{nj} the formula

$$C_{nj} = \int_0^1 (s_n(x) - P^{-1} Q c_n(x)) X_j(x) dx, \qquad (n = 0, 1, 2, \ldots). \tag{2.12}$$

Since

$$s_0(x) = c_0(x) = u_0(x),$$

substituting the initial function $u_0(x) \in C^m[0,1]$ in (2.12) as $n = 0$ produces the coefficients C_{0j}, and putting them together with $u_0(x)$ in (2.10) as $n = 0$ gives the solution $u_0(x,t)$ of BVP (2.1)-(2.3) on the interval $0 \le t < h/\lambda$. Since $u_0(x,h) = c_1(x)$ and $u_0(x, h/\lambda) = s_1(x)$, we can find from (2.12) the numbers C_{1j} and then substitute them along with $c_1(x)$ in (2.10) as $n = 1$, to obtain the solution $u_1(x,t)$ on $h/\lambda \le t \le 2h/\lambda$. This method of steps allows to extend the solution to any interval $nh/\lambda \le t \le (n+1)h/\lambda$. Furthermore, continuity of the solution $u(x,t)$ implies

$$u_n(x, (n+1)h/\lambda) = u_{n+1}(x, (n+1)h/\lambda) = s_{n+1}(x),$$

hence, at $t = (n+1)/h\,\lambda$ we get the recursion relations

$$s_{n+1}(x) = \sum_{j=1}^{\infty} C_{nj} e^{-\mu_j h/\lambda} X_j(x) + P^{-1} Q c_n(x). \tag{2.13}$$

Finally, from (2.11) and (2.13) we obtain

$$s_{n+1}(x) = s_n(x) - \sum_{j=1}^{\infty} C_{nj}(1 - e^{-\mu_j h/\lambda}(X_j(x).$$

This concludes the proof of the following theorem:

THEOREM 2.1. Formula (2.10), with coefficients C_{nj} defined by recursion relations (2.12), represents a formal solution of BVP (2.1)-(2.3) in $[0,1] \times [nh/\lambda, (n+1)h/\lambda]$, for $n = 0, 1, \ldots$, if BVP (2.8) has a countable number of eigenvalues μ_j and a complete orthonormal set of eigenfunctions $X_j(x) \in C^m[0,1]$ and the initial function $u_0(x) \in C^m[0,1]$ satisfies (2.2).

The solution of Eq. (2.1) on $nh/\lambda \le t < (n+1)h/\lambda$ can be also sought in the form

$$u_n(x,t) = \sum_{j=1}^{\infty} X_j(x) T_{nj}(t), \tag{2.14}$$

where $X_j(x)$ are the eigenfunctions of the operator P. Upon multiplying (2.14) by $X_k(x)$, then integrating between 0 and 1 and changing k to j, we obtain

$$T_{nj}'(t) + \mu_j T_{nj}(t) = q_{nj},$$

$$q_{nj} = \int_0^1 X_j(x) Q(d/dx) c_n(x) dx,$$

$$c_n(x) = u(x, nh),$$

whence

$$T_{nj}(t) = T_{nj}(nh/\lambda)e^{-\mu_j(t-nh/\lambda)} + \mu_j^{-1}q_{nj}\left(1 - e^{-\mu_j(t-nh/\lambda)}\right),$$

$$T_{nj}(nh/\lambda) = \int_0^1 s_n(x)X_j(x)dx,$$

$$s_n(x) = u(x, nh/\lambda).$$

The principal role of the operator P emerges from these methods of constructing the solution. Let

$$Py = \sum_{j=0}^{m} p_j y^{(m-j)},$$

where p_j are real-valued functions of classes C^{m-j} on $0 \le x \le 1$ and $p_0(x) \ne 0$ on $[0,1]$. Assuming $C^m[0,1]$ is embedded in $L^2[0,1]$ with the inner product

$$(y,z) = \int_0^1 y(x)z(x)dx,$$

BVP (2.8) is called self-adjoint if

$$(Py,z) = (y,Pz),$$

for all $y, z \in C^m[0,1]$ that satisfy the boundary conditions

$$Ly = Lz = 0.$$

If BVP (2.8) is self-adjoint, then all its eigenvalues are real and form at most a countable set without finite limit points. The eigenfunctions corresponding to different eigenvalues are orthogonal. The proof of the following theorem is omitted since it parallels the proof of Theorem 2.3 in [6].

THEOREM 2.2. BVP (2.1)-(2.3) has a solution in $[0,1] \times [nh/\lambda, (n+1)h/\lambda]$, for each $n = 0, 1, \ldots,$ given by formula (2.10) if the following hypotheses hold true.

(i) BVP (2.8) is self-adjoint, all its eigenvalues μ_j are positive.

(ii) For each μ_j, the roots of the equation $P(s) - \mu_j = 0$ have non-positive real parts.

(iii) The initial function $u_0(x) \in C^m[0,1]$ satisfies (2.2).

EXAMPLE 2.1. The solution $u_n(x,t)$ of the equation

$$u_t(x,t) = a^2u_{xx}(x,t) + bu(x,[\lambda t/h]h) \tag{2.15}$$

in $[0,1] \times [nh/\lambda, (n+1)h/\lambda]$, with the boundary conditions $u_n(0,t) = u_n(1,t) = 0$ and initial condition $u_n(x, nh/\lambda) = s_n(x)$, is sought in form (2.14). Separation of variables produces $X_j(x) = \sqrt{2}\sin(\pi jx)$ and

$$T_{nj}'(t) = -a^2\pi^2j^2T_{nj}(t) + bT_j(nh), \quad (nh/\lambda \le t < (n+1)h/\lambda) \tag{2.16}$$

whence

$$T_{nj}(t) = C_{nj}e^{-a^2\pi^2j^2(t-nh/\lambda)} + \frac{b}{a^2\pi^2j^2}T_j(nh). \tag{2.17}$$

The following remark is in order. The subindex n is omitted from the term $T_j(nh)$ in (2.16) and (2.17) because the point $t = nh$ does not belong to the interval $[nh/\lambda, (n+1)h/\lambda]$. Since $0 < \lambda < 1$, the delay $t - nh$ in Eq. (2.16) becomes infinite as $t \to +\infty$. As mentioned above, $u_n(x,t)$ is the restriction of the solution $u(x,t)$ of problem (2.1)-(2.3) to the interval $[nh/\lambda, (n+1)h/\lambda]$. Therefore, if $u(x,t)$ is sought in form (2.14), $T_{nj}(t)$ is the restriction of $T_j(t)$ to the indicated interval. Furthermore, putting $t = nh/\lambda$ in (2.17) gives

$$T_{nj}(nh/\lambda) = C_{nj} + \frac{b}{a^2\pi^2 j^2} T_j(nh),$$

whence

$$C_{nj} = T_{nj}(nh/\lambda) - \frac{b}{a^2\pi^2 j^2} T_j(nh)$$

and

$$T_{nj}(t) = T_{nj}(nh/\lambda) e^{-a^2\pi^2 j^2 (t - nh/\lambda)} + \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2 (t - nh/\lambda)}\right) T_j(nh). \tag{2.18}$$

At $t = (n+1)h/\lambda$ we get from (2.18)

$$T_{nj}(h(n+1)/\lambda) = e^{-a^2\pi^2 hj^2/\lambda} T_{nj}(nh/\lambda) + \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 hj^2/\lambda}\right) T_j(nh). \tag{2.19}$$

We denote

$$A_j = e^{-a^2\pi^2 hj^2/\lambda}, \qquad B_j = \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 hj^2/\lambda}\right),$$

$$T_{nj}(nh/\lambda) = t_{nj}, \qquad T_{nj}(nh) = s_j .$$

Since continuity of the solution implies

$$T_{nj}(h(n+1)/\lambda) = T_{n+1,j}(h(n+1)/\lambda),$$

equation (2.19) becomes

$$t_{n+1,j} = A_j t_{nj} + B_j s_{nj} . \tag{2.20}$$

The difference equation (2.20) with respect to t_{nj} is of unbounded order because it contains $s_{nj} = T_j(nh)$, where $nh \in [Nh/\lambda, (N+1)h/\lambda)$, that is, N is the integral part of $n\lambda$.

For instance, if $\lambda = 1/2$, then $s_{nj} = T_j(nh)$, $t_{nj} = T_{nj}(2nh) = s_{2n,j}$, and in this case Eq. (2.20) becomes

$$s_{2n+2,j} = A_j s_{2n,j} + B_j s_{nj} .$$

Furthermore, at $t = (2n+1)h$, formula (2.18) gives

$$s_{2n+1,j} = e^{-a^2\pi^2 hj^2} s_{2n,j} + \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 hj^2}\right) s_{nj} .$$

THEOREM 2.3. If $|b| < \pi^2 a^2$, then all functions $T_j(t)$ in the expansion (2.14) for the solution of equation (2.15) with homogeneous boundary conditions exponentially tend to zero as $t \to +\infty$.

PROOF. Denote

$$M_n^{(j)} = \max |T_j(t)|, \qquad t \in [(n-1)h, \infty),$$

$$q = e^{-a^2\pi^2 h/\lambda} + \pi^{-2} a^{-2} |b| \left(1 - e^{-a^2\pi^2 h/\lambda}\right),$$

then

$$|s_{n-1,j}| \le M_n^{(j)}, \quad |t_{n-1,j}| \le M_n^{(j)}, \quad A_j + |B_j| \le q ,$$

and from (2.20) we get $|t_{nj}| \le qM_n^{(j)}$, while the condition $|b| < \pi^2 a^2$ implies $q < 1$. By induction, we conclude from (2.20) that $|t_{n+i,j}| \le qM_n^{(j)}$, $i \ge 1$. Furthermore, it follows from (2.16) that on every interval $[nh/\lambda, (n+1)/\lambda]$ the function $|T_{nj}(t)|$ attains its maximum at an endpoint of this interval. Hence, the inequality $|t_{[n/\lambda],j}| \le qM_n^{(j)}$ leads to $M_{[n/\lambda]}^{(j)} \le qM_n^{(j)}$. Therefore, $M_{[n/\lambda]}^{(j)} \le q^2 M_{[n\lambda]}^{(j)}$, and the proof is completed by lowering the subindex $[1/\lambda]$ times successively. We also note that the functions $T_j(t)$ decay slower for equation (2.15) than for the equation without delay

$$u_t(x,t) = a^2u_{xx}(x,t) + bu(x,t). \tag{2.21}$$

THEOREM 2.4. If $0 < b < \pi^2a^2$, then the functions $T_j(t)$ tend to zero monotonically as $t \to +\infty$, and none of them has a zero in $(0,\infty)$.

PROOF. Assuming, for instance, $T_{0j}(0) > 0$ we resort to equation (2.16) and the condition $b < \pi^2a^2$ to show that the function $T_{0j}(t)$ is monotonically decreasing on $[0,h/\lambda)$. Moreover, since $b > 0$ we conclude from (2.18) that $T_{0j}(t) > 0$ on $[0,h/\lambda]$. Hence, $t_{1j} > 0$, $s_{1j} > 0$, and from (2.20) we see that $t_{2j} > 0$. Therefore, $T_{1j}(t)$ is decreasing and positive on $[h/\lambda, 2h/\lambda]$, and it remains to use (2.20) successively to obtain the same result on each interval $[nh/\lambda, (n+1)h/\lambda]$.

THEOREM 2.5. For $b < 0$, each function $T_j(t)$ is oscillatory, that is, it has infinitely large zeros.

PROOF. Assume that a certain function $T_j(t)$ is nonoscillatory, say, positive for large t. Then t_{nj} and s_{nj} are positive for large n, and therefore it follows from (2.20) that $t_{n+1,j} < A_jt_{nj}$, with $0 < A_j < 1$. Hence, t_{nj} tends to zero faster than A_j^n as $n \to \infty$, whereas s_{nj} decays at a slower rate of $A_j^{n\lambda}$ as $n \to \infty$. This contradicts (2.20) and proves that $T_j(t)$ is oscillatory. This theorem reveals a striking difference between the behavior of the functions $T_j(t)$ for equations (2.15) and (2.21) when $b < 0$: for equation (2.21) without delay, the $T_j(t)$ are always nonoscillatory.

THEOREM 2.6. If $b > \pi^2a^2m^2$, then the functions $T_1(t), \dots, T_m(t)$ are unbounded.

EXAMPLE 2.2. For the equation

$$u_t(x,t) = a^2u_{xx}(x,t) + bu_{xx}(x,[\lambda t/h]h), \tag{2.22}$$

the functions $T_j(t)$ satisfy the relation

$$T_{nj}'(t) = -a^2\pi^2j^2T_{nj}(t) - b\pi^2j^2s_{nj},$$

from which the following conclusion can be derived.

THEOREM 2.7. If $|b| < a^2$, then all functions $T_j(t)$ for equation (2.22) exponentially tend to zero as $t \to +\infty$. If $-a^2 < b < 0$, then all $T_j(t)$ tend to zero monotonically as $t \to +\infty$, and none of them has a zero in $(0,\infty)$. For $b > 0$, each function $T_j(t)$ is bounded and oscillatory.

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