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TRANSIENT DEVELOPMENT OF GRAVITY WAVES FOR TWO LAYERED FLUIDS

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ABSTRACT. The transient gravity waves generated by a harmonically oscillating wave maker immersed in two incompressible fluids, the upper fluid having a free surface, is considered. The resulting linearized initial value problem is solved using the method of generalized functions, and asymptotic analysis for large time and distance are given for the elevation.

KEY WORDS AND PHRASES. Internal waves, transient waves.
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1. INTRODUCTION.

The two-dimensional problem of gravity waves generated by moving oscillating surface pressure distributions in a fluid which is unbounded in both horizontal directions has been studied by Kaplan [1] and Debnath and Rosenblat [2] in infinite depth and in finite depth respectively. Pramanik [3] considered the initial value problem of waves generated by a moving oscillating surface pressure against a vertical cliff and a uniform asymptotic analysis was given for the unsteady case. Debnath and Basu [4] treated the same problem taking into account the effect of surface tension. Faltas [5] investigated the initial value problem of surface waves generated by a harmonically oscillating vertical wave maker immersed in an infinite incompressible fluid of finite constant depth. It is the purpose of this paper to discuss the transient development of two-dimensional linearized waves at the free surface and at the interface between two fluids. The waves are produced by a harmonically oscillating wave maker immersed vertically in both fluids. The integral representations of free and interface elevations are obtained through an application of the Laplace and the generalized cosine Fourier transforms of the equations of motion. Then the application of the stationary phase method combined with the contour integration method leads to the asymptotic waves valid for large time and distance.

2. FORMULATION AND SOLUTION OF THE PROBLEMS.

We are concerned with the transient development of two dimensional infinitesimal wave motion of two superimposed immiscible non-viscous and incompressible fluids separated by a common interface, where the upper fluid has a free surface. The waves are generated by a harmonically oscillating wave maker immersed vertically in the two fluids.

Take the origin O at the mean level fo the interface and the axis Oy to be vertically downwards along the wave maker. The upper fluid is of finite constant height with mean level at

$y = -h$, while the lower fluid has infinite depth. If the motion is generated originally from rest by the oscillations of the wavemaker, it will be irrotational throughout all time and we may describe the motion by velocity $0 < y < \infty$, and $0 < x < \infty$, $-h < y < 0$ of the lower and upper fluids respectively. The unsteady motion is produced in the two fluids by the continuous oscillations of the wave maker. Let it oscillate horizontally with velocity $U(y,t)$ given by

$$U(y,t) = u(y)e^{i\omega t}H(t) \tag{2.1}$$

where $u(y)$ is an arbitrary function of y, ω is the frequency, and $H(t)$ is the unit step function. The functions ϕ_j satisfy an initial boundary value problem in which

$$\nabla^2\phi_j = 0 . \tag{2.2}$$

Neglecting surface tension, the linearized pressure and kinematical boundary conditions at the interface of the two fluids are respectively

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial t} - s\frac{\partial\phi_2}{\partial t} &= (1-s)g\eta_2 \\ \frac{\partial\eta_1}{\partial t} = \frac{\partial\phi_1}{\partial y} &= \frac{\partial\phi_2}{\partial t} \end{aligned}\right\} y = 0 \tag{2.3}$$

while the corresponding conditions at the free surface of the upper fluid are

$$\left.\begin{aligned} \frac{\partial\phi_2}{\partial t} &= g\eta_2 \\ \frac{\partial\eta_2}{\partial t} &= \frac{\partial\phi_2}{\partial y} \end{aligned}\right\} y = -h \tag{2.4}$$

where $s(0 < s < 1)$ is the ratio of the densities of the upper and lower fluids and $n_j = n_j(x,t)$ are the wave elevations associated with the lower and upper fluids. Also,

$$\frac{\partial\phi_1}{\partial y} \to 0 \qquad \text{as } y \to \infty . \tag{2.5}$$

At the wave maker

$$\frac{\partial\phi_j}{\partial y} = U(y,t) \qquad \text{on } x = 0 \tag{2.6}$$

and the initial conditions are

$$\phi_j = \eta_j = 0 \qquad \text{when } t = 0 . \tag{2.7}$$

We suppose also that ϕ_j, η_j are treated as the generalized function in the sense of Lighthill [6].

We introduce the Fourier cosine transform with respect to x and the Laplace transform with respect to t as

$$\overline{F}_c(k,y,r) = \sqrt{\tfrac{2}{\pi}}\int_0^\infty \cos kx\,dx \int_0^\infty e^{-rt}F(x,y,t)\,dt \tag{2.8}$$

where the suffix c and the bar in the transformed function refer to the cosine Fourier and Laplace transform respectively. Application of (2.8) to the system (2.2) - (2.7) gives

$$\frac{d^2}{dy^2}\overline{\phi}_{jc} - k^2\overline{\phi}_{jc} = \sqrt{\tfrac{2}{\pi}}\overline{U}(y,t), \qquad r > 0 \tag{2.9a,b}$$

$$\left.\begin{aligned} \overline{\phi}_{1c} - s\overline{\phi}_{2c} &= \frac{g}{r}(1-s)\overline{\eta}_{1c} \\ \frac{d}{dy}\overline{\phi}_{1c} = \frac{d}{y}\overline{\phi}_{2c} &= r\overline{\eta}_{1c} \end{aligned}\right\} \text{on } y = 0, r > 0 \tag{2.10}$$

$$\left.\begin{aligned}\bar{\phi}_{2c} &= \frac{g}{r}\bar{\eta}_{2c} \\ \frac{d}{dy}\bar{\phi}_{2c} &= r\bar{\eta}_{2c}\end{aligned}\right\} \text{on } y = -h, r > 0 \tag{2.11}$$

$$\frac{d}{dy}\phi_{1c} \to 0 \qquad \text{as } y \to \infty \tag{2.12}$$

The solutions of (2.9a) and (2.9b) satisfying condition (2.1) are respectively

$$\bar{\phi}_{1c} = A(k,r)e^{-ky} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty k^{-1}e^{k(y-z)}\bar{U}(z,r)\,dz + \frac{2}{\pi}\int_0^\infty k^{-1}\,\text{sinhk}\,(y-z)\bar{U}(z,r)\,dz\,, \tag{2.13}$$

$$\bar{\phi}_{2c} = B(k,r)e^{ky} + C(k,r)e^{-ky} + \sqrt{\frac{2}{\pi}}\int_0^y k^{-1}\,\text{sinhk}\,(y-z)\bar{U}(z,r)\,dz\,, \tag{2.14}$$

where $A(k,r)$ $B(k,r)$ and $C(k,r)$ are functions to be determined. The transformed boundary conditions (2.10) are satisfied if

$$\begin{aligned} A &= -\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-kz}}{k}\bar{U}\,dz - \frac{r}{k}\bar{\eta}_{1c}\,, \\ B &= -\frac{1}{2s}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-kz}}{k}\bar{U}\,dz - \frac{(1-s)(r^2+gk)}{2skr}\bar{\eta}_{1c}\,, \\ C &= -\frac{1}{2s}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-kz}}{k}\bar{U}\,dz - \frac{(1+s)r^2+gk(1-s)}{2skr}\bar{\eta}_{1c} \end{aligned} \tag{2.15}$$

We are interested in the waves after a large time and large distance. To investigate the principal feature of the wave motion it suffices to work only with the elevation η_j. From (2.11), (2.13) - (2.15) we get

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\,\bar{\eta}_{1c} = \frac{-rc}{(r^2+m_1^2)(r^2+m_2^2)}\Big\{&r^2[\text{coshkh}\int_0^\infty e^{-kz}\,\bar{U}\,dz + s\int_0^{-h}\text{sinh k}\,(h+z)\bar{U}\,dz] \\ &+ gk[\text{sinhk}\int_0^\infty e^{-kz}\bar{U}\,dz + s\int_0^{-h}\text{cosh kh}\,(h+z)\bar{U}\,dz]\Big\}\end{aligned} \tag{2.16}$$

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\,\bar{\eta}_{2c} = \frac{-rc}{(r^2+m_1^2)(r^2+m_2^2)}\Big\{&r^2[\int_0^\infty e^{-kz}\,\bar{U}\,dz - \int_0^{-h}\text{coshkz} + s\,\text{sinhkz}\,\bar{U}\,dz] \\ &- gk(1-s)\int_0^{-h}\text{coshkz}\,\bar{U}\,dz\Big\}\,,\end{aligned} \tag{2.17}$$

where $1/c = \text{coshkh} + s\,\text{sinkh}$ and $m_1^2(k) = gk$, $m_2^2(k) = \dfrac{(1-s)\,gk}{s+\text{cothkh}}$. Assuming the particular form of $U(y,t)$ as given by (2.1), the inverse of Laplace and cosine Fourier transforms with the convolution theorem for Laplace transform give

$$\eta_1(x,t) = -\frac{2}{\pi}\sum_{p=1}^{2}\int_0^\infty \beta_p(k)\,\text{coskxdk}\int_0^t e^{i\omega t}\cos \text{m}_p(t-\tau)\,d\tau\,, \tag{2.18}$$

$$\eta_2(x,t) = -\frac{2}{\pi}\sum_{p=1}^{2}\int_0^\infty \gamma_p(k)\,\text{coskxdk}\int_0^t e^{i\omega t}\cos \text{m}_p(t-\tau)\,d\tau\,, \tag{2.19}$$

where

$$\beta_1(k) = \frac{c\, m_1^2}{(m_2^2 - m_1^2)}\left[s\int_0^{-h} e^{-k(h+z)}u(z)dz - \int_0^{\infty} e^{-k(h+z)}u(z)dz \right],$$

$$\beta_2(k) = \frac{c}{(m_2^2 - m_1^2)}\ [s\int_0^{-h}(m_2^2 \text{ sinhk } (h+z) - m_1^2 \text{ coshk } (h+z))u(z)dz$$

$$+ (m_2^2 \text{ coshkh } - m_1^2 \text{ sinhkh })\int_0^{\infty} e^{-kz}u(z)dz],$$

$$\gamma_1(k) = \frac{c\, m_1^2}{(m_2^2 - m_1^2)}\ [s\int_0^{-h} e^{kz}u(z)dz - \int_0^{\infty} e^{-kz}u(z)dz,$$

$$\gamma_2(k) = \frac{-c}{(m_2^2 - m_1^2)}\ [s\int_0^{-h}(m_2^2 \text{ sinh kz} + m_1^2 \text{ coshkz} + (m_2^2 - m_1^2) \text{ coshkz } u(z)dz\ - m_2^2\int_0^{\infty} e^{-kz}u(z)dz].$$

Carrying out the integral in (2.18) or (2.19), we get,

$$\eta_1(x,t) = \frac{2}{\pi}\sum_{p=1}^{2}\int_0^{\infty}\frac{\beta_p(k)}{m_p^2 - \omega^2}\left[i\omega \text{ cosm}_p t - m \text{ sinm}_p t - i\omega\, e^{i\omega t}\right] \text{coskx dk}, \tag{2.20}$$

and a similar expression for $\eta_2(x,t)$.

3. ASYMPTOTIC ANALYSIS OF SOLUTION.

To evaluate the integral (2.20) or the corresponding one for η_2 for large values of x and t we shall use formulae developed by Lighthill [6] and Jones [7]. Write $\eta_1 = I + I' + J + J'$, where

$$I = \frac{-2}{\pi} i\omega\, e^{i\omega t}\int_0^{\infty}\frac{\beta_1(k)}{m_1^2 - \omega^2} \text{coskx dk}, I' = \frac{-2i\omega}{\pi} e^{i\omega t}\int_0^{\infty}\frac{\beta_2(k)}{m_2^2 - \omega^2} \text{coskx dk} \tag{3.1}$$

$$\left.\begin{aligned} J &= \frac{2}{\pi}\int_0^{\infty}\frac{\beta_1(k)}{m_1^2 - \omega^2}(i\omega \text{ cosm}_1 t - m_1 \text{ sinm}_1 t) \text{ coskx dk},\\ J' &= \frac{2}{\pi}\int_0^{\infty}\frac{\beta_1(k)}{m_2^2 - \omega^2}(i\omega \text{ cosm}_2 t - m_2 \text{ sinm}_2 t) \text{ coskx dk}\end{aligned}\right\} \tag{3.2}$$

The first two integrals (3.1), represent the steady state solution while the second two (3.2) represent the transient solution. It is convenient to rewrite (3.1), (3.2) as follows

$$I = \frac{i}{2\pi} e^{i\omega t}\sum_{n=1}^{2} I_n, \quad I' = \frac{i}{2\pi} e^{i\omega t}\sum_{n=1}^{2} I'_n, \quad J = \frac{i}{2\pi}\sum_{n=1}^{4} J_n, \quad J' = \frac{i}{2\pi}\sum_{n=1}^{4} J'_n$$

where

$$I_1, I_2 = \pm\int_0^{\infty}\frac{\beta_1(k)}{m_1 \mp \omega}(e^{ikx} + e^{-ikx})dx, \quad J_1, J_2 = \int_0^{\infty}\frac{\beta_1(k)}{m_1 - \omega} e^{i(\omega t \pm kx)}dk$$

$$J_3, J_4 = -\int_0^{\infty}\frac{\beta_1(k)}{m_1 + \omega} e^{i(\omega t \mp kx)}dk, \quad I_1, I'_2 = \mp\int_0^{\infty}\frac{\beta_2(k)}{m_2 \mp \omega}(e^{ikx} + e^{-ikx})dk,$$

$$J'_1, J'_2 = \int_0^{\infty}\frac{\beta_2(k)}{m_2 - \omega} e^{i(\omega t \pm kx)}dk, \quad J'_3, J'_4 = -\int_0^{\infty}\frac{\beta_2(k)}{m_2 + \omega} e^{i(\omega t \mp kx)}dk.$$

We follow the method of Debnath and Rosenblat [2] to evaluate these wave integrals. The main contribution to the asymptotic value of the above integrals comes from the poles and stationary points of the integrals. It is noted that each I_1, J_1, and J_2 contains one pole at $k = k_0$ where $k_0 = \omega^2/g$, and each of I'_1, J'_2, J'_2 contains one pole at $k = k'_0$, where k'_0 is the only real positive root of the equation

$$\sqrt{\frac{(1-s)gk}{s + \text{cothkh}}} = \omega \,.$$

In addition, the integrals J_2, J_3 contain one stationary point at $k = k_1$, which is the root of the equations

$$\frac{dm_1}{dk} = \frac{x}{t} \qquad \text{i.e., } k_1 = \frac{gt^2}{4x^2}\,,$$

also, the integrals J'_2, J'_1 contain one stationary point at $k = k'_1$ which is the root of the equation

$$\frac{dm_2}{dk} = \frac{x}{t}\,. \tag{3.3}$$

We note that

$$\frac{d^2m}{dk^2} = \frac{m(k)}{4k^2}(s\ \text{sinh2kh} + \text{cosh2kh} - 1)^{-1}[(4h^2k^2 - \sinh^2 2kh)$$
$$+ 4kh\,(\text{sinh2kh} - \text{2khcosh2kh} - s^2\,(\text{cosh2kh} - 1)^2$$
$$+ s(-8h^2k^2\,\text{sinh2kh} + 2(2kh - \sinh^2 2kh)\,(\text{cosh2kh} - 1))] < 0\,.$$

Therefore dm/dk decreases monotonically from $\sqrt{gh(1-s)}$ to 0 as k varies from 0 to ∞. Hence equation (3.3) has only one real root k'_1. On the other hand, the integrals I_2, J_4, I'_2 and I'_4 contains neither poles nor stationary points in the range of integration.

Now the contribution from the poles k_0, k'_0 can be evaluated using the formula (24) for the asymptotic development as stated by Debnath and Rosenblat [2]. It then follows that as $x \to \infty$,

$$I \sim \frac{\beta_1(k_0)}{2m'_1(k_0)}\, e^{i\omega t}(e^{ik_0x} - e^{-k_0x})\,, \qquad I' \sim \frac{\beta_2(k'_0)}{2m'_2(k'_0)}\, e^{i\omega t}(e^{ik'_0x} - e^{ik'_0x}) \tag{3.4ab}$$

where $m'_1(k_0), m'_2(k'_0)$ are the derivatives of $m_1(k)$ at $k = k'_0$ and $m_2(k)$ at $k = k'_0$ respectively.

The method of stationary phase (Jones [7]) can be used to evaluate the transient component of J (that is the contribution from the stationary points)

$$J_{tr} \sim \frac{i}{2\pi}\beta_1(k_1)\sqrt{\frac{2\pi}{t\,|\,m''_1(k_1)\,|}}\left[\frac{\exp[i\{tm_1(k_1) - k_1x - \frac{\pi}{4}\}]}{m_1(k_1) + \omega} - \frac{\exp[-i\{tm_1(k_1) - k_1x - \frac{\pi}{4}\}]}{m_1(k_1) + \omega}\right] \tag{3.5}$$

$$J'_{tr} \sim \frac{i}{2\pi}\beta_1(k'_1)\sqrt{\frac{2\pi}{t\,|\,m''_2(k'_1)\,|}}\left[\frac{\exp[i\{tm_2(k'_1) - k'_1x - \pi/4\}]}{m_2(k'_1) - \omega} - \frac{\exp[i\{tm_2(k'_1) - k'_1x - \pi/4\}]}{m_2(k'_1) + \omega}\right] + 0(1/t), \tag{3.6}$$

where J_{tr}, J'_{tr} denote the transient parts of J and J' respectively for large t.

Finally we calculate the contribution to J and J' from their polar singularity. This can be easily estimated by formula (24), as stated in Debnath and Rosenblat [2].

$$J_{\text{polar}} \sim -\frac{\beta_1(k_0)}{2m'_1(k_0)}\, e^{i\omega t}(e^{ik_0x} + e^{ik_0x})\,, \qquad J'_{\text{polar}} \sim -\frac{\beta_2(k_0)}{2m'_2(k'_0)}\, e^{i\omega t}(e^{ik'_0x} + e^{-ik'_0x})\,. \tag{3.7ab}$$

We write $\eta_1 = \eta_{st} + \eta'_{st} + \eta_{tr} + \eta'_{tr}$ where η_{st}, η'_{st} are the steady state components of η_1 and η_{tr}, η'_{tr} are the transient components. The first term in η_1, is the polar contribution to I and J and the second term is the polar contribution to I' and J' which are given by

$$\eta_{st} = -\frac{\beta_1(k_0)}{m_1'(k_0)} e^{i(\omega t - k_0 x)} + 0\left(\frac{1}{x}\right), \quad \eta_{st}' = -\frac{\beta_2(k_0')}{m_2'(k_0')} e^{i(\omega t - k_0' x)} + 0\left(\frac{1}{x}\right) \tag{3.8ab}$$

and the transient components η_{tr}, η_{tr}' are given respectively by (3.5) and (3.6).

So far the entire analysis of the asymptotic behavior has been carried out for $\eta_1(x,t)$. A similar asymptotic analysis can be obtained for $\eta_2(x,t)$. It is clear that there are two modes of waves spreading at each of the free surface of the upper fluid and in the interface of the two fluids and of course one of them will dominate on the other. The above analysis reveals the fact that the transient solution decays rapidly to zero as time $t \to \infty$. The ultimate steady state is established in the limit. Solutions (3.8ab) represent outgoing waves propagating with phase velocity ω/k_0 and ω/k_0' respectively. These results justify the use by previous authors of the condition at infinity known as the Sommerfield radiation condition when investigating steady-state harmonic surface waves problem. The application of this condition instead of the boundedness condition at infinity was necessary to render the solution unique.

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