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RESEARCH NOTES

A NOTE ON BEST APPROXIMATION AND INVERTIBILITY OF OPERATORS ON UNIFORMLY CONVEX BANACH SPACES

by

JAMES R. HOLUB

Department of Mathematics
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia 24061

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ABSTRACT

It is shown that if X is a uniformly convex Banach space and S a bounded linear operator on X for which $\|I-S\|=1$, then S is invertible if and only if $\|I-\frac{1}{2}S\|<1$. From this it follows that if S is invertible on X then either (i) $dist(I,[S])<1$, or (ii) 0 is the unique best approximation to I from $[S]$, a natural (partial) converse to the well-known sufficient condition for invertibility that $dist(I,[S])<1$.

Key Words and Phrases: uniformly convex space, invertible operator, unique best approximation.

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§1. **Introduction.** It is well-known [3, p. 584] that if S is a bounded linear operator on a Banach space X for which $\|I-S\|<1$ then S is invertible. Equivalently, if $[S]$ denotes the subspace of $\mathcal{L}(X)$ spanned by S, then S is invertible if $dist(I,[S])<1$. Simple examples show that in the "extreme" case when $\|I-S\|=1$ the operator S may, or may not, be invertible.

In this paper we characterize the invertible operators S on X for which $\|I-S\|=1$ in the case where X is a uniformly convex space (Theorem 1). As a consequence of this result we derive a necessary condition for invertibility of an operator on a uniformly convex space in terms of best approximation to the identity operator in $\mathcal{L}(X)$ which is a natural complement to the sufficient condition cited above (Theorem 2).

The terminology and notation used here is standard (e.g. [3]). For simplicity the word "operator" will be used to mean "bounded linear operator", the word "space" to mean "Banach space", and the symbol $\mathcal{L}(X)$ to denote the space of all operators on X. Finally, we recall that a space X is called uniformly convex [2] if for each $0<\epsilon\leq 2$ there exists $0<\delta<1$ so that if $\|x\|\leq 1$, $\|y\|\leq 1$, and $\|x-y\|\geq\epsilon$ in X, then $\|x+y\|<2(1-\delta)$; e.g., it is well-known that every $L^{\rho}(\mu)$-space with $1<\rho<+\infty$ is uniformly convex [2].

§2. Our results are based on the following recent result of Abramovich, Aliprantis, and Burkinshaw concerning Daugavet's equation in uniformly convex spaces:

THEOREM (A–A–B) [1]. : *If X is a uniformly convex space, an operator T on X satisfies the equation $\|I+T\| = 1+\|T\|$ if and only if $\|T\|$ is in the approximate point spectrum of T (i.e. there is a sequence $\{x_n\}$ in X with $\|x_n\| = 1$ for all n for which $\|Tx_n - \|T\|x_n\| \to 0$).*

From this we have:

PROPOSITION 1. *Let X be a uniformly convex space and T an operator on X for which $\|T\| = 1$. Then $\|I+T\| < 2$ if and only if $I - T$ is invertible on X.*

PROOF: If $I - T$ is invertible then $1 = \|T\|$ is not in the approximate point spectrum of T, so by Theorem (A–A–B) above $\|I+T\| < 2$.

On the other hand, if $\|T\| = 1$ and $\|I+T\| < 2$ then by Theorem (A–A–B) the number 1 is not in the approximate point spectrum of T so the operator $I - T$ must be bounded below on the unit sphere $\{x \| \|x\| = 1\}$ in X, and hence $I - T$ is an isomorphism from X onto the closed subspace $ran(I-T)$ of X. If this range of $I - T$ were a proper subspace of X then there would exist a functional $f \in X^*$ for which $\|f\| = 1$ and $(I - T^*)(f) = 0$; but then $T^*f = f$, so $\|I+T\| = \|I+T^*\| \geq \|(I+T^*)(f)\| = 2$, a contradiction. Therefore it must be that $ran(I-T) = X$, and $I - T$ is invertible.

Now, as we remarked earlier, it is well–known that if S is an operator on a space X for which $\|I - S\| < 1$ then S is invertible, but if $\|I - S\| = 1$ no conclusion is possible. However we now show that in contrast to the general case, if X is uniformly convex we can characterize exactly which such operators are invertible.

THEOREM 1. *Let X be a uniformly convex space and S an operator on X for which $\|I - S\| = 1$. Then the following are equivalent:*

(i) S is invertible.

(ii) $\|I - \frac{1}{2}S\| < 1$.

(iii) $\|I - tS\| < 1$ for all $0 < t < 1$.

PROOF: (i) ⇒ (ii). Suppose S is invertible, but $\|I - \frac{1}{2}S\| \geq 1$. Since $\|I - S\| = 1$ it follows that $\|I - \frac{1}{2}S\| = \frac{1}{2}\|I + (I - S)\| \leq 1$ as well, so $\|I - \frac{1}{2}S\| = 1$ and hence $\|I + (I - S)\| = \|2I - S\| = 2$. But then by Proposition 1 (with $T = I - S$) we have that $S = I - (I - S)$ is not invertible, a contradiction. Therefore, if S is invertible it must be that $\|I - \frac{1}{2}S\| < 1$.

(ii) ⇒ (iii). Suppose $\|I - \frac{1}{2}S\| < 1$ but $\|I - t_0S\| \geq 1$for some $0 < t_0 < 1$. Again, this implies $\|I - t_0S\| = 1$, and hence that $\|(1 - t_0)I + t_0(I - S)\| = \|I\| = \|I - S\| = 1$. By the Hahn–Banach

Theorem it follows easily that $||(1-t)I + t(I-S)|| = 1$ for all $0 < t < 1$ as well, a contradiction to (ii) when $t = \frac{1}{2}$, so (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). If $||I - tS|| < 1$ for all $0 < t < 1$, then for any such t the operator tS must be invertible by the condition cited above, implying S itself is invertible.

In terms of the geometry of the space $\mathcal{L}(X)$ Theorem 1 has the equivalent formulation:

COROLLARY 1. *If X is uniformly convex, $S \in \mathcal{L}(X)$, and $||I - S|| = 1$, then S is invertible if and only if the open segment $(I, I - S)$ in the unit ball B of $\mathcal{L}(X)$ contains no boundary point of B.*

Recall, too, that if X is any Banach space and $T \in \mathcal{L}(X)$ satisfies $||I - T|| < 1$, then not only is T invertible, but T^{-1} has the representation

$$T^{-1} = I + \sum_{n=1}^{\infty}(I - T)^n,$$

where this series converges absolutely in $\mathcal{L}(X)$ [3, p.584]. Using this result and Theorem 1 we get the same sort of representation for the inverse of an invertible operator S on a uniformly convex space even when $||I - S|| = 1$.

COROLLARY 2. *Let X be a uniformly convex space and S an invertible operator on X for which $||I - S|| = 1$. Then*

$$S^{-1} = 2I + 2\sum_{n=1}^{\infty}(I - \frac{1}{2}S)^n,$$

where this series converges absolutely in $L(X)$.

PROOF: Since S is invertible, by Theorem 1 $||I - \frac{1}{2}S|| < 1$. It follows (as above) that $\frac{1}{2}S$ is invertible and $(\frac{1}{2}S)^{-1} = I + \sum_{n=1}^{\infty}(I - \frac{1}{2}S)^n$, from which the result follows.

Remark: While the assumption of uniform convexity in Theorem 1 is sufficient to imply the conclusions of that theorem, it is possible to weaken this requirement somewhat and still obtain the same results. For example, one can show that if X is only assumed to have a Kadec–Klee norm [4] and X^* is strictly convex then Theorem 1 still holds. On the other hand, the fact that some fairly strong geometric conditions must be imposed on X in order to obtain the conclusion of Theorem 1 can be easily seen by examples such as the following:

Example: Let $S : l^1 \to l^1$ be defined by $S(e_1) = \frac{1}{2}e_1 + \frac{1}{2}e_2$ and $S(e_n) = e_n$ for $n \geq 2$, where $\{e_n\}_{n=1}^{\infty}$ denotes the standard basis for l^1. Clearly S is invertible, $||I - S|| = \sup ||(I - S)e_n|| = 1$, and yet $||I - \frac{1}{2}S|| = \sup ||(I - \frac{1}{2}S)e_n|| = ||e_1 - \frac{1}{2}Se_1|| = 1$ also, so Theorem 1 fails to hold for operators on l^1.

Now let us return to a consideration of the criterion $||I - S|| < 1$ for invertibility of an operator S on an arbitrary Banach space X. Since S is invertible if and only if λS is inveryible for some

$\lambda \neq 0$, this condition admits the following interpretation in terms of approximation in $\mathcal{L}(X)$:

If $[S]$ denotes the subspace of $\mathcal{L}(X)$ spanned by S, and if $dist(I,[S]) < 1$, then S is invertible.

In general, of course, the converse of this result need not hold; however, if X is uniformly convex we can apply Theorem 1 to obtain an interesting partial converse which reveals further the relationship between invertibility of an operator S and best approximation to I from the subspace $[S]$ of $\mathcal{L}(X)$.

THEOREM 2. *Let X be a uniformly convex space and $S \in \mathcal{L}(X)$. If S is invertible on X then either*
(i) $dist(I,[S]) < 1$, or
(ii) 0 is the unique best approxiamtion to I from $[S]$.

PROOF: Suppose S is invertible on X and $dist(I,[S]) \geq 1$. Since $dist(I,[S]) \leq 1$ it must then be that $dist(I,[S]) = 1 = \|I - 0\|$, so 0 is a best approximation to I from $[S]$.

If 0 is not the unique best approximation there is some $\lambda \neq 0$ for which $\|I - \lambda S\| = 1$ as well. Since S is assumed to be invertible, λ, S is invertible and by Theorem 1 it follows that $\|I - \frac{1}{2}(\lambda S)\| < 1$. But this is a contradiction to the fact that $dist(I,[S]) = 1$, so 0 must, in fact, be the unique best approximation, and the result follows.

Remark: Again, the operator S of the example above shows that, in general, Theorem 2 need not hold for an arbitrary space X. Exact conditions on X for the validity of Theorem 2 are not known.

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