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AN EXTENSION OF HELSON-EDWARDS THEOREM TO BANACH MODULES

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ABSTRACT. An extension of the Helson-Edwards theorem for the group algebras to Banach modules over commutative Banach algebras is given. This extension can be viewed as a generalization of Liu-Rooij-Wang's result for Banach modules over the group algebras.

KEY WORDS AND PHRASES. Multiplier, Banach modules, bounded approximate identity, compact abelian group, completely regular.

1. INTRODUCTION.

Let A be a commutative complex Banach algebra with a bounded approximate identity $\{u_\lambda\}$ of norm β and denote by Φ_A the class of all nonzero homomorphisms of A into the field of complex numbers. The space Φ_A, with the Gelfand topology, is called the carrier space of A. Let X be a Banach left A-module. A continuous module homomorphism of A into X is called a multiplier of X. We introduce a family $\{X_\phi: \phi \in \Phi_A\}$ of Banach A-modules such that any multiplier T of X can be represented as a function T on Φ_A with $T(\phi) \in X_\phi$ for each $\phi \in \Phi_A$. In this setting we give an extension of the Helson-Edwards theorem for the group algebras to Banach modules. We also observe that this extension can be viewed as a generalization of Liu-Rooij-Wang's result for Banach modules over the group algebras. We further consider a local property of multipliers when A is completely regular.

2. REPRESENTATION THEOREM OF MULTIPLIERS.

For each $\phi \in \Phi_A$, let M_ϕ denote the maximal modular ideal of A corresponding to ϕ and define

$$X^\phi = \overline{sp}\{M_\phi X + (1 - e_\phi)X\},$$

where $\overline{sp}$ denotes the closed linear span and e_ϕ is an element of A with $\phi(e_\phi) = 1$. Note that X^ϕ does not depend on the choise of e_ϕ.

Throughout the remainder of this note we will assume

$$\bigcap_{\phi \in \Phi_A} \overline{sp}(M_\phi) = \{0\}. \tag{2.1}$$

In the case of X = A, the condition (2.1) is equivalent to the semisimplicity of A. The space $\overline{sp}(AX)$ is called the essential part of X and is denoted by X_e. Since A has a bounded approximate identity, it follows that $X_e = AX$ from the Cohen-Hewitt factorization theorem (see Doran-Wichman [1]). We also have

$$X^\phi \quad X_e = \overline{sp}(M_\phi X) \tag{2.2}$$

for all $\phi \in \Phi_A$. In fact, let $\phi \in \Phi_A$, $x \in X^\phi \quad X_e$ and $\varepsilon > 0$. Since $x \in X^\phi$, there exist $a_1,\dots,a_n \quad M_\phi$ and $x_1,\dots,x_n$, $y \quad X$ such that

$$||x - \sum_{i=1}^{n} a_i x_i - (1 - e_\phi)y|| < \varepsilon/\beta.$$

Therefore for each λ, we have

$$||u_\lambda x - (\sum_{i=1}^{n} u_\lambda a_i x_i + (u_\lambda - u_\lambda e_\phi)y)|| < \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we obtain that $u_\lambda x \in \overline{sp}(M_\phi X)$ for all λ. Since $x \in X_e$, $\lim u_\lambda x = x$. Consequently, we have that $x \in \overline{sp}(M_\phi X)$ and hence $X^\phi \cap X_e \subset \overline{sp}(M_\phi X)$. The reverse inclusion is immediate.

We denote by M(A, X), or simply M(X), the class of all multipliers of X. Then M(X) also becomes a Banach A-module under the module multiplication defined by $(aT)b = a(Tb)$. For each $x \in X$, the mapping τ_x of A into X defined by $\tau_x(a) = ax$ is a multiplier of X, so that τ becomes a module homomorphism of X into M(X). Also it can be easily observed that

$$TA \subset X_e \text{ and } TM_\phi \subset \overline{M_\phi X} \tag{2.3}$$

for all $T \in M(X)$ and $\phi \in \Phi_A$, where the bar denotes the norm closure.

Now, for each $\phi \in \Phi_A$, let $X_\phi = X/X^\phi$ be the quotient of X by X^ϕ. So X_ϕ becomes a Banach A-module under the natural module structure and the quotient norm. For each $x \in X$, let $x(\phi) = x + X^\phi$ be the natural image of x in X^ϕ. A vector field on Φ_A is a function σ defined on Φ_A with $\sigma(\phi) \in X_\phi$ for each $\phi \in \Phi_A$. Of course, $\hat{x}(x \in X)$ is a vector field on Φ_A. Denote by ΠX_ϕ the class of all vector fields on Φ_A and so it becomes an A-module under the module multiplication defined by $(a\sigma)(\phi) = \hat{a}(\phi)\sigma(\phi)$, where a denotes the Gelfand transform of a A. Define

$$\Pi^b X_\phi = \{\sigma \quad \Pi X_\phi : ||\sigma||_\infty = \sup_{\phi \in \Phi_A} ||\sigma(\phi)|| < +\infty\}.$$

Then $\Pi^b X_\phi$ beomes a Banach A-module under the norm $||\ ||_\infty$ and $X = \{\hat{x} : x \in X\} \subset \Pi^b X_\phi$.

With the above notations, we have the following representation theorem of multipliers.

THEOREM 2.1. (i) If $T \in M(X)$, then there exists a unique vector field $\hat{T}$ on Φ_A such that $\widehat{Ta} = a\hat{T}$ for all $a \in A$. (ii) The mapping $T \to \hat{T}$ is a continuous module isomorphism of $M(X)$ into $\Pi^b X_\phi$.

PROOF. Let $T \in M(X)$, $a \in A$ and $\phi \in \Phi_A$. Since $e_\phi a u_\lambda - \hat{a}(\phi) e_\phi u_\lambda \in M_\phi$ for all λ, it follows from (2.3) that $T(e_\phi a u_\lambda) - \hat{a}(\phi)T(e_\phi u_\lambda) \in \overline{M_\phi X}$ for all λ. Hence, after taking the limit with respect to λ, we obtain $T(e_\phi a) - \hat{a}(\phi)Te_\phi \in \overline{M_\phi X}$. Note also that $Ta \in X_e$ from (2.3). Then there exist $c \in A$ and $y \in X$ such that $Ta = cy$, so that

$$Ta - T(e_\phi a) = Ta - e_\phi Ta = (c - e_\phi c)y \in M_\phi X. \quad \text{We therefore have}$$

$$Ta - \hat{a}(\phi)Te_\phi = (Ta - Te_\phi a) + (Te_\phi a - \hat{a}(\phi)Te_\phi) \in M_\phi X + \overline{M_\phi X} \subset \overline{sp}(M_\phi X) \subset X^\phi.$$

Setting $\hat{T}(\phi) = \widehat{Te_\phi}(\phi)$, we obtain that $\widehat{Ta}(\phi) = \hat{a}(\phi)\hat{T}(\phi) = (a\hat{T})(\phi)$. In other words, $\widehat{Ta} = a\hat{T}$ for all $a \in A$. If $\sigma \in \Pi X_\phi$ such that $\widehat{Ta} = a\sigma$ for all $a \in A$, then $\hat{T}(\phi) = \widehat{Te_\phi}(\phi) = \hat{e}_\phi(\phi)\sigma(\phi) = \sigma(\phi)$ for all $\phi \in \Phi_A$, so that $\hat{T} = \sigma$. This proves (i). It is immediate from (i) that $T \to \hat{T}$ is a continuous module homomorphism of $M(X)$ into $\Pi^b X_\phi$. To show that this mapping is injective, let $T \in M(X)$ with $\hat{T} = 0$. Then $\widehat{TA} = A\hat{T} = \{0\}$ from (i), so $TA \subset \bigcap_{\phi \in \Phi_A} X^\phi$. Also $TA \subset X_e$ from (2.3). Therefore, by (2.2) and our assumption (2.1),

$$TA \subset \bigcap_{\phi \in \Phi_A} X_e \cap X^\phi = \bigcap_{\phi \in \Phi_A} \overline{sp}(M_\phi X) = \{0\}.$$

We thus obtain $T = 0$, and (ii) is proved.

A Banach left A-module X is said to be order-free if for every $x \in X$ with $x \neq 0$ there exists $a \in A$ with $ax \neq 0$.

COROLLARY 2.2. Let $x \in X$. If either $x \in X_e$ or X is order-free, then $\hat{x} = 0$ implies $x = 0$.

PROOF. Note first that

$$\widehat{ax} = a\hat{x}, \; a \in A, \; x \in X. \tag{2.4}$$

In fact, for each $\phi \in \Phi_A$,

$$ax - \hat{a}(\phi)x = (a - \hat{a}(\phi)e_\phi)x - \hat{a}(\phi)(1 - e_\phi)\, x \in M_\phi X - (1-e_\phi)\, X \subset X^\phi.$$

This implies (2.4). Now let $x \in X$ with $\hat{x} = 0$. By the above theorem and (2.4), we have

$$\hat{\tau}_x(\phi) = \hat{e}_\phi(\phi)\hat{\tau}_x(\phi) = (e_\phi\hat{\tau}_x)(\phi) = \hat{\tau}_x e_\phi(\phi) = \widehat{e_\phi x}(\phi) = \hat{e}_\phi(\phi)\hat{x}(\phi) = \hat{x}(\phi)$$

for all $\phi \in \Phi_A$, so that $\hat{\tau}_x = \hat{x}$. Then $\hat{\tau}_x = 0$ and hence $Ax = \{0\}$. Accordingly, if either $x \in X_e$ or X is order-free, then $x = 0$.

3. EXTENSION OF HELSON-EDWARDS THEOREM.

We give a characterization of multipliers of an order-free Banach A-module which is similar to [2, Theorem 1.2.4] and Liu, van Rooij, and Wang [3, Lemma 1.3].

COROLLARY 3.1. Let X be order-free and T a mapping of A into X. Then the following conditions are equivalent.

(i) $T \in M(X)$.

(ii) T is linear and continuous; $TM_\phi \subset X^\phi$ for every $\phi \in \Phi_A$.

(iii) $T(ab) = aTb$ for all $a,b \in A$.

PROOF. (i) ==> (ii) follows immediately from (2.3). (ii) ==> (iii). Let $a,b \in A$ and $\phi \in \Phi_A$. Since $abu_\lambda - \hat{a}(\phi)bu_\lambda \in M_\phi$ for all λ, it follows from (ii) that $T(abu_\lambda) - \hat{a}(\phi)T(bu_\lambda) \in TM_\phi \subset X^\phi$ for all λ. Hence, after taking the limit with respect to λ, we obtain $T(ab) - \hat{a}(\phi)Tb \in X^\phi$. Then, by (2.4), $\widehat{T(ab)} = \widehat{aTb} = \hat{a}\widehat{Tb}$, so that $T(ab) = aTb$ by Corollary 2.2.

(iii) ==> (i). To show that T is linear, let $a,b \in A$ and α,β scalars. Then

$$cT(\alpha a + \beta b) = T(\alpha ac - \beta bc) = (\alpha a + \beta b)Tc = \alpha aTc + \beta bTc$$

$$= \alpha cTa + \beta cTb = c(\alpha Ta + \beta Tb)$$

for all $c \in A$. Since X is order-free, $T(\alpha a + \beta b) = \alpha Ta + \beta Tb$.

To show the continuity of T, let $\lim a_n = a \in A$ and $\lim Ta_n = x \in X$. Then

$$bTa = aTb = \lim a_nTb = \lim bTa_n = bx$$

for all $b \in A$. So $Ta = x$ and hence T is continuous by the closed graph theorem.

Let $\widehat{M(A)} = \{\hat{T}: T \in M(X)\}$. The following result is an extension of the Helson-Edwards theorem for the group algebra of a locally compact Abelian group (see Rudin [4, Theorem 3.8.1]).

THEOREM 3.2. Let $\sigma \in \Pi X_\phi$. Then, $A\sigma \subset \widehat{M(X)}$ if and only if $\sigma \in \widehat{M(X)}$.

PROOF. Note first that $\hat{\tau}_x = \hat{x}$ for all $x \in X$ as observed in the proof of Corollary 2.2. If $T \in M(X)$ with $\hat{T} = \sigma$, then, by Theorem 2.1, $a\sigma = a\hat{T} = \widehat{Ta} = \hat{\tau}_{Ta} \in \widehat{M(X)}$ for all a A.

Suppose conversely that $A\sigma \subset \widehat{M(X)}$. Let $a \in A$. By the Cohen-Hewitt factorization theorem, a can be written as $a = bc$ for some $b, c \in A$. Choose $S \in M(X)$ with $c\sigma = \hat{S}$. Then, $a\sigma = bc\sigma = b\hat{S} = \widehat{Sb} \in \hat{X}_e$ from (2.3). Hence, by Corollary 2.2, there is a unique element of X_e, say Ta, such that $a\sigma = \widehat{Ta}$. If a,b are arbitrary elements of A, then $\widehat{T(ab)} = ab\sigma = a(b\sigma) = a\widehat{Tb} = \widehat{aTb}$ by (2.4). Since TA X_e, $T(ab) = aTb$ by Corollary 3.1. Note that X_e is an order-free Banach A-module. Then, by Corollary 3.1, $T \in M(A, X_e) \subset M(A, X) = M(X)$. Consequently, $\sigma = \hat{T} \in \widehat{M(X)}$ and the theorem is proved.

We will observe that Theorem 3.2 can be viewed as a generalization of Liu-Rooij-Wang's result [3, Theorem 2.3].

Let G be a compact Abelian group and X a Banach L^1 (G)-module. Let $X_\gamma = \gamma X$ for each $\gamma \hat{G}$ the dual group of G. Also denote by πX_γ the class of all mappings ρ of $\hat{G}$ into X such that $\rho(\gamma) \in X_\gamma$ for every $\gamma \in G$. Set $\o_\gamma(f) = \hat{f}(\gamma)$ $(\gamma \in \hat{G}, f \in L^1(G))$, where $\hat{f}$ is the Fourier transform of f. Note that for each $\gamma \in \hat{G}, X^{\o\gamma} = (1-\gamma)X$ and $X^{\o\gamma}$ is isometrically module-isomorphic to X_γ. Also since $\overline{\mathrm{sp}}\ \hat{G} = L^1(G)$, it follows that

$$\bigcap_{\gamma \in G} X^{\phi_\gamma} = \{0\}$$

and hence X satisfies (2.1). For each $x \in X$, denote by $\tilde{x}$ the restriction of τ_x to $\hat{G}$ and set $\tilde{X} = \{\tilde{x}: x \in X\}$.

COROLLARY (Liu-Rooij-Wang). $\rho \in \Pi X_\gamma$ can be extended to a multiplier of X if and only if $\hat{f}\rho \in \tilde{X}$ for every $f \in L^1(G)$.

PROOF. Clearly $\hat{f}(\gamma)\gamma = \gamma * f$ $(\gamma \in \hat{G},\ f \in L^1(G))$. So if $\rho = T|\hat{G}$ for some $T \in M(X)$, then $\hat{f}\rho = \widetilde{Tf} \in \tilde{X}$ for every $f \in L^1(G)$. Suppose conversely that $\rho \in \Pi X_\gamma$ and $\hat{f}\rho \in \tilde{X}$ for every $f \in L^1(G)$. Then for each $f \in L^1(G)$, choose $x_f \in X$ with $\hat{f}\rho = \tilde{x}_f$. set $\sigma(\phi_\gamma) = \widehat{\rho(\gamma)}(\phi_\gamma)$ for each $\gamma \in \hat{G}$. We then have

$$(f\sigma)(\phi_\gamma) = \hat{f}(\gamma)\widehat{\rho(\gamma)}(\phi_\gamma) = (\tilde{x}_f(\gamma))\ (\phi_\gamma)$$
$$= \widehat{\gamma x_f}(\phi_\gamma) = \hat{x}_f(\phi_\gamma)$$

for all $\gamma \in \hat{G}$ and $f \in L^1(G)$. Thus $f\sigma = \hat{x}_f \in \hat{X}$ for all $f \in L^1(G)$ and hence $\sigma = \hat{T}$ for some $T \in M(X)$ from Theorem 3.2. Therefore,

$$\widehat{\rho(\gamma)}(\phi_\gamma) = \hat{T}(\phi_\gamma) = (\widehat{\gamma T})(\phi_\gamma) = \widehat{T\gamma}(\phi_\gamma),$$

so that $\rho(\gamma) - T\gamma \in X^{\phi_\gamma} = (1 - \gamma)X$ for all $\gamma \in \hat{G}$. But $\rho(\gamma)$, $T\gamma \in \gamma X$ and so $\rho(\gamma) - T\gamma \in \gamma X$ for all $\gamma \in \hat{G}$. Consequently, $\rho = T|\hat{G}$.

4. LOCAL PROPERTIES OF MULTIPLIERS.

We will consider local properties of multipliers. To do this, we introduce the following notation which is exactly similar to one given in Rickart [5, 2.7.13].

DEFINITION. Let $\sigma \in \Pi X_\phi$ and $\Sigma \subset \Pi X_\phi$. Then σ is said to belong to Σ near a point $\phi \in \Phi_A$ (or at infinity) provided there exists a neighborhood V of ϕ (or infinity) and an element σ' Σ such that $\sigma|V = \sigma'|V$. If σ belongs to near every point of Φ_A and at infinity, then σ is said to belong locally to Σ.

The following result is similar to one given in [5, 2.7.16] and we refer to the proof of one.

THEOREM 4.1. Assume A to be completely regular and let Σ be a submodule of ΠX_ϕ. If $\sigma \in \Pi X_\phi$ belongs locally to Σ, then $\sigma \in \Sigma$.

PROOF. Since σ belongs to Σ at infinity, there exists an open set U_0 of Φ_A with compact complement K and $\sigma_0 \in \Sigma$ with $\sigma_0|U_0 = \sigma|U_0$. Also since σ belongs to Σ near every point of K, there exists a finite open covering $\{U_1,\dots,U_n\}$ of K and a finite subset $\{\sigma_1,\dots,\sigma_n\}$ of Σ with $\sigma_i|U_i = \sigma|U_i$ $(i = 1,\dots,n)$. Note that A admits a partion of the identity (cf. [5, Theorem 2.7.12]). Then there exists $e_1,\dots,e_n$ A such that $e = e_1 + \dots + e_n$ is an identity for A modulo ker K and $e_i \in \ker(\Phi_A - U_i)$ $(i = 1,\dots, n)$, where ker K denotes the kernel of K. Set

$$\sigma' = (1 - e)\sigma_0 + e_1\sigma_1 + \dots + e_n\sigma_n .$$

Then σ' is obviously in Σ. We further assert $\sigma' = \sigma$. In fact, if $\phi \in U_0$, then we have

$$\sigma'(\phi) = (1 - \hat{e}(\phi))\sigma(\phi) + \sum_{\phi \in U_i} \hat{e}_i(\phi)\sigma(\phi)$$

$$= (1 - \hat{e}(\phi) + \sum_{\phi \in U_i} \hat{e}_i(\phi))\sigma(\phi)$$

$$= \sigma(\phi).$$

If $\phi \in K$, then $\hat{e}(\phi) = 1$ and $\{i: 1 \leq i \leq n, \phi \in U_i\} \neq \emptyset$, so that

$$\sigma'(\phi) = \sum_{\phi \in U_i} \hat{e}_i(\phi)\sigma_i(\phi) = \sum_{\phi \in U_i} \hat{e}_i(\phi)\sigma(\phi)$$

$$= \hat{e}(\phi)\sigma(\phi) = \sigma(\phi).$$

Consequently, $\sigma' = \sigma$ and the theorem is proved.

Because M(X) is a submodule of ΠX_ϕ, we obtain the following local property of multipliers from the preceding theorem.

COROLLARY 4.2. Assume A to be completely regular. If $\sigma \in \Pi X_\phi$ belongs locally to M(X), then $\sigma \in M(X)$.

Let A contain local identities (cf. [5, 3.6.11]) and $T \in M(X)$. The closure of $\{\phi \in \Phi_A: \hat{T}(\phi) \neq 0\}$ is called the support of T and is denoted by supp T. If supp T is compact, then there exists a unique $x \in X_e$ with $T = \tau_x$. In fact, by [5, Theorem 3.6.13], A has an identity for A modulo ker(supp $\hat{T}$), say e. Set $x = Te$. So the desired result follows from Theorem 2.1 and Corollary 2.2.

Similarly, we obtain that for each compact set K of Φ_A, there exists $x \in X_e$ with $\hat{T}|K = \hat{x}|K$.

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