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INVERSES OF MEASURES ON A CLASS OF DISCRETE GROUPS

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ABSTRACT. We examine a class of groups G, having a certain growth condition. We given an estimate for the norm of the inverse of an element in $l_1(G)$ in terms of the spectral radius and the cardinality of the support.

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1. INTRODUCTION.

Throughout this work G is a discrete group and M(G) is the usual measure algebra on G: we write δ for the unit of M(G), $\mu * v$ for the (convolution) product of two measures μ, $v \in M(G)$.

The purpose of this paper is to investigate the following problems: Let $\mu \in M(G)$ have finite support and the (convolution) inverse of $(\delta-\mu)$ exist in M(G). Is it possible to estimate the norm $||(\delta-\mu)^{-1}||$ of it?

One can easily realize that this problem becomes more interesting in the "limit" case, where the support of μ is infinite. One also can realize the connection of this with the general problem of the invertibility in M(G); namely the characterization of the class of Hermitian groups G (see [1]) or the equality of different norms and spectrums in M(G) (see [2], [3] and [4]).

Our work here can be separated into two parts. In the first part we consider the class of groups A and we give an estimate of $||(\delta-\mu)^{-1}||$ and of $||\exp\mu||$ in terms of the spectral radius $r(\mu)$ and of the cardinality of the support of μ.

In the second part we examine the relation of the class A, with the class of nilpotent groups and [FC] groups (groups having finite conjugacy class) (see [1]). We show that nilpotent groups are A-groups and that the class A is closed under finite extention. We should note that be Gromov's well known result, any finitely generated group G with polynomial growth is a finite extention of a nilpotent group, and so it is an A-group. It is remarkable to note that all known Hermitian groups, as they are referred to in [1], are A-groups. We shall complete this introduction with some definitions and notations.

We say that G is an A-group if there is a map $\kappa\colon J \to N$, where J is the set of all finite subsets of G, and a $\lambda \in N$ such that for any $F \in J$

$$\#(F^n) \leq \binom{\kappa(F)+n-1}{n}^{\lambda},\ n \in N$$

where $F^n = F\,F \dots F$ (n-times) and $\#F$ is the candinality of E.

An n-word in the elements of F is any (reduced) word of length n. We denote by $(F^n)'$ a collection of n-words in the elements of F which as a subset of G consists of all distinct elements of F^n. Finally we shall denote by μ^n the convolution product $\mu*\mu\dots*\mu$ (n-times); the spectral radius $r(\mu)$ of μ is the limit $\lim_{n\to\infty} ||\mu^n||^{1/n}$.

2. NORMS OF CERTAIN INVERTIBLE MEASURES.

We are going to show the following:

THEOREM 2.1. Let G be an A-group and let μ be an element in M(G), with finite support F and $r(\mu) < 1$. Then $\delta-\mu$ is invertible such that

$$||(\delta-\mu)^{-1}|| \leq (1-r(\mu))^{-\kappa(F)^{\lambda}} \tag{2.1}$$

where $\kappa(F)$ and λ are constants determined from the A-group structure. Furthermore if $\exp\mu = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}$ we have

$$||\exp\mu|| \leq \exp(K(F)^{\lambda} r(\mu)) \tag{2.2}$$

PROOF. First we observe that for any $n \in N$

$$\begin{aligned} ||\mu^n|| &= \sum_{x \in G} |\mu^n(x)| \\ &\leq \#\,(F^n)^{1/2}\,||\mu^n||_2 \\ &\leq \#\,(F^n)^{1/2} \sup ||\mu^n * g||_2, (g \in l_2(G), ||g||_2 = 1) \\ &\leq \#\,(F^n)^{1/2} \rho(\mu)^n \end{aligned}$$

where $||\cdot||_2$ is the norm in $l_2(G)$ and $\rho(\mu)$ is the norm of the left regular representation, i.e. the norm of the operator $\mu\colon l_2(G) \to l_2(G)\colon f \to \mu*f$. Since $\rho(\mu) \leq r(\mu)$ we have

$$||\mu^n|| \leq \#(F^n)^{1/2}\ r(\mu)^n. \tag{2.3}$$

Now by (2.3)

$$||\delta+\mu+\mu^2+\ \dots\ || \leq 1 + \#(F)^{1/2}r(\mu) +$$

$$+ \# (F^2)^{1/2}r(\mu)^2+ \dots$$

$$< \sum_{n=0}^{\infty} \binom{\kappa(F)+n-1}{n}^{\lambda} r(\mu)^n$$

Now let $K = \kappa(F)$ we see that $\binom{K+n-1}{n}^{\lambda} < \binom{K^{\lambda}+n-1}{n}$. In fact,

$$\binom{K+n-1}{n}^{\lambda} = \prod_{j=1}^{n} \left(\frac{K-1}{j} + 1\right)^{\lambda}$$

$$= \prod_{j=1}^{n} \left(1 + \sum_{m=1}^{\lambda} \binom{\lambda}{m}\left(\frac{K-1}{j}\right)^m\right)$$

$$< \prod_{j=1}^{n} \left(\frac{1}{j} [(K-1)+1]^{\lambda} + \frac{j-1}{j}\right)$$

$$< \prod_{j=1}^{n} \left(\frac{K^{\lambda}-1}{j} + 1\right)$$

$$< \binom{K^{\lambda}+n-1}{n}$$

Thus

$$||\delta+\mu+\mu^2+\dots || < \sum_{n=0}^{\infty} \binom{\kappa(F)^{\lambda}+n-1}{n} r(\mu)^n$$

and since $r(\mu) < 1$, by the binomial formula, $(\delta-\mu)^{-1}$ exists and we obtain (2.1).

To see (2.2) we observe that since $\kappa \geq 1$ and $j \geq 1$, $\left(\frac{\kappa-1}{j} + 1\right) \leq \kappa$ and

$$\binom{\kappa(F)+n-1}{n}^{\lambda} \leq \prod_{j=1}^{n} \left(\frac{\kappa(F)-1}{j} + 1\right)^{\lambda}$$

$$< \kappa(F)^{n\lambda} \tag{2.4}$$

Thus by (2.3) and (2.4)

$$||\exp\mu|| < \sum_{n=0}^{\infty} \frac{\#(F^n)}{n!} r(\mu)^n$$

$$< \sum_{n=0}^{\infty} \frac{1}{n!} \binom{\kappa(F)+n-1}{n}^{\lambda} r(\mu)^n$$

$$< \sum_{n=0}^{\infty} \frac{1}{n!} (K(F)^{\lambda}r(\mu))^n$$

$$= \exp(K(F)^{\lambda}r(\mu)).$$

3. THE CLASS OF GROUPS A.

In this section, we show that [FC] groups and nilpotent groups are A-groups. We also show that the class A is closed under finite extention.

First we examine the growth of [FC] groups.

PROPOSITION 3.1. Let G be an [FC] group and F be a finite subset of G. Then

$$\# (F^r) \leq \binom{k+r-1}{r} \qquad (r \in N)$$

where $k = \# (\bigcup_{f \in F}[f])$, [f] is the conjugacy class of f.

PROOF. We show that if $[F] = \bigcup_{f \in F} [f]$

$$\#(F^2) \leq \#([F]^2) \leq \binom{k+1}{2}$$

Given f, g ε F, f ≠ g, there is an element g_1 (say) in $[g] \subset [F]$ such that $fgf^{-1}=g_1$, and so $fg = g_1 f$.

Hence any 2-word in the elements of [F] consisting of two different letters is equal to another 2-word in the elements of [F].

Thus $[F]^2$ has no more than k elements f^2 and k(k-1)/2 elements fg where f, g ε F, f ≠ g. It is clear that

$$\# (F^2) \leq k + \frac{k(k-1)}{2} = \binom{k+1}{2}.$$

We suppose that the Theorem is true for any r < n, we show that it is also true for r = n.

We denote by $\phi(g)$ the number of all appearances of a g ε [F] in the words of $([F]^n)'$.

If all elements of [F] had the same chance to appear in $([F^n])'$ then for each g ε [F], $\phi(g) = \frac{n}{k} ([F]^n)$.
Thus we may consider a g ε [F] such that

$$\# ([F]^n) \leq \frac{k}{n} \phi(g) \tag{3.1}$$

Since for any f ε [F] f ≠ g, there is some f_1 (say) such that $fg = gf_1$. Hence without loss of generality we may assume that in any word of $([F]^n)'$, either there is no g or all g's keep the left place of the word. Now from each word of $([F]^n)'$, where g appears, cancel one g. The resulting (n-1)-words form a subset of distint elements of $[F]^{n-1}$; we denote this set by $g^{-1}([F]^n)'$. Hence from our hypothesis it is clear that

$$\phi(g) \leq \binom{k+n-2}{n-1} + \phi_1(g) \tag{3.2}$$

where $\phi_1(g)$ is the number of all appearances of g in $g^{-1}([F]^n)'$. Suppose that

$$\phi_1(g) \leq \frac{n-1}{k} \binom{k+n-2}{n-1} \tag{3.3}$$

then by (3.1) and (3.2) we obtain

$$\# ([F]^n) \leq \frac{k}{n} (1 + \frac{n-1}{k})\binom{k+n-2}{n-1} \leq \binom{k+n-1}{n}$$

and so in this case there is nothing to show.
If the inequality (3.3) does not occur, from (3.1) and (3.2) we have

$$([F]^n) \leq \frac{k}{n}\left(\frac{k}{n-1} + 1\right)\phi_1(g) \leq \frac{k(k+n-1)}{n(n-1)}\,\phi_1(g)$$

In a similar way we define $\phi_i(g)$ $(1 \leq i \leq n-1)$ i.e. the number of appearances of g in the collection $g^{-i}([F]^n)'$.

As in (3.2)

$$\phi_{i-1}(g) \leq \binom{k+n-i-1}{n-i} + \phi_i(g) \qquad (i=2,3,\ldots,n-1)$$

and if for some $i \leq n-1$

$$\phi_i(g) \leq \frac{n-i}{k}\binom{k+n-i-1}{n-i} \tag{3.4}$$

it is nothing to show. If the inequality (3.4) does not occur for any $i \leq n-i$ we observe that

$$\phi_{n-1}(g) \leq \frac{1}{k}\binom{k}{1}$$

In this case we write

$$\#\,([F]^n) \leq \frac{k(k+n-1)\,\ldots\,(k+1)}{n.(n-1)\,\ldots\,1} = \frac{(k+n-1)!}{(k-1)!n!}$$

and this completes the proof.

COROLLARY 3.1. If G is abelian and F is a finite subset of G then,

$$(F^r) \leq \binom{\#\,F+r-1}{r}, \quad r \in N$$

PROOF. Clear

LEMMA 3.1. Let G be a discrete group with a normal subgroup K such that G/K is abelian, and let π be the canonical map $\pi: G \to G/K$. If $F \subset G$ is such that $\#F = \#\,\pi(F)$, then the number of all r-words in the elements of $F(r \in N)$, in a given class of G modulo K can not be greater than

$$\binom{\#F + r-1}{r}.$$

PROOF. Let $\tilde{x} \in G/K$ fixed and $J_r = \pi^{-1}(\tilde{x}) \cap F^r$

Note that J_r and F^r in this proof mean collections of r-words, which as elements of G may not be distinct.

We shall denote by $\#\,J_r$ the cardinality of J_r, and we shall show that

$$\#\,([F]^n) \leq \binom{\#F + 1}{2} \tag{3.5}$$

Let $x,y \in F$ and $xy \in J_2$, then $yx \in J_2$; in fact since G/K is abelian $\pi(xy) = \pi(x)\pi(y) = \pi(yx)$. Now suppose that there is a $z \in F$ such that x,z (or z,x) is in J_2, i.e. $\pi(x,y) = \pi(x,z)$ and so $\pi(y) = \pi(z)$ and $\#\,\pi(F) < \#\,F$-contradiction. Hence it is clear that $\#J_2 < \#\,F$ and (3.5) follows.

We suppose that Lemma 3.1 is true for any $r > n-1$ and we show this for r=n. For

for some $\tilde{g} \in G/K$, let

$$\mathcal{J}_n = \pi^{-1}(\tilde{g}) \cap F^n.$$

If all elements of F had the same chance to appear in $\mathcal{J}_n$, then the number of all appearances $\phi(x)$, $x \in F$, of x in the words of $\mathcal{J}_n$ should be

$$\phi(x) = \frac{n}{\# F} \# (\mathcal{J}_n)$$

We consider a $x \in F$ such that

$$\# \mathcal{J}_n \leq \frac{\# F}{n} \phi(x) \tag{3.6}$$

From the set of all words in $\mathcal{J}_n$ where x has at least one entry we cancel one x. We denote by ${}_x\mathcal{J}_n$ the collection of all the resulting (n-1)-words. We show that ${}_x\mathcal{J}_n$ is in a class of G modulo K.

Let w_1, w_2, w'_1, w'_2, be words in the elements of F such that $w_1 \times w_2$, $w'_1 \times w'_2 \in \mathcal{J}_n$. Since $\pi(w_1 \times w_2) = \pi(w'_1 \times w'_2)$, we have $\pi(w_1 w_2) = \pi(w'_1 w'_2)$ and ${}_x\mathcal{J}_n$ is as we claimed.

Now, the set ${}_x\mathcal{J}_n$ by our inductive hypothesis has cardinality no greater than $\binom{\# F+n-2}{n-1}$ and so

$$\phi(x) \leq \binom{\# F+n-2}{n-1} + \phi_1(x)$$

where $\phi_1(x)$ is the number of appearances of x in ${}_x\mathcal{J}_n$.

As in Proposition (3.1), (3.7) in the case where

$$\phi_1(x) \leq \frac{n-1}{\# F} \binom{\# F+n-2}{n-1}$$

it is nothing to show. If the inequality above is not true by (3.6) and (3.7) we obtain

$$\# \mathcal{J}_n \leq \frac{\# F}{n} \frac{\# F+n-1}{n-1} \phi_1(x)$$

We complete the proof in the same arguments as in Proposition 3.1.

PROPOSITION 3.2. Any nilpotent group G is an A-group with $\kappa(F) = \# F$ and $\lambda = 2q-1$, where q is the index of G and F is a finite set.

PROOF. Let $G = A_0 \supset A_1 \supset \ldots \supset A_{q-1} \supset A_q = \{e\}$ be the normal series of a nilpotent group G of index q. We note that A_{i-1}/A_i is the center of G/A_i $(1 \leq i \leq q-1)$ and we denote by π_i the canonical map $G \to G/A_i$.

It is obvious that for any $F \subset G$ and $r \in N$

$$\#(F^r) \leq \#(\pi_1(F^r)) \max \{\# \pi^{-1}(\tilde{g}) \cap F^r : \tilde{g} \in G/A_1\}. \tag{3.8}$$

We denote by $|F^r|_1$ the RHS of (3.8).

We shall show that there are q positive integers $m_1, m_2, \ldots, m_q$ such that

$\#(F) = m_1+m_2+\ldots+m_q$ and

$$|F^r|_1 \leq \binom{m_1+r-1}{r}^2 \binom{m_2+r-1}{r}^2 \ldots \binom{m_{q-1}+r-1}{r}^2 \binom{m_q+r-1}{r} \tag{3.9}$$

If q=1, i.e. G is abelian. Corollary (3.1) implies (3.9). We suppose that (3.9) is true for each nilpotent group of index $q = p - 1$.

Let G be nilpotent of index p. Since G/A_1 is abelian by Lemma 3.1 if $\# F \leq \# \pi_1(F)$.

We have, $|F^r|_1 \leq \binom{\# F + r-1}{r}^2$.

Let $\# (\pi_1(F)) = m_1 < \# (F)$, then F can be written as

$$F = \{x_i \alpha_j : 1 \leq i \leq m_1,\ 1 \leq j \leq \#(F)-m_1\}$$

where $\pi_1(x_i) \neq \pi_1(x_j)$ $i \neq j$, $1 \leq j \leq m_1$, and all α_j's are in A_1. By (3.8) we obtain,

$$|F^r|_1 \leq \binom{m_1+r-1}{r} \# J_1 \tag{3.10}$$

where $J_1 = {}_1^{-1}(g)\ F^r$, for some $g \in G/A_1$. Any element of J_1 can be written as

$$x_{i_1}\alpha_{j_1} x_{i_2}\alpha_{j_2}\ldots x_{i_r}\alpha_{j_r} \tag{3.11}$$

where $(x_{i_1} x_{i_2} \ldots x_{i_r}) \in \tilde{g}$, each i_t $(1 \leq t \leq r)$ is one of $1,2,\ldots,m_1$ and each j_t is one of $1,2,\ldots,\#F-m_1$.

By Lemma (3.1) the cardinality of all $x_{i_1}\ldots x_{i_r}$ in $\tilde{g}$ is $\leq \binom{m_1+r-1}{r}$ and so

$$\# J_1' \leq \binom{m_1+r-1}{r} \# J_1'$$

where

$$J_1' = x_{i_1}F_1 x_{i_2}F_1 \ldots x_{i_r}F_1$$

$x_{i_1}x_{i_2}\ldots x_{i_r}$ is fixed suitably choosen from (3.11) and belongs to $\tilde{g}$;

$$F_1 = \{\alpha_j : 1 \leq j \leq \# F-m_1\}$$

Note that if $q = 2$, then A_1 is the center of G, all α_j's commute with x_i's and by Corollary (3.1), $|F_1^r| \leq \binom{\# F-m_1+r-1}{r}$; thus by (3.10) and (3.12), (3.9) follows.

As in (3.8), for some $\tilde{g} \in G/A_2$ we have

$$\# J_1' \leq \# \pi_2(J_1')\ \#\ \{\pi_2^{-1}(\tilde{g}) \cap J_1'\} \tag{3.13}$$

Since A_1/A_2 is the center of G/A_2 and $F_1 \subset A_1$, by (3.10), (3.12) and (3.13) we obtain

$$|F^r|_1 \leq \binom{m_1+r-1}{r}^2 \cdot |F_1^r|_1 \tag{3.14}$$

where $|F_1^r|_1$ is defined as $|F^r|_1$.

Since A_1 is a nilpotent of index $\rho-1$ we apply our inductive hypothesis in (3.14) and for $q = p$ we obtain (3.9).

Now if we replace $m_1,\dots,m_q$ in (3.9) by #F we see that G is an A group with constants $k = 1$ and $\lambda = 2q-1$.

PROPOSITION 3.3. The class of A-groups is closed under extensions by finite groups.

PROOF. We may write $G/A = \{d_1A, d_2A, \dots, d_sA\}$ where $d_1,d_2,\dots,d_s$ are s representatives of all the different classes of G/A; without loss of generality let $d_s = e$ the unit of G. We may also write

$$d_id_j = \alpha(i,j)\ d(i,j) \qquad (1 \leq i,\ j \leq s)$$

where each $d(i,j)$ is one of $d_1,\dots,d_s$ and each $a(i,j)$ is in A.

Let $F = \{d_{i_1}x_1,\dots,d_{i_m}x_m\}$, #F=m, each d_{i_t} is one of $d_1,\dots,d_s$ and $x_t \in A$ $(1 \leq t \leq m)$.

Let $\langle x_i\rangle = \{d_j^{-1}x_id_j : j = 1,2,\dots,s\}$ $(1 \leq i \leq m)$ and

$$d_{j_1}x_{i_1}d_{j_2}x_{i_2}\dots d_{j_r}x_{i_r} \tag{3.15}$$

be a typical r-word in the elements of F. Each word in (3.15) is in the set

$\langle x_{i_1}\rangle\ \alpha(j_1,j_2)\ d(j_1,j_2)x_{i_2}\dots d_{j_r}x_{i_r}$ or in

$\langle x_{i_1}\rangle\ \alpha(j_1,j_2)\ \langle x_{i_2}\rangle\ \alpha(\cdot,j_3)\ \dots\ d_{j_r}x_{i_r}$ or finally in

$$\langle x_{i_1}\rangle\ \langle x_{i_2}\rangle\ \dots\ \langle x_{i_r}\rangle\ a.d. \tag{3.16}$$

where $a \in A$ and $d \in \{d_1,d_2,\dots,d_s\}$.

It is clear that the cardinality of the r-words in the elements of F, as in (3.15) is less than the candinality of the words in (3.16) in the elements of $\{\langle x_1\rangle, \dots, \langle x_m\rangle\}$, which is a subset of the A-group, A. Thus G inherits the growth of A.

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