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# ON PERVIN'S EXAMPLE CONCERNING THE CONNECTED-OPEN TOPOLOGY

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ABSTRACT. Irudayanathan and Naimpally [1] introduced a topology for function spaces (called the "connected-open" topology) which has the property that the connected functions form a closed set provided that the codomain is completely normal. Pervin [2] gave an example showing that the proviso cannot be weakened to normality. The purpose of this note is to point out a lacuna in his demonstration, and to re-establish the validity of the example.

KEY WORDS AND PHRASES. *Function space, connected-open topology, complete normality.*
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## 1. INTRODUCTION.

Let X and Y denote topological spaces, and F the set of all mappings from X to Y. For each connected subset K of X and each pair U, V of open subsets of Y denote by W(K; U, V) the subset

$$\{f \in F : f(K) \subseteq U \cup V,\ f(K) \cap U \neq \emptyset \neq f(K) \cap V\}$$

of F. The collection $S$ of all these sets W(K; U, V) is a subbase for the connected-open topology T on F, introduced by Irudayanathan and Naimpally in [1] where it is proved that the collection $C^{-2}$ of all connected (Darboux) functions is T-closed if Y is completely normal.

To show that normality of Y is not sufficient for this result, Pervin [2] took Y as a modification of the Tychonoff plank, with an open interval of reals interpolated between each ordinal and its successor in the construction; appealed to cardinality to obtain a function f from the unit interval X = [0,1] onto a subset A* ∪ B* of Y, where A* and B* were separated but had no disjoint neighbourhoods in Y, and where $f^{-1}(\{y\})$ was dense in X for every y in A* ∪ B*; and proved that any member W(K; U, V) of $S$ which contained f must also contain a connected function. However, this does not suffice to establish that the (non-connected) function f belongs to the T-closure of $C^{-2}$, it being perfectly possible for every subbasic neighbourhood of a point to intersect a set without every basic neighbourhood doing so. We shall show that f is, nevertheless, a limit of connected (indeed, of continuous) functions.

## 2. PERVIN'S EXAMPLE REVISITED.

Let J denote the connected, compact $T_2$ space formed from the second uncountable ordinal $\overline{W}_\Omega$ by interpolating a copy of (0,1) between each element (other than the maximum) and its successor, and imposing the order topology on the resulting chain; and consider the product space $Y = J\times[0,1]$. (The space $\overline{W}^*_\omega$ used here by Pervin instead of [0,1] is homeomorphic to [0,1].) Denote by a and b (respectively) the least and greatest elements of J, and by A* and B* the following subsets of Y:

$$A^* = [a,b)\times\{1\},\ B^* = \{b\}\times[0,1).$$

(Pervin's definition of these sets is incompatible with his assertion that they are connected; the above is presumably what was intended.) Considerations of cardinality establish the existence of a mapping f from [0,1] onto $A^* \cup B^*$ such that the preimage of each singleton is dense. It will now be shown that every neighbourhood of f contains a connected function.

Consider a typical basic T-neighbourhood

$$G = \cap\{W(K_i;\ U_i,\ V_i) : i = 1, 2, \dots, n\}$$

of f, where (for each i) $K_i$ is a connected subset of [0,1], $U_i$ and $V_i$ are open in Y, and $f(K_i)$ is contained in the union of $U_i$ and $V_i$ and meets them both. No loss of generality will be incurred by assuming that the sets $K_i$ are distinct, since

$$W(K;\ U,\ V) \cap W(K;\ U',\ V') \supseteq W(K;\ U \cap U',\ V \cap V').$$

Denoting by j the number of degenerate intervals amongst the $K_i$, where $0 \le j \le n$, we can arrange the labelling so that $K_i$ is a singleton for $i \le j$ and is non-degenerate for $i > j$. The strategy of the proof is to determine a subset Z of Y of the form suggested by $\alpha\beta\gamma\delta\epsilon\beta$ in the diagram below (which see), where x is chosen to ensure that Z is contained in $U_i \cup V_i$ for all $i > j$, and z is selected so that Z includes at least one point of $U_i \cap V_i$ for each i; a path-connectedness argument within Z will then produce a continuous function belonging to G.

For $i > j$, $f(K_i)$ is the whole of $A^* \cup B^*$ and is contained in $U_i \cup V_i$. Thus for each positive integer n the product of compact sets

$$\{b\}\times[0,1-2^{-n}]$$

is contained in $U_i \cup V_i$, and a lemma of A.D. Wallace (see [3], p.142) allows us to find $x_{i,n} \in [a,b)$ such that

$$(x_{i,n},b]\times[0,1-2^{-n}] \subseteq U_i \cup V_i.$$

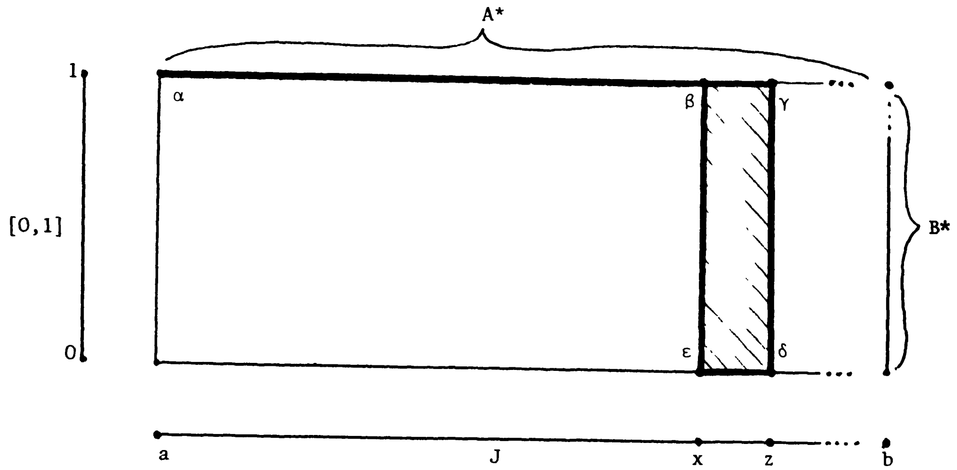


Figure 1.

Now $[a,b)$ inherits from its cofinal subset $\overline{W}_\Omega \setminus \{b\}$ the property that each countable subset is bounded above: choosing then a strict upper bound $x_i < b$ for the sequence $(x_{i,n})$ we see that

$$[x_i,b]\times[0,1) \subseteq U_i \cup V_i;$$

and so if $x$ denotes the maximum of the elements $x_i$ here chosen, we have

$$[x,b]\times[0,1) \subseteq U_i \cup V_i \qquad \text{for all } i > j. \tag{2.1}$$

(In the event that $j = n$, i.e. that all the $K_i$ are degenerate, (2.1) may be obtained by an arbitrary choice of $x < b$.)

Still considering the case $i > j$, we see from (2.1) that the connected set $A^* \cup (x,b]\times[0,1)$ is contained in the union of $U_i$ and $V_i$ and intersects them both; so it must be possible to choose a point $t(i) = (t(i)_1, t(i)_2)$ of $U_i \cap V_i$ such that either $t(i) \in A^*$, or else $t(i) \in (x,b]\times[0,1)$: and in the latter case, the observations that $U_i \cap V_i$ is a neighbourhood of $t(i)$ and that $b$ is not isolated in $J$ will allow us to assume that $x < t(i)_1 < b$. Turning now to the case $i \le j$, $f(K_i)$ is here a single point of $(A^* \cup B^*) \cap U_i \cap V_i$; if this point lies in $A^*$ we denote it by $t(i) = (t(i)_1, t(i)_2)$, while if it belongs to B*, an argument like that above will yield $t(i) = (t(i)_1, t(i)_2)$ in $U_i \cap V_i$ satisfying $x < t(i)_1 < b$. Lastly let z denote the maximum of $t(1)_1, t(2)_1, \ldots, t(n)_1$: the consequence of the choices of x and of z is that the set

$$Z = [a,z]\times\{1\} \cup [x,z]\times[0,1]$$

includes the point t(i) of $U_i \cap V_i$ for every i, and is contained in $U_i \cup V_i$ for those values of i (if any) for which $K_i$ is non-degenerate.

Now since $z < b$, the interval (x,z) contains only countably many elements of $\overline{W}_\Omega$ and only countably many interpolated real intervals or parts thereof, so it possesses a countable dense subset. It is routine to verify that it contains a supremum and an infimum for each of its bounded subsets, and it has no least nor greatest element and no gaps. Thus by a well-known characterization due to Hausdorff ([4], p. 54) it is homeomorphic to the real line. Then [x,z] and, similarly, [a,z] are homeomorphic to bounded closed real intervals; and Z, being essentially the unit square in the real plane with a line segment attached to one corner, is path-connected. Choosing distinct elements $k_i$ in $K_i$ for each i, which will be possible since the $K_i$ are themselves distinct intervals, this guarantees the existence of a continuous (and therefore connected) function $g : [0,1] \to Z$ such that $g(k_i) = t(i)$ for each i. Regarding g as a mapping into Y, we see that it is common to all the sets $W(K_i;\ U_i,\ V_i)$ and the demonstration is complete.

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