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EIGENFUNCTION EXPANSION FOR A REGULAR FOURTH ORDER EIGENVALUE PROBLEM WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITIONS

E.M.E. ZAYED

Department of Mathematics
Faculty of Science
Zagazig University
Zagazig, Egypt

and

S.F.M. IBRAHIM

Department of Mathematics
Faculty of Education
Ain Shams University
Helipolis, Cairo, Egypt

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1. INTRODUCTION.

The regular right-definite eigenvalue problems for second order differential equations with eigenvalue parameter in the boundary conditions, have been studied in Walter [1], Fulton [2] and Hinton [3].

The object of this paper is to prove the expansion theorem for the following regular fourth order eigenvalue problem:

$$\left.\begin{aligned} \tau u: \quad &= (Ku'')'' - (Pu')' + qu = \lambda u \quad, x\varepsilon[a,b] \\ &u(a) = (Pu')(a) = (Ku'')(a) = 0 \\ &(Ku''')(b) - (Pu')(b) = -\lambda u(b) \end{aligned}\right\} \qquad (1.1)$$

where P,q and K are continuous real-valued functions on [a,b]. We assume that $P(x) > 0$, $q(x) > 0$, and $K(x) > 0$ while λ is a complex number.

Recently, Zayed [4] has studied the special case of the problem (1.1) wherein $K(x) = \alpha^2$, α^2 is a constant and $q(x) = 0$.

Further, problem (1.1), in general, describes the transverse motion of a rotating beam with tip mass, such as a helicopter blade (Ahn [5]) or a bob pendulum suspended from a wire (Ahn [6]).

Ahn [7] has shown that the set of eigenvalues of problem (1.1) is not empty, has no finite accumulation points and is bounded from below. He used an integral-equation approach.

In this paper, our approach is to give a Hilbert space formulation to the problem (1.1) and self-adjoint operator defined in it such that (1.1) can be considered as the eigenvalue problem of this operator.

2. HILBERT SPACE FORMULATION.

We define a Hilbert space H of two-component vectors by

$$H = L^2(a,b) \oplus C;$$

with inner product

$$\langle f,g\rangle = \int_a^b f_1\bar{g}_1 dx + f_2\bar{g}_2 \qquad ,f,g \in H \tag{2.1}$$

and norm

$$||f||_H^2 = \int_a^b |f_1|^2\, dx + |f_2|^2 \tag{2.2}$$

where

$$f = (f_1,\ f_2) = (f_1(x),\ f_1(b)) \in H$$

and

$$g = (g_1,\ g_2) = (g_1(x),\ g_1(b) \in H .$$

We can define a linear operator $A:D(A) \to H$ by

$$Af = (\tau f_1,\ -(Kf_1''')(b) + (Pf_1')(b)) \quad \forall\ f = (f_1,f_2) \in D(A) \tag{2.3}$$

where the domain D(A) of A is a set of all $f = (f_1,f_2) \in H$ which satisfy the following:

(i) f_1, f_1', f_1'' and f_1''' are absolutely continuous with

$\tau f_1 \in L^2(a,b)$ and $\int_a^b (K|f_1''|^2 + P|f_1'|^2 + q|f_1|^2)dx < \infty.$

(ii) $f_1(a) = (Pf_1')(a) = (Kf_1'')(a) = 0$

(iii) $f_2 = f_1(b)$.

REMARK 2.1. The parameter λ is an eigenvalue of (1.1) and f_1 is a corresponding eigenfunction of (1.1) if and only if

$$f = (f_1,f_1(b)) \in D(A) \qquad \text{and} \qquad Af = \lambda f \tag{2.4}$$

Therefore, the eigenvalues and the eigenfunctions of problem (1.1) are equivalent to the eigenvalues and the eigenfunctions of operator A.

We consider the following assumptions:

$$\text{(i)}\quad \lim_{x\to b}\,[K'(x)f_1(x) - K(x)f_1'(x)] = 0,$$
$$\text{(ii)}\quad \lim_{x\to b}\,[K'(x)\bar{g}_1(x) - K(x)\bar{g}_1'(x)] = 0\,. \tag{2.5}$$

LEMMA 2.1. The linear operator A in H is symmetric.

PROOF. On using the boundary conditions of (1.1) we get,

$$\langle Af,g\rangle = \int_a^b (\tau f)\bar{g}_1 dx + [-(Kf_1'')(b) + (Pf_1')(b)]\bar{g}_1(b)$$
$$= \int_a^b (Kf_1'')''\bar{g}_1 dx - \int_a^b (Pf_1')'\bar{g}_1 dx + \int_a^b qf_1\bar{g}dx - (Kf_1'')(b)\bar{g}_1(b)$$
$$+ (Pf_1')(b)\bar{g}_1(b) \tag{2.6}$$

Integrating the first term of (2.6) by parts four times and integrating the second term of (2.6) by parts twice, we get

$$\langle Af,g\rangle = \int_a^b f_1[(\overline{Kg_1''})'' - (\overline{Pg_1'})' + \overline{qg}_1]dx + f_1(b)[-(\overline{Kg_1''})(b)+(\overline{Pg_1'})(b)]$$
$$+ f_1''(b)\,[K'(b)\bar{g}_1(b) - K(b)\bar{g}_1'(b)] - \bar{g}_1''(b)\,[K'(b)f_1(b)-K(b)f_1'(b)]$$

Applying the conditions (2.5) and using the boundary conditions of (1.1),we obtain

$$\langle Af,g\rangle = \int_a^b f_1(\overline{\tau g_1})dx + f_1(b)\,[-(\overline{Kg_1''})(b) + (\overline{Pg_1'})(b)] = \langle f,Ag\rangle.$$

REMARK. 2.2. For all $f = (f_1,f_2)$ in D(A) and $f_2 = f_1(b) \neq 0$, the domain D(A) is dense in H.

Since the operator A in H is symmetric and dense in H, A is self-adjoint.

3. THE BOUNDEDNESS.

We shall show that the self-adjoint operator A is unbounded from above and bounded from below. We also show that A is strictly positive.

LEMMA 3.1.

(i) If f, f' are absolutely continuous with $f(a) = 0$ and $P(x) > 0$ in $[a,b]$, then we have $P(x) \geq c_1$ for some constant $c_1 > 0$ such that

$$\int_a^b P(x)|f'(x)|^2\,dx \geq c_1|f(b)|^2\,.$$

(ii) For $f\in C^2[a,b]$, there exists a positive constant c_2 such that

$$\int_a^b |f(x)|^2\,dx \leq c_2 \int_a^b |f''(x)|^2 dx$$

PROOF.

(i) Since $P(x) > 0$ in $[a,b]$, we have $P(x) \geq c_1$ for some $c_1 > 0$.

Consequently, on using Schwartz's inequality, we get

$$\int_a^b P(x)|f'(x)|^2\,dx \geq c_1 \int_a^b |f'(x)|^2\,dx \geq c_1 [\int_a^b |f'(x)|dx]^2 \geq c_1|f(b)|^2$$

where $\int_a^b f'(x)dx = f(b) - f(a) = f(b)$, Since $f(a) = 0$.

(ii) By using Theorem 2 in [8, p.67], we have for $f(x) \in C^1[a,b]$,

$$\int_a^b |f(x)|^2\,dx \leq 4(b-a)^2 \int_a^b \left| \frac{d|f(x)|}{dx} \right|^2 dx$$

Since $\left| \frac{d|f(x)|}{dx} \right|^2 \leq 4\left| \frac{df(x)}{dx} \right|^2$,

then

$$\int_a^b |f(x)|^2\,dx \leq 4(b-a)^2 \int_a^b \left|\frac{d|f(x)|}{dx}\right|^2 dx \leq 16(b-a)^2 \int_a^b |f'(x)|^2 dx \qquad (3.1)$$

Applying (3.1) again for $|f'(x)|$, we get

$$\int_a^b |f'(x)|^2\,dx \leq 16(b-a)^2 \int_a^b |f''(x)|^2 dx \qquad (3.2)$$

from (3.1) and (3.2) we get

$$\int_a^b |f(x)|^2\,dx \leq c_2 \int_a^b |f''(x)|^2\,dx \text{ where the constant } c_2=256(b-a)^4.$$

LEMMA 3.2. The linear operator A is bounded from below.

PROOF. On using the boundary conditions of (1.1) we get

$$\langle Af,f\rangle = \int_a^b (\tau f_1)\overline{f}_1 dx + [-(Kf_1'')(b) + (Pf_1')(b)]\overline{f}_1(b)$$

$$= \int_a^b (Kf_1'')''\overline{f}_1 dx - \int_a^b (Pf_1')'\overline{f}_1 dx + \int_a^b qf_1\overline{f}_1 dx - (Kf_1'')(b)\overline{f}_1(b)$$

$$+ (Pf_1')(b)\overline{f}_1(b). \qquad (3.3)$$

Integrating (3.3) by parts twice and using the boundary conditions of (1.1), we obtain

$$\langle Af,f\rangle = f_1''(b)\ [K'(b)\overline{f}_1(b) - K(b)\overline{f}_1'(b)] + \int_a^b K|f_1''|^2 dx$$

$$+ \int_a^b P|f_1'|^2\,dx + \int_a^b q|f_1|^2 dx\ .$$

On using (2.5) (ii) and lemma (3.1), we get

$$\langle Af,f\rangle \geq \int_a^b \frac{K(x)}{c_2}|f_1(x)|^2\,dx + c_1|f_1(b)|^2 + \int_a^b q(x)|f_1(x)|^2\,dx$$

$$= \int_a^b [\frac{K(x)}{c_2} + q(x)]|f_1(x)|^2\,dx + c_1|f_2|^2$$

$$\geq c_3 \int_a^b |f_1(x)|^2 \, dx + c_1 |f_2|^2$$

where

$$c_3 = \inf_{x \in [a,b]} \left[\frac{K(x)}{c_2} + q(x)\right]$$

Therefore

$$\langle Af, f\rangle \geq c \, ||f||^2 \tag{3.4}$$

where the constant $c = \min (c_3, c_1)$.

It follows, from (3.4), that the operator A is bounded from below. Since $c_1 > 0$, $K(x) > 0$, $q(x) > 0$, $c_2 > 0$ and $c = \min (c_3, c_1)$ then the constant c is positive ($c > 0$) and hence A is strictly positive.

REMARK 3.1.

(i) Since A is a symmetric operator (from lemma 2.1) then A has only real eigenvalues.

(ii) By Lemma 3.2, we deduce that the set of all eigenvalues of A is also bounded from below.

(iii) Since A is strictly positive, then the zero is not an eigenvalue of A.

By using theorem 3 in [8, p.60] we can state that:

Since A in H is symmetric and bounded from below, then for every eigenvalue λ_i of A in H, $\lambda_i \geq c$ where the constant c is the same as in (3.4). This means that $0 < c \leq \lambda_1 \leq \lambda_2 \leq \dots\dots\dots\dots \leq \lambda_i$ according to the size and $\lambda_i \to \infty$ as $i \to \infty$. This implies that the set of all eigenvalues of A is unbounded from above.

REMARK 3.2. Since the operator A is self-adjoint, then A has only real eigenvalues and the eigenfunctions of A are orthonormal. By using theorem 3 in [8, p.30], the density of the domain D(A) in H gives us the completeness of the orthonormal system of eigenfunctions $\Phi_1, \Phi_2, \Phi_3, \dots$ of A.

4. THE EIGENFUNCTIONS OF THE OPERATOR A.

We suppose $\phi_\lambda(x)$, $\psi_\lambda(x)$, $\chi_\lambda(x)$ and $\gamma_\lambda(x)$, where $\lambda \in C$ is not an eigenvalue of A, are the fundamental set of solutions of the fourth order differential equation of (1.1) with the initial conditions:

$$\phi_\lambda(a) = 0, \quad (P\phi'_\lambda)(a) = 0, \quad \phi''_\lambda(a) = 1, \quad (K\phi'''_\lambda)(a) = 0 \tag{4.1}$$

$$\psi_\lambda(a) = 0, \quad (P\psi'_\lambda)(a) = 0, \quad \psi''_\lambda(a) = 0, \quad (K\psi'''_\lambda)(a) = 1 \tag{4.2}$$

$$\chi_\lambda(b) = 0, \quad (P\chi'_\lambda)(b) = 1, \quad \chi''_\lambda(b) = 0, \quad (K\chi'''_\lambda)(b) = 1 \tag{4.3}$$

$$\gamma_\lambda(b) = 1, \quad (P\gamma'_\lambda)(b) = 1+\lambda, \quad \gamma''_\lambda(b) = 0, \quad (K\gamma'''_\lambda)(b) = 1 \tag{4.4}$$

Therefore the Wronskian is

$$W = \lim_{x \to b} [\, \chi_\lambda(x)(P\gamma'_\lambda)(x) - (P\chi'_\lambda)(x)\gamma_\lambda(x)] = -1 \neq 0$$

Thus the solutions $\phi_\lambda(x), \psi_\lambda(x), \chi_\lambda(x)$ and $\gamma_\lambda(x)$ are linearly independent of $\tau u = \lambda u$. Putting $x = b$, we obtain the Wronskian in the form:

$$W = \psi''_\lambda(b)\,[\lambda\phi_\lambda(b) - (P\phi'_\lambda)(b) + (K\phi'''_\lambda)(b)]$$

$$- \phi''_\lambda(b)\,[\lambda\psi_\lambda(b) - (P\psi'_\lambda)(b) + (K\psi'''_\lambda)(b)] \neq 0 \qquad (4.5)$$

Now, for $f = (f_1, f_2) \in H$, we define $\Phi = (\Phi_1, \Phi_2) \in D(A)$ as the unique solution of $(\lambda I - A)\Phi = f$.

Application of variation of parameter method yields the unique solution $\Phi \in D(A)$ of $(\lambda I - A)\Phi = f$, $f \in H$ with:

$$\begin{aligned} &(\lambda I - \tau)\,\Phi_1 = f_1 \\ &\lambda\Phi_1(b) - (P\Phi'_1)(b) + (K\Phi'''_1)(b) = f_2 \end{aligned} \qquad (4.6)$$

Therefore

$$\Phi_1(x) = \int_a^b \left[\frac{\phi_\lambda(x)\alpha_1(t) + \psi_\lambda(x)\alpha_2(t)}{W}\right] f_1(t)dt$$

$$+ \int_a^b \left[\frac{\chi_\lambda(x)\alpha_3(t) + \gamma_\lambda(x)\alpha_4(t)}{W}\right] f_1(t)dt$$

$$+ d_1\phi_\lambda(x) + d_2\psi_\lambda(x) + d_3\chi_\lambda(x) + d_4\gamma_\lambda(x), \qquad (4.7)$$

where

$$\alpha_1(t) = \frac{-P(t)}{K(t)} \begin{vmatrix} \psi_\lambda(t) & \chi_\lambda(t) & \gamma_\lambda(t) \\ \psi'_\lambda(t) & \chi'_\lambda(t) & \gamma'_\lambda(t) \\ \psi''_\lambda(t) & \chi''_\lambda(t) & \gamma''_\lambda(t) \end{vmatrix}$$

$$\alpha_2(t) = \frac{P(t)}{K(t)} \begin{vmatrix} \phi_\lambda(t) & \chi_\lambda(t) & \gamma_\lambda(t) \\ \phi'_\lambda(t) & \chi'_\lambda(t) & \gamma'_\lambda(t) \\ \phi''_\lambda(t) & \chi''_\lambda(t) & \gamma''_\lambda(t) \end{vmatrix}$$

$$\alpha_3(t) = \frac{-P(t)}{K(t)} \begin{vmatrix} \phi_\lambda(t) & \psi_\lambda(t) & \gamma_\lambda(t) \\ \phi'_\lambda(t) & \psi'_\lambda(t) & \gamma'_\lambda(t) \\ \phi''_\lambda(t) & \psi''_\lambda(t) & \gamma''_\lambda(t) \end{vmatrix}$$

and

$$\alpha_4(t) = \frac{P(t)}{K(t)} \begin{vmatrix} \phi_\lambda(t) & \psi_\lambda(t) & \chi_\lambda(t) \\ \phi'_\lambda(t) & \psi'_\lambda(t) & \chi'_\lambda(t) \\ \phi''_\lambda(t) & \psi''_\lambda(t) & \chi''_\lambda(t) \end{vmatrix}$$

while d_1, d_2, d_3 and d_4 are constants.

Calculation of $\Phi_1(b), \Phi'_1(b)$ and $\Phi'''_1(b)$ from (4.7) and substitution into (4.6) with the initial conditions (4.3) and (4.4), we can get the constants d_1, d_2, d_3 and d_4 as follows:

$$d_1 = \frac{1}{W}\ [- f_2\psi''_\lambda(b) + \int_a^b \alpha_1(t)f_1(t)dt],$$

$$d_2 = \frac{1}{W}\ [f_2\phi''_\lambda(b) + \int_a^b \alpha_2(t)f_1(t)dt]$$

and $d_3 = d_4 = 0$.

Consequently, we deduce that

$$\Phi_1(x) = \frac{f_2}{W}\ [\psi_\lambda(x)\phi''_\lambda(b) - \phi_\lambda(x)\psi''_\lambda(b)] + \int_a^b G(x,t,\lambda)f_1(t)dt \tag{4.8}$$

and

$$\Phi_2 = \Phi_1(b)$$

where $G(x,t,\lambda)$ is the Green's function defined by:

$$G(x,t,\lambda) = \left\{\begin{array}{ll} \dfrac{\phi_\lambda(x)\alpha_1(t) + \psi_\lambda(x)\alpha_2(t)}{W} & a < x < t < b \\ \\ \dfrac{\chi_\lambda(x)\alpha_3(t) + \gamma_\lambda(x)\alpha_4(t)}{W} & a < t < x < b \end{array}\right. \tag{4.9}$$

The form of equations (4.8) and (4.9) shows that the inverse operator $(\lambda I - A)^{-1}$ is actually compact; for details of argument of theorem 5 in [8, p.120] can be used.

5. EXPANSION THEOREM.

We now arrive at the problem of expanding an arbitrary function $f(x) \in H$ for $x \in [a,b]$ in terms of the eigenfunctions of (1.1). The results of our ivestigations are summarized in the following theorem:

THEOREM 5.1. The operator A in H has unbounded set of real eigenvalues of finite multiplicity, (they have at most multiplicity four), without accumulation points in $(-\infty, \infty)$ and they can be ordered according to the size, $0 < c < \lambda_1 < \lambda_2 < \ldots < \lambda_i$ with $\lambda_i \to \infty$ as $i \to \infty$. If the corresponding eigenfunctions $\Phi_1, \Phi_2, \Phi_3, \ldots$ form a complete orthonormal system, then for any function $f(x) \in H$, we have the expansion:

$$f(x) = \sum_{i=1}^{\infty} \langle f, \Phi_i \rangle \Phi_i \tag{4.10}$$

which is a uniformly convergent series.

The above theorem has some interesting corollaries for particular choices of f.

COROLLARY 4.1. If $f_1 \in L^2(a,b)$ and $f = (f_1, 0) \in H$, then we have

(i) $f_1 = \sum_{i=1}^{\infty} (\int_a^b f_1\Phi_{i1}dx)\Phi_{i1}(x)$

(ii) $0 = \sum_{i=1}^{\infty} (\int_a^b f_1\Phi_{i1}dx)\Phi_{i2}$

COROLLARY 4.2. If $\Phi_i = (\Phi_{i1}(x), \Phi_{i2}) \in D(A)$ and $f = (0,1) \in H$, we have:

(i) $$0 = \sum_{i=1}^{\infty} \Phi_{i2}\Phi_{i1}(x) = \sum_{i=1}^{\infty} \Phi_{i1}(b)\, \Phi_{i1}(x).$$

(ii) $$1 = \sum_{i=1}^{\infty} [\Phi_{i2}]^2 = \sum_{i=1}^{\infty} [\Phi_{i1}(b)]^2.$$

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