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ON THE NUMBER OF SOLUTIONS OF SOME INTEGRAL EQUATIONS ARISING IN RADIATIVE TRANSFER

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ABSTRACT. We discuss the number of solutions of some nonlinear integral equations arising in the theories of radiative transfer, neutron transport and in the kinetic theory of gases.

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1. INTRODUCTION. In the theories of radiative transfer [1], [2] and neutron transport [3], [4] an important role is played by nonlinear integral equations of the form

$$H(x) = 1 + xH(x)\int_0^1 \frac{\psi(t)H(t)}{x+t}\,dt. \tag{1.1}$$

The known function ψ is assumed to be nonegative, bounded, and measurable on $[0,1]$, and a positive, continuous solution H of (1.1) is sought.

Chandrasekhar's treatment of (1.1) can be found in [2]. The first proof however of the existence of a solution of (1.1) was given by M. Crum, who considered the equation in the complex plane [5]. Crum also showed that if $\int_0^1 \psi(t)dt < 1/2$, then (1.1) has at most two solutions which are bounded in $[0,1]$ and in case $\int_0^1 \psi(t)dt = 1/2$, there is only one such solution. C. Fox [6] solved simpler equations in order to prove existence of solutions of (1.1). But the solution of Fox's equation are not necessarily solutions of (1.1)[1]. C. Stuart [7] gave a nonconstructive existence proof for (1.1) using the Leray-Schauder degree theory but did not discuss the number or location of solutions. B. Cahlon and M. Eskin [3] used a theorem of Darbo for a set contraction map to prove a nonconstructive existence theorem for (1.1).

Finally, C. Kelley [8] had solved some interesting generalizations of (1.1) using the solutions of finite rank approximations of solutions of (1.1).

Here we consider the generalized equation:

$$H(x) = 1 + xH(x)\int_0^1 k(x,t)\psi(t)H(t)dt. \tag{1.2}$$

The known kernal function $k(x,t)$ is a measurable function on $[0,1] \times [0,1]$ satisfying

(a) $0 < k(x,t) < 1$ for all $x, t \in [0,1]$,

and

(b) $k(x,t) + k(t,x) = 1$ for all $x, t \in [0,1]$.

We show that whenever $\int_0^1 \psi(t)dt \leq 1/2$, a minimal solution H can be found using a specific iteration.

Finally, under the same assumption, we provide a way of constructing new nonminimal solutions H of (1.1) in terms of the minimal solution.

2. BASIC RESULTS.

We denote by $C[0,1]$ the Banach space of all real continuous functions on $[0,1]$ with the maximum norm

$$u = \max_{0 \leq t \leq 1} |u(t)|.$$

We now list the following well-known theorem whose proof can be found in [2, pp. 106-107].

THEOREM 1. If H is a solution of (1.2), then either

$$\int_0^1 \psi(t)H(t)dt = 1 - [1 - 2\int_0^1 \psi(t)dt]^{1/2} \tag{2.1}$$

or

$$\int_0^1 \psi(t)H(t)dt = 1 + [1 - 2\int_0^1 \psi(t)dt]^{1/2}. \tag{2.2}$$

A necessary condition that (1.2) has a solution is that

$$\int_0^1 \psi(t)dt \leq 1/2. \tag{2.3}$$

A function $H \in C[0,1]$ satisfies the equation

$$H(x)^{-1} = [1 - 2\int_0^1 \psi(t)dt]^{1/2} + \int_0^1 k(t,x)\psi(t)H(t)dt \tag{2.4}$$

if and only if H satisfies (1.2) and (2.1).

Chandrasekhar in [2], after proving that a solution H of (1.1) satisfies either (2.1) or (2.2), claims that, in fact, H must satisfy (2.1). This claim is not true because as we show, there always exists a solution H satisfying (2.1), but in many cases there exists a second solution $\tilde{H}$ satisfying (2.2) and not (2.1).

Let $\leq$ be the natural partial ordering on $C[0,1]$, that is, if $p_1, p_2 \in C[0,1]$, then $p_1 \leq p_2$ if $p_1(x) \leq p_2(x)$ for all $x \in [0,1]$ and define the following:

$$d = [1 - 2\int_0^1 \psi(t)dt]^{1/2},$$

$$D = \{p \in C[0,1]/p(x) \geq d,\ x \in [0,1]\},$$

the operator

$$R: D \to C[0,1] \text{ by}$$
$$R(p(x)) = 1 + p(x)\int_0^1 k(x,t)\psi(t)p(t)dt,\ p \in D$$

and for $d > 0$, define the operator $F : D \to C[0,1]$ by

$$F(p(x)) = d + \int_0^1 k(t,x)\psi(t)(p(t))^{-1}dt, \quad p \in D.$$

It is routine to verify that R is isotone, that is if $p_1 \le p_2$ then $R(p_1) \le R(p_2)$ and F is antitone, that is if $p_1 \le p_2$ then $F(p_2) \le F(p_1)$.

Finally, denote by 1 (respectively, d) the function with constant value 1 (respectively, d).

We can now prove the proposition:

PROPOSITION. Assume that the kernel function k(x,t) is as in the introduction and satisfies the condition

$$|k(x,t) - k(y,t)| \le b|x - y| \text{ for all } x, y, t \in [0,1]$$

and some $b > 0$. Then the sequence $R^n(1)$, n = 1, 2, ... is equicontinuous.

PROOF. Let H be a solution satisfying (1.2) and (2.1) and
$A = \{p \in C[0,1]/1 \le p \le H\}$.
Define $Q : A \to C[0,1]$ by

$$Q(p(x)) = \int_0^1 k(x,t)\psi(t)p(t)dt, \quad x \in [0,1], \; p \in A.$$

Let $\varepsilon > 0$, then there exists a, $0 < a < 1$, such that

$\int_0^a \psi(t)H(t)dt < \frac{\varepsilon}{4}$ and $\int_a^1 \psi(t)A(t)dt > 0$. Then for x, y $\in$ [0,1],

$$\begin{aligned} |Q(p(x)) - Q(p(y))| &= \left|\int_0^1 (k(x,t) - k(y,t))\psi(t)p(t)dt\right| \\ &\le \int_0^a (|k(x,t)| + |k(y,t)|)\psi(t)H(t)dt + \int_a^1 |k(x,t) - k(y,t)|\psi(t)H(t)dt \\ &\le 2\int_0^a \psi(t)H(t)dt + b\int_a^1 |x - y|\psi(t)\psi(t)H(t)dt \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

by choosing, $|x - y| < \delta = \frac{\varepsilon}{2b}(\int_a^1 \psi(t)H(t)dt)^{-1}$. That is the set Q(A) is equicontinuous.

Let $p \in A$ and $x \in [0,1]$, then

$$Q(p(x)) = \int_0^1 k(x,t)\,\psi(t)p(t)dt \le \int_0^1 k(x,t)\psi(t)H(t)dt$$

$$< \int_0^1 \psi(t)H(t)dt = 1 - d \le 1.$$

Therefore there exists c, $0 < c < 1$, such that $Q(p(x)) < c$ for all $p \in A$ and $x \in [0,1]$.

For any $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that for every $g \in Q(A)$, $|g(x) - g(y)| \le ||H||^{-1}(1 - c)\varepsilon_0$ if $|x - y| < \delta_0$ (since Q(A) is equicontinuous). The function R(1) is continuous and hence uniformly continuous, therefore there exists $0 < \delta_1 \le \delta_0$ such that

$$|R(1(x)) - R(1(y))| < \varepsilon_0 \text{ if } |x - y| < \delta_1 .$$

We shall show that the same δ_1 works for ε_0 and $R^{n+1}(1)$ if

$$|s^k(1(x)) - s^k(1(y))| < \varepsilon_0 \text{ if } |x - y| < \delta_1 \text{ for } k = 1, 2, \ldots n.$$

Set $p = R^n(1)$. Then if $|x - y| < \delta_1$,

$$
\begin{aligned}
|R(p(x)) - R(p(y))| &= |p(x)Q(p(x)) - p(y)Q(p(y))| \\
&\leq |p(x)Q(p(x)) - p(x)Q(p(y))| + |p(x)Q(p(y)) - p(y)Q(p(y)))| \\
&\leq p(x)|Q(p(x)) - Q(p(y))| + Q(p(y))|p(x) - p(y)| \\
&\leq ||H||\,||H||^{-1}(1 - c)\,\varepsilon_0 + c\,\varepsilon_0 = \varepsilon_0,
\end{aligned}
$$

that is,

$$|R^{n+1}(1(x)) - R^{n+1}(1(y))| < \varepsilon_0 \quad \text{if } |x - y| < \delta_1$$

which completes the induction and the proof of the proposition.

THEOREM. 2. Assume that the kernel function $k(x,t)$ is as in the proposition. Then the following are true:

(a) equation (1.2) has exactly one solution H satisfying (2.1) if and only if (2.3) holds. Moreover, the increasing sequence $R^n(1)$, $n = 0, 1, 2, \ldots$ converges to H; and

(b) if inequality holds in (2.3), the sequence $F^n(d)$, $n = 0, 1, 2, \ldots$ converges to H^{-1} and

$$|H^{-1}(x) - F^n(d(x))| \leq |F^n(d(x)) - F^{n+1}(d(x))|, \quad x \in [0,1]. \tag{2.5}$$

PROOF. (A). If (1.2) has a solution H, then by Theorem 1, $\int_0^1 \psi(t)dt \leq 1/2$.

CASE 1. Assume $\int_0^1 \psi(t)dt < 1/2$. It can easily be verified that since F is antitone:

$$d \leq F^2(d) \leq F^4(d) \leq F^6(d) \leq \ldots \leq F^7(d) \leq F^5(d) \leq F^3(d) \leq F(d).$$

Working as in the proposition we can easily show that the bounded set

$$N = \{F(p)/d \leq p \leq F(d)\}$$

is equicontinuous. Then the sequences $F^{2n}(d)$, $n = 1, 2, \ldots$ and $F^{2n+1}(d)$, $n = 0, 1, 2, \ldots$ have convergent subsequences converging to the functions v and w respectively. From the monotonicity of the above sequences and the continuity of F we obtain

$$F^{2n}(d) \to v,$$

$$F^{2n+1}(d) \to w,$$

$$d \leq v \leq w,$$

$$F(v) = w$$

and

$$F(w) = v.$$

The function v has minimum value greater than zero, so that there exists a largest number q, $0 < q \leq 1$, with $qw \leq v$. If $q = 1$, then $w \leq v \leq w$, that is $v = w$.

If $q < 1$, define on the domain of F the operator F_1 by

$$F_1(p) = F(p) - d.$$

Then

$$v = d + F_1(w) \geq d + F_1(q^{-1}v) = d + qF_1(v)$$

$$= (1 - q)d + q(d + F_1(v)) = (1 - q)d + qw$$

$$\leq ew + qw = (e + q)w,$$

for some $e > 0$. But this contradicts the maximality of q. Therefore,

$$F(v) = v = w,$$
$$H \equiv v^{-1}$$

is a solution of (1.2), satisfying (2.1), and the sequence $F^n(d)$, $n = 0, 1, 2, \ldots$ converges to H^{-1}. Inequality (2.5) follows from the fact that $F^{2k}(d) \leq H^{-1} \leq F^{2k+1}(d)$, for $k = 1, 2, 3, \ldots$.

CASE 2. Assume that $\int_0^1 \psi(t)dt = 1/2$. Let $\{c_n\}$, $n = 1, 2, \ldots$ be a strictly increasing sequence of positive numbers converging to 1, and consider the functions $c_n\psi$, $n = 1, 2, 3, \ldots$.

Since $\int_0^1 c_n\psi(t)dt = 1/2\, c_n < 1/2$, it follows from Case 1 that the equation

$$H(x) = 1 + H(x)\int_0^1 k(x,t)c_n\psi(t)H(t)dt$$

has a solution H_n for $n = 1, 2, 3, \ldots$. Then for each $x \in [0,1]$ $h_n(x) \geq 1$ and

$$(H_n(x))^{-1} = [1 - 2\int_0^1 c_n\psi(t)dt]^{1/2} + \int_0^1 k(t,x)\ c_n\psi(t)H_n(t)dt$$

$$\geq c_n\int_0^1 k(t,x)\psi(t)dt \geq c_1\int_0^1 k(t,x)\psi(t)dt.$$

Therefore, there exists $r > 0$ such that $(H_n(x))^{-1} \geq r$ for each $x \in [0,1]$ and each $n = 1, 2, 3, \ldots$.

Set $M = \{p \in C[0,1)/r \leq p(x) \leq 1,\ x \in [0,1]\}$. Then $H_n^{-1} \in M$, $n = 1, 2, \ldots$. Define $F\colon M \to C[0,1]$ by

$$F(p(x)) = \int_0^1 k(t,x)\Psi(t)(p(t))^{-1}dt,\ p \in M.$$

It is easy to verify that the set $F(M)$ is bounded and equicontinuous.

Also, for each n,

$$H_n^{-1}(x) = [1 - 2\int_0^1 c_n\psi(t)dt]^{1/2} + \int_0^1 k(t,x)\ c_n\ \psi(t)H_n(t)dt$$

$$= [1 - 2\int_0^1 c_n\psi(t)dt]^{1/2} + c_nF(H_n^{-1})(x).$$

Since $F(H_n^{-1}) \in F(M)$ for each n, some subsequence $F(H_{n_j}^{-1})$, $j = 1, 2, \ldots$, of $F(H_n^{-1})$, $n = 1, 2, \ldots$ converges in $C[0,1]$ to some point H_0^{-1}, so that $H_{n_j}^{-1} \to H_0^{-1}$. Then the sequence $F(H_{n_j}^{-1})$, $j = 1, 2, \ldots$ converges to $F(H_0^{-1})$ and H_0^{-1}; that is,

$$F(H_0^{-1}) = H_0^{-1} .$$

Then H_0 satisfies (1.2), (2.1) and (2.2).

Therefore there exists a positive function H satisfying (1.2) and (2.1) whenever ψ satisfies (2.3).

(B). Assume (2.3) holds, and suppose H satisfies (1.2) and (2.1). Since $1 \leq H$ and $1 \leq R(1)$, it follows from the fact that R is isotone that

$$1 \leq R(1) \leq R^2(1) \leq R^3(1) \leq \ldots \leq H.$$

Since the sequences $R^n(1)$, n = 1, 2, ... is uniformly bounded and equicontinuous there is a convergent subsequence, say $R^{n_k} \to h \leq H$, and, since the sequence $R^n(1)$, n = 0, 1, 2, ... is nondecreasing, the entire sequence converges to h. It follows from the continuity of R that R(h) = h. Now h must satisfy either (2.1) or (2.2), and since $0 \leq h \leq H$, h must satisfy (2.1). Therefore, for $x \in [0,1]$,

$$h^{-1}(x) = [1 - 2\int_0^1 \psi(t)dt]^{1/2} + \int_0^1 k(t,x)\psi(t)h(t)dt$$

$$\leq [1 - 2\int_0^1 \psi(t)dt]^{1/2} + \int_0^1 k(t,x)\psi(t)H(t)dt = H^{-1}(x),$$

that is, $h^{-1} \leq H^{-1}$. Together with the inequality $h \leq H$, this implies h = H.

We have proved that H is the only function satisfying both (1.2) and (2.1), and that the increasing sequence $R^n(1)$, n = 0, 1, 2, ... converges to H which completes the proof of the theorem.

COROLLARY. Suppose that ψ_1 and ψ_2 are nonnegative, bounded, measurable functions on [0,1] such that $\psi_1(t) \leq \psi_2(t)$ almost everywhere in [0,1] and such that $\int_0^1 \psi(t)dt \leq 1/2$, i = 1, 2. Let H_i be the unique solution of equations (1.2) and (2.1) corresponding to $\psi = \psi_i$, i = 1, 2.

Then,

$$H_1 \leq H_2 .$$

PROOF. Define $R_i : C[0,1] \to C[0,1]$, i = 1, 2, by

$$R_i(p(x)) = 1 + p(x)\int_0^1 k(x,t)\psi_i(t)p(t)dt, \quad p \in C[0,1].$$

If p_1 and p_2 are nonnegative functions in C[0,1] with $p_1 \leq p_2$, then

$R_1(p_1) \leq R_2(p_2)$. Hence $R_1(1) \leq R_2(1)$, $R_1^2(1) \leq R_2^2(1)$, and in general,

$R_1^n(1) \leq R_2^n(1)$. Since the increasing sequence $R_i^n(1)$, converges to H_i, i = 1, 2, it follows that $H_1 \leq H_2$.

Note that if $\int_0^1 \psi(t)dt = 1/2$, it follows from the previous results that the function H satisfying (1.2) and (2.1) is the unique solution of (1.2), since, in this case (2.1) and (2.2) reduce to the same equation. However, if $\int_0^1 \psi(t)dt < 1/2$, equation (1.2) may have two distinct solutions.

THEOREM 3. Assume:

(a) $\int_0^1 \Psi(t)dt < 1/2$ and H is the unique solution of (1.2) and (2.1);

(b) the following estimate is true

$$\int_0^1 \frac{\psi(t)}{1-t} H(t)dt > 1 \tag{2.6}$$

and

(c) there exists functions $\varphi_1, \varphi_2, \varphi_3 \in C[0,1]$ such that

$$k(x,t)[\varphi_2(x)(1 - kt) - (1 + \varphi_1(t))] + \varphi_3(x) = 0, \text{ for all } x, t \in [0,1] \tag{2.7}$$

$$\varphi_1(x) + kx > 0, \text{ for all } x \in (0,1], \ \varphi_1(0) = 0 \tag{2.8}$$

and

$$(1 + \varphi_1(x))[\varphi_2(x)(H(x) - 1) + \varphi_3(x)H(x)] =$$

$$(H_1(x) - 1)(1 - kx) \text{ for all } x \in [0,1] \tag{2.9}$$

where k is the unique number in (0,1) for which

$$\int_0^1 \frac{\psi(t)}{1-kt} H(t)dt = 1 \tag{2.10}$$

and the function H_1 is given by

$$H_1(x) = \frac{1+ \varphi_1(x)}{1-kx} H(x), \ x \in [0,1]. \tag{2.11}$$

Then H_1 is a solution of (1.2) and (2.2) and

$$H_1(x) > H(x), \ x \in (0,1], \ H_1(0) = H(0).$$

PROOF. By the monotone convergence theorem

$$\lim_{k\to 1^-} \int_0^1 \frac{\psi(t)}{1-kt} H(t)dt = \int_0^1 \frac{\psi(t)}{1-t} H(t)dt$$

since $(1 - kt)^{-1}$ increases monotonically with k, $0 < k < 1$,
If (2.6) holds, since

$$\int_0^1 \frac{\psi(t)}{1-0\cdot t} H(t)dt = 1 - [1 - 2\int_0^1 \psi(t)dt]^{1/2} < 1,$$

and since the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(k) = \int_0^1 \frac{\psi(t)}{1-kt} H(t)dt$$

is strictly increasing, there exists a unique $k \in (0,1)$, for which (2.10) holds. Let H_1 be defined as in (2.11). Applying a trick used in [5], [9], we find that for each $x \in [0,1]$

$$\int_0^1 k(x,t)\psi(t)H_1(t)dt = \varphi_2(x)\int_0^1 k(x,t)\psi(t)H(t)dt + \varphi_3(x)\int_0^1 \frac{H(t)}{1-kt} \psi(t)dt$$

$$= \varphi_2(x)\int_0^1 k(x,t)\psi(t)H(t)dt + \varphi_3(x)$$

$$= \varphi_2(x)[1 - \frac{1}{H(x)}] + \varphi_3(x) = 1 - \frac{1}{H_1(x)},$$

(by (2.7) and (2.9)) that is, H_1 satisfies (1.2). Since H_1 must satisfy either (2.1) or (2.2) and since $H_1(x) > H(x)$, $x \in [0,1]$ (by (2.8)), H_1 satisfies (2.2) and the proof of the theorem is completed.

REMARK. By choosing the kernel function k(x,t) to be

$$k(x,t) = \frac{x}{x+t},\ x,\ t \in [0,1]$$

we observe that the conditions (a) and (b) in the introduction are satisfied and that the equaion (1.2) reduces to equation (1.1).

Moreover the conditions (2.7), (2.8), and (2.9) can then be satisfied if we choose

$$\varphi_1(x) = kx,$$
$$\varphi_2(x) = \frac{1-kx}{1+kx},$$

and

$$\varphi_3(x) = \frac{2kx}{1+kx}.$$

REFERENCES

1. BUSBRIDGE, L.W. The Mathematics of Radiative Transfer. Cambridge Publ. Cambridge, England, 1960.
2. CHANDRASEKHAR, S. Radiative Transfer, Oxford Univ. Press, London, 1950.
3. CAHLON, B. AND ESKIN, M. Existence Theorems for an Integral Equation of the Chandrasekhar H-equation with Perturbation. J. Math. Anal. and Applic. 83(1981), 159-171.
4. ARGYROS, I.K. Quadratic Equations and Applications to Chandrasekhar's and Related Equations. Bull. Austral. math. Sco., Vol. 32, 2(1985), 275-292.
5. CRUM, M. On an Integral Equation of Chandrasekhar. Quart. J. Math., 18(1947), 244-252.
6. FOX, C. A Solution of Chandrasekhar's Integral Equation. Trans. Amer. Math. Soc., 99(1961), 285-291.
7. STUART, C.A. Existence Theorems for a Class of Nonlinear Integral Equations. Math. Z. 137(1974), 49-66.
8. KELLEY, C.T. Approximation of Solutions of Some Quadratic Integral Equations in Transport Theory. J. Integr. Eq., 4(1982), 221-237.
9. LEGGETT, R.W. A New Approach to the H-equation of Chandrasekhar. S.I.A.M. J. Math., 7, 4(1976), 542-550.
10. CASE, K.M. & ZWEIFEL, P.F. Linear Transport Theory, Addison-Wesley Publ., MA, 1967.
11. KANTOROVICH, L.V. AND AKILOV, G.P. Functional Analysis in Normed Spaces. Pergamon Press, London, 1964.
12. KURATOWSKI, C. Sur les Espaces Complets. Fund. Math., 15(1930), 301-309.