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CONFORMAL VECTOR FIELDS IN SYMMETRIC AND CONFORMAL SYMMETRIC SPACES

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ABSTRACT. Consequences of the existence of conformal vector fields in (locally) symmetric and conformal symmetric spaces, have been obtained. An attempt has been made for a physical interpretation of the consequences in the framework of general relativity.

KEY WORDS AND PHRASES: Symmetric spaces, Conformal symmetric spaces, Conformal vector field, Null Killing vector field, Killing horizon.
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1. INTRODUCTION.

Let M denote a semi-Riemannian manifold with metric tensor g_{ab} of arbitrary signature. All the geometric objects defined on M are assumed sufficiently smooth. Although our treatment is local, nevertheless we shall drop the term 'locally', for example in 'locally symmetric'. We denote the Christoffel symbols by Γ^a_{bc} and the covariant differentiation by a semi-colon ; . We say that M is symmetric in Cartan's sense if the Riemann curvature tensor R^a_{bcd} is covariant constant, i.e. $R^a_{bcd;e} = 0$. We say that M is conformal symmetric [1] if its Weyl conformal curvature tensor $C^a{}_{bcd}$ is covariant constant, i.e. $C^a_{bcd;e} = 0$. Thus a symmetric space is conformal symmetric but the converse is not necessarily true.

A vector field ξ on M is said to be conformal if

$$L_\xi g_{ab} = 2\sigma g_{ab} \tag{1.1}$$

where L_ξ denotes the Lie-derivative operator via ξ and σ denotes a scalar function on M. In particular, if $\sigma = 0$, ξ is called a Killing vector field and if σ is a non-zero constant, ξ is called a homothetic vector field. It is known that a conformal vector field ξ satisfies:

$$L_\xi C^a{}_{bcd} = 0 \tag{1.2}$$

Equation (1.1) implies

$$L_\xi \Gamma^c_{ab} = \delta^c_a \sigma_{;b} + \delta^c_b \sigma_{;a} - g_{ab} g^{cd} \sigma_{;d} \tag{1.3}$$

but the converse is not necessarily true. However, we know [2] that (1.3) is equivalent to

$$L_\xi g_{ab} = 2\sigma g_{ab} + h_{ab} \tag{1.4}$$

where h_{ab} is a covariant constant tensor field. A vector field ξ satisfying (1.3) or (1.4) is said to be affine conformal [2] and is said to generate a one-parameter group of conformal collineations [3,4]. An affine conformal vector field with constant σ (i.e. $L_\xi \Gamma^a_{bc} = 0$) is known as an affine Killing vector field (which preserves the geodesics). For (i) a compact orientable positive definite Riemannian manifold without boundary, (ii) an irreducible positive definite Riemannian manifold and (iii) an $n(n > 2)$ - dimensional non-flat space-form; an affine conformal vector field reduces to conformal vector field. For a non-Einstein conformally flat space of dimension > 2; Levine and Katzin [5] proved that h_{ab} is a linear combination of g_{ab} and the Ricci tensor R_{ab}.

Conformal motion (generated by a conformal vector field) is a natural symmetry of the space-time manifolds in general relativity, inherited by its causality-preserving [6] character. But sometimes, it is desirable to consider conformal motions which provide covariant conservation law generators. It was pointed out by Katzin et al [7] that there is a fundamental symmetry called curvature collineation (CC) defined by a vector field ξ satisfying

$$L_\xi R^a{}_{bcd} = 0 \tag{1.5}$$

Komar's identities [8] (which define a conservation law generator) follow naturally by the existence of CC. A conformal vector which also generate CC, is called a special conformal vector. A conformal vector is special conformal vector iff

$$\sigma_{;ab} = 0 \tag{1.6}$$

The purpose of this paper is to study the consequences of the existence of (i) a special conformal vector field in a symmetric space and (ii) a conformal vector field in a conformal symmetric space; and indicate the physical interpretation of the consequences within the framework of general relativity.

2. SYMMETRIC AND CONFORMAL SYMMETRIC SPACES.

Here we prove two theorems as follows:

THEOREM 1. *Let a non-flat symmetric space* M *of dimension* $n > 4$, *admit a special conformal vector field* ξ. *Then either* (i) M *has zero scalar curvature and grad* σ *is a null Killing vector field, or* (ii) ξ *reduces to a homothetic vector field.*

(Note that the above theorem is valid also for an affine conformal vector field, in which case the alternative conclusion (ii) would be: ξ reduces to an affine Killing vector field. The proof is common).

PROOF. We have the following identity [9]:

$$L_\xi(R^b{}_{cde;a}) - (L_\xi R^b{}_{cde})_{;a} = (L_\xi \Gamma^b_{af})R^f{}_{cde} - (L_\xi \Gamma^f_{ac})R^b{}_{fde} - (L_\xi \Gamma^f_{ad})R^b{}_{cfe} - (L_\xi \Gamma^f_{ae})R^b{}_{cdf}$$

By our hypothesis the left hand side vanishes and consequently, in view of (1.3) the above equation assumes the form:

$$\begin{aligned}\delta^b_a \sigma_f R^f{}_{cde} + \sigma_a R^b{}_{cde} - \sigma^b R_{acde} &= \sigma_c R^b{}_{ade} \\ &+ \sigma_a R^b{}_{cde} - g_{ac} R^b{}_{fde} \sigma^f + \sigma_d R^b{}_{cae} + \sigma_a R^b{}_{cde} \\ &- g_{ad} R^b{}_{cfe} \sigma^f + \sigma_e R^b{}_{cda} + \sigma_a R^b{}_{cde} - g_{ae} R^b{}_{cdf} \sigma^f\end{aligned} \tag{2.1}$$

where σ_a stands for $\sigma_{;a}$. Taking the product of both sides with σ^a yields

$$(\sigma_a \sigma^a) R^b{}_{cde} = 0$$

As per our hypothesis, M is not flat and therefore the above equation shows

$$\sigma_a \sigma^a = 0$$

which implies that either (i) grad σ is null (a non-zero vector of zero norm), or (ii) σ is constant. We first take up case (i). Successive contractions of (2.1) lead to

$$(n - 4)\sigma_f R^f{}_e = R \sigma_e \tag{2.2}$$

Two subcases arise: If $n = 4$, then (2.2) gives $R = 0$. If $n > 4$, then using the condition (1.6) obtains $R^f{}_e \sigma_f = 0$. This, substituted in (2.2), shows that $R = 0$. Thus, in case (i) grad σ is null and Killing (in virtue of (1.6)) and the scalar curvature R vanishes identically. In case (ii) ξ is homothetic or affine Killing according as ξ is conformal or affine conformal. This proves the theorem.

THEOREM 2. *Let a conformal symmetric space* M (dim M > 3) *admit a conformal vector field* ξ. *Then one of the following holds:*

(i) M *is conformally flat*

(ii) grad σ *is a null vector*

(iii) ξ *reduces to a homothetic vector field.*

In particular, if ξ *were non-homothetic special conformal vector field and* M *were not conformally flat, then* grad σ *would have been null and Killing too.*

PROOF. Consider the identity [9]:

$$\begin{aligned}L_\xi(C^b{}_{cde;a}) - (L_\xi C^b{}_{cde})_{;a} &= (L_\xi \Gamma^b_{af})C^f{}_{cde} - (L_\xi \Gamma^f_{ac})C^b{}_{fde} \\ &- (L_\xi \Gamma^f_{ad})C^b{}_{cfe} - (L_\xi \Gamma^f_{ae})C^b{}_{cdf}\end{aligned}$$

Observe that the left hand side vanishes because M is conformal symmetric and (1.2) holds for a conformal vector field. By use of (1.3) in the above equation and contracting at a and b we obtain (noting dim M > 3)

$$\sigma^a \sigma_a C^b{}_{cde} = 0$$

Therefore we conclude that either (i) $C^b{}_{cde} = 0$ meaning M is conformally flat, or $\sigma^a \sigma_a = 0$ so that (ii) σ^a is a null vector or (iii) σ is constant. Thus we have proved that one of the following is true: (i) M is conformally flat, (ii) grad σ is a null vector, (iii) ξ is homothetic. In particular, if ξ were non-homothetic special conformal vector field and M not conformally flat then, of course, (ii) holds. Moreover, in this case grad σ would be Killing in virtue of the condition (1.6) for special conformal vector field. This completes the proof.

REMARK 1. The conclusion (i) of Theorem 2 can be highlighted by saying that, if a conformal symmetric space M admits a one-parameter group of conformal motions (such that grad σ is neither null nor zero) then M is conformally flat. This can be compared with the standard result:[9] If an n-dimensional semi-Riemannian manifold M admits a maximal, i.e. $\frac{1}{2}(n+1)(n+2)$ - parameter group of conformal motions, then M is conformally flat.

REMARK 2. The conclusion (i) of Theorem 1 can be interpreted in the context of general relativity as follows. Let M be the space-time manifold of general relativity and satisfy the hypothesis of Theorem 1. M with zero scalar curvature, is a space-time carrying pure radiation [10] (e.g. massless scalar fields, neutrino fields or high frequency gravitational waves) and Einstein-Maxwell field. M with the gradient of conformal scalar field as a null Killing field, has a Killing horizon [11] defined by the null hypersurfaces of transitivity, σ = constant.

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