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MIXED FOLIATE CR-SUBMANIFOLDS IN A COMPLEX HYPERBOLIC SPACE ARE NON-PROPER

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ABSTRACT. It was conjectured in [1 II] (also in [2]) that mixed foliate CR-submanifolds in a complex hyperbolic space are either complex submanifolds or totally real submanifolds. In this paper we give an affirmative solution to this conjecture.

KEY WORDS AND PHRASES: CR-submanifolds, complex hyperbolic space, mixed foliate.
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1. INTRODUCTION.

A submanifold M of a Kaehler manifold $\tilde{M}$ is called a *CR-submanifold* if (1) the maximal complex subspace $\mathcal{D}_x$ of the tangent space $T_x\tilde{M}$ containing in T_xM, $x \in M$, defines a differentiable distribution $\mathcal{D}$, called the holomorphic distribution, and (2) the orthogonal complementary distribution $\mathcal{D}^\perp$ of $\mathcal{D}$ in TM is a totally real distribution, i.e., $J\mathcal{D}_x^\perp \subset T_x^\perp M$, where J denotes the almost complex structure of $\tilde{M}$ and $T_x^\perp M$ the normal space of M at x. Complex submanifolds and totally real submanifolds of $\tilde{M}$ are trivial examples of CR-submanifolds. A CR-submanifold is called *proper* if it is neither a complex submanifold nor a totally real submanifold. The totally real distribution $\mathcal{D}^\perp$ of a CR-submanifold of a Kaehler manifold is always integrable [1,3]. A CR-submanifold M is called *mixed foliate* if (a) the holomorphic distribution $\mathcal{D}$ is integrable, and (b) the second fundamental form σ^0 of M in $\tilde{M}$ satisfies $\sigma^0(\mathcal{D},\mathcal{D}^\perp) = \{0\}$.

It is known that mixed foliate CR-submanifolds in $\mathbb{C}^m$ are exactly CR-products in $\mathbb{C}^m$ [1 I] and mixed foliate CR-submanifolds in $\mathbb{C}P^m$ are non-proper [4]. It was conjectured in [1 II] (also in [2]) that mixed foliate CR-submanifolds in a complex hyperbolic space H^m are non-proper too.

In this paper, we solve this conjecture completely to give the following

THEOREM 1. *Let M be a mixed foliate CR-submanifold of* H^m. *Then M is either a complex submanifold or a totally real submanifold.*

2. PRELIMINARIES.

For simplicity, we assume that H^m is the (complex) m-dimensional complex hyperbolic space with constant holomorphic sectional curvature −4. Let M be a mixed foliate CR-submanifold of H^m. Then, by definition, the holomorphic distribution $\mathcal{D}$ of M is integrable and the second fundamental form σ^o of M in H^m satisfies $\sigma^o(\mathcal{D},\mathcal{D}^\perp) = \{0\}$. We denote by $\langle\ ,\ \rangle$ the metric tensor of H^m as well as the induced one on M. Let D^o and A^o denote the normal connection and the Weingarten map of M in H^m, respectively. If N is a leaf of $\mathcal{D}$, then N is a complex submanifold of H^m. Denote by σ, D, A and ∇ the second fundamental form, the normal connection, the Weingarten map and the Levi-Civita connection of N (in H^m), respectively, and by σ', D', A' the corresponding quantities for N in M. Then we have $\sigma(X,Y) = \sigma'(X,Y) + \sigma^o(X,Y)$ for X,Y tangent to N. Since $\sigma^o(\mathcal{D},\mathcal{D}^\perp) = \{0\}$, we also have $A_{JZ} = A^o_{JZ}$, on TN, for Z in $\mathcal{D}^\perp$. Since N is a complex submanifold of H^m, the almost complex structure J satisfies $\sigma(JX,Y) = J\sigma(X,Y) = \sigma(X,JY)$, $A_{J\xi} = JA_\xi$, $JA_\xi = -A_\xi J$, for X,Y tangent to N and ξ normal to N.

For any vector X tangent to M, we put $JX = PX + FX$ where PX and FX are the tangential and the normal components of JX, respectively. For a vector ξ normal to M, we put $J\xi = t\xi + f\xi$, where $t\xi$ and $f\xi$ are the tangential and the normal components of $J\xi$, respectively.

Since H^m is of constant holomorphic sectional curvature −4, the curvature tensor $\tilde{R}$ of H^m is given by

$$\tilde{R}(X,Y)Z = \langle X,Z\rangle Y - \langle Y,Z\rangle X + \langle JX,Z\rangle JY - \langle JY,Z\rangle JX - 2\langle X,JY\rangle JZ \tag{2.1}$$

for X, Y, Z tangent to H^m.

We need the following result of [1 II] for later use.

LEMMA 1. *Let M be a mixed foliate CR-submanifold of* H^m. *Then* (a) $D_X JZ = D^o_X JZ = F\nabla^o_X Z$, (b) $D_X Z = D'_X Z = -tD^o_X JZ$, (c) $\operatorname{Im}\sigma = \mathcal{D}^\perp \oplus J\mathcal{D}^\perp$, (d) $A_Z, A_{JZ} \in O(2h)$, and (e) $A_Z A_W + A_W A_Z = 0$, for X tangent to N and *orthonormal vectors* Z *and* W *in* $\mathcal{D}^\perp$.

LEMMA 2. *Under the hypothesis of Lemma 1, if M is proper, then* (a) *each leaf* N *of* $\mathcal{D}$ *lies in a complex (h+p)-dimensional totally geodesic complex submanifold* H^{h+p} *of* H^m *and* (b) $h+1 \geq p \geq 2$ *and* $h \geq 2$ *where* $p = \dim_{\mathbb{R}} \mathcal{D}^\perp$ *and* $h = \dim_{\mathbb{C}} \mathcal{D}$.

3. MORE LEMMAS.

Let M be a mixed foliate CR-submanifold of H^m. If M is non-proper, there is nothing to prove. Thus we may assume that M is proper. By Lemma 2, $p \geq 2$. From Lemma 1, we have

$$A_Z A_W + A_W A_Z = 0 \tag{3.1}$$

for orthonormal vectors Z,W in $\mathcal{D}^\perp$. Let $Z_1,\ldots,Z_p$ be an orthonormal frame of $\mathcal{D}^\perp$. We put

$$A_\alpha = A_{Z_\alpha}, \quad A_{\alpha^*} = A_{JZ_\alpha}, \qquad \alpha = 1,\dots,p . \tag{3.2}$$

From property (d) of Lemma 1, each A_{α^*} has eigenvalues 1 and -1 with the same multiplicity h. Let $X_1,\dots,X_h$ be h orthonormal eigenvectors of A_{α^*} with eigenvalue 1. Then $JX_1,\dots,JX_p$ are eigenvectors of A_{α^*} with eigenvalue -1. With respect to the basis $\{X_1,\dots,X_h, JX_1,\dots,JX_h\}$, we have

$$A_{\alpha^*} = \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -I_h \\ I_h & 0 \end{pmatrix}, \tag{3.3}$$

where I_h denotes the $h \times h$ identity matrix. Thus, by (2.1), we have

$$A_\alpha = \begin{pmatrix} 0 & -I_h \\ -I_h & 0 \end{pmatrix}. \tag{3.4}$$

In particular, if we choose $\alpha = 1$, we obtain

$$A_1 = \begin{pmatrix} 0 & -I_h \\ -I_h & 0 \end{pmatrix}, \qquad A_{1^*} = \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix}. \tag{3.5}$$

From (2.1) and (3.1) we have

$$A_\alpha A_{\beta^*} - A_{\beta^*}A_\alpha = 0, \qquad \alpha \neq \beta, \qquad \alpha,\beta = 1,\dots,p. \tag{3.6}$$

Using (3.1), (3.5) and (3.6) we may get

$$A_2 = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, \qquad A_{2^*} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}. \tag{3.7}$$

Since $A_2 \in O(2h)$ (Lemma 1), we also have

$$B \in O(h), \qquad {}^tB = B, \tag{3.8}$$

where ${}^tB$ denotes the transpose of B.

LEMMA 3. *If M is a proper mixed foliate CR-submanifold of* H^m, *then* $p \geq 3$.

PROOF. Under the hypothesis, Lemma 2 shows that if $p < 3$, then $p = 2$. If $p = 2$, we may choose an orthonormal frame $X_1,\dots,X_h, JX_1,\dots,JX_h, Z_1, Z_2, JZ_1, JZ_2$ such that, with respect to this frame, A_1, A_2, A_{1^*} and A_{2^*} take the forms of (3.5), (3.7) and (3.8).

We put

$$V = \mathrm{Span}\{X_1,\dots,X_h\}. \tag{3.9}$$

Then $TN = V \oplus JV$. Since $B \in O(h)$ with ${}^tB = B$, we may further choose $\{X_1,\dots,X_h\}$ such that with respect to it, B has the form:

$$B = \begin{pmatrix} I_r & 0 \\ 0 & -I_{h-r} \end{pmatrix} \tag{3.10}$$

for some r, $0 \leq r \leq h$.

CASE 1: $r = h$. In this case we have

$$-A_1 = A_{2*} = \begin{pmatrix} 0 & I_h \\ I_h & 0 \end{pmatrix}, \quad A_{1*} = A_2 = \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix}. \tag{3.11}$$

So, if we put

$$W = \frac{1}{\sqrt{2}}(Z_1 + JZ_2), \tag{3.12}$$

then $A_W = A_{JW} = 0$, which contradicts statement (c) of Lemma 1.

CASE 2: $r = 0$. This case is impossible by applying an argument similar to Case 1.

CASE 3: $r > 0$ and $h > r$. In this case we can decompose V and JV into orthogonal decompositions:

$$V = V' \oplus V'', \qquad JV = JV' \oplus JV'', \tag{3.13}$$

where V' and V'' are eigenspaces of B (defined by (3.10)) with eigenvalues 1 and -1, respectively. By (3.5), (3.7), (3.10) and Lemma 1 we have

$$\begin{aligned} \sigma(X,T) &= \langle JX,T\rangle(JZ_2-Z_1) + \langle X,T\rangle(JZ_1+Z_2), \\ \sigma(Y,T) &= -\langle JY,T\rangle(JZ_2+Z_1) + \langle Y,T\rangle(JZ_1-Z_2) \end{aligned} \tag{3.14}$$

for $X \in V'$, $Y \in V''$ and $T \in TN$.

By Lemma 1 we have

$$DZ_1 = \lambda Z_2, \quad DZ_2 = -\lambda Z_1, \quad DJZ_1 = \lambda JZ_2, \quad DJZ_2 = -\lambda JZ_1, \tag{3.15}$$

for some 1-form λ on N. Since N is a complex submanifold of H^m, the equation of Codazzi gives

$$(\bar\nabla_X\sigma)(Y,Z) = (\bar\nabla_Y\sigma)(X,Z) \tag{3.16}$$

where $(\bar\nabla_X\sigma)(Y,Z) = D_X\sigma(Y,Z) - \sigma(\nabla_XY,Z) - \sigma(Y,\nabla_XZ)$ for X,Y,Z tangent to N.

In particular, if $X \in V'$, $Y \in V''$ and $W \in JV'$, then by applying (3.14), (3.15) and (3.16), we see that the Z_2-components of both sides of (3.16) yield

$$0 = \lambda(Y)\langle JX,W\rangle - \langle W,\nabla_YX\rangle + \langle X,\nabla_YW\rangle. \tag{3.17}$$

Because $\langle X,W\rangle = 0$, (3.17) implies

$$2\langle\nabla_YX,W\rangle = \lambda(Y)\langle JX,W\rangle. \tag{3.18}$$

Similarly, if $X \in V'$, $Y \in V''$ and $W \in JV'$, the JZ_1-components yield

$$2\langle\nabla_YX,W\rangle - \lambda(Y)\langle JX,W\rangle = 2\langle\nabla_XY,W\rangle. \tag{3.19}$$

Combining (3.18) and (3.19) we find

$$\langle\nabla_XY,W\rangle = 0 \tag{3.20}$$

which also implies $\langle\nabla_XW,Y\rangle = 0$. Therefore

$$\nabla_{V'}V'' \perp JV', \qquad \nabla_{V'}JV' \perp V''. \tag{3.21}$$

Since J is parallel, this also gives

$$\nabla_{V'}JV'' \perp V', \qquad \nabla_{V'}V' \perp JV''. \tag{3.22}$$

Similarly, we may obtain

$$\nabla_{V'}V' \perp V'', \quad \nabla_{V'}JV' \perp JV'', \tag{3.23}$$

$$\nabla_{V'}V'' \perp V', \quad \nabla_{V'}JV'' \perp JV'. \tag{3.24}$$

Let $U' = V' \oplus JV'$ and $U'' = V'' \oplus JV''$. Then (3.21) - (3.24) show that

$$\nabla_{V'}U' \perp U'', \quad \nabla_{V'}U'' \perp U'. \tag{3.25}$$

In a similar way we may also obtain $\nabla_{JV'}U' \perp U''$ and $\nabla_{JV'}U'' \perp U'$. Therefore, we see that U' and U'' are both integrable and parallel distributions. Thus N is locally the Riemannian product of two Kaehler manifolds. This is a contradiction since H^m admits no complex submanifold which is a product of two Kaehler manifolds (cf. [1 I]). (Q.E.D.)

LEMMA 4. *Let* M *be a proper mixed foliate CR-submanifold of* H^m. *If* $p = \dim_{\mathbb{R}} \mathcal{D}^{\perp} \geq 3$, *then* $h = \dim_{\mathbb{C}} \mathcal{D} = 2r$ *is even and with respect to a suitable orthonormal frame* $X_1,\dots,X_h$, $JX_1,\dots,JX_h$, $Z_1,\dots,Z_p$, $JZ_1,\dots,JZ_p$, *we have*

$$A_1 = \begin{pmatrix} 0 & -I_h \\ -I_h & 0 \end{pmatrix}, \quad A_{1^*} = \begin{pmatrix} I_h & 0 \\ 0 & -I_h \end{pmatrix},$$

$$A_2 = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, \quad A_{2^*} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \quad B = \begin{pmatrix} I_r & 0 \\ 0 & -I_r \end{pmatrix}, \tag{3.27}$$

$$A_3 = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \quad A_{3^*} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}.$$

If $p \geq 4$, *then, for* $\alpha \geq 4$, *we also have*

$$A_\alpha = \begin{pmatrix} D_\alpha & 0 \\ 0 & -D_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} 0 & D_\alpha \\ D_\alpha & 0 \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} 0 & E_\alpha \\ {}^tE_\alpha & 0 \end{pmatrix} \tag{3.28}$$

for some $E_\alpha \in O(r)$ *such that* ${}^tE_\alpha = -E_\alpha$.

PROOF. Under the hypothesis, there is a suitable orthonormal frame $X_1,\dots,X_h$, $JX_1,\dots,JX_h$, $Z_1,\dots,Z_p$, $JX_1,\dots,JX_p$ such that A_1, A_2, A_{1^*} and A_{2^*} take the desired forms (cf. (3.5), (3.7) and (3.10)). Since $A_\alpha A_1 + A_1 A_\alpha = 0$ for $\alpha \geq 3$, we also have

$$A_\alpha = \begin{pmatrix} D_\alpha & 0 \\ 0 & -D_\alpha \end{pmatrix}, \qquad A_{\alpha^*} = \begin{pmatrix} 0 & D_\alpha \\ D_\alpha & 0 \end{pmatrix}, \tag{3.29}$$

where $D_\alpha \in O(h)$ with ${}^tD_\alpha = D_\alpha$. From Lemma 1 we also have

$$A_2A_\alpha + A_\alpha A_2 = 0, \qquad A_2A_{\alpha^*} - A_{\alpha^*}A_2 = 0. \tag{3.30}$$

From this we see that each D_α takes the following form:

$$D_\alpha = \begin{pmatrix} 0 & E_\alpha \\ {}^tE_\alpha & 0 \end{pmatrix}, \qquad \alpha \geq 3, \tag{3.31}$$

where each E_α is a $(r\times(h-r))$-matrix. Since $D_\alpha \in O(h)$, this implies

$$E_\alpha {}^tE_\alpha = I_r \quad \text{and} \quad {}^tE_\alpha E_\alpha = I_{h-r}. \tag{3.32}$$

It is clear that this is impossible unless E_α is a square matrix. Therefore, we have $r = 0$, $h = r$, or $h = 2r$. However, the first two cases cannot occur since, for instance, if $r = 0$, then $A_2 = -A_{1*}$ which implies $A_\alpha = 0$ by virture of (3.30). This contradicts to (c) of Lemma 1. Similar argument works for the second case. Consequently, $h = 2r$ which is even. Now, let $X_1,\ldots,X_h$ be chosen in such a way that

$$X_{r+1} = A_3X_1, \ \ldots \ ,X_h = A_3X_r.$$

Then A_3 and A_{3*} are expressed in the forms given in (3.31). Finally, for each $\alpha \geq 4$, by using the properties $A_3A_\alpha + A_\alpha A_3 = 0$ and $D_\alpha \in O(h)$, we may conclude that D_α is in the desired form. (Q.E.D.)

LEMMA 5. *Let* M *be a proper mixed foliate CR-submanifold of* H^m. *If* $p \geq 4$, *then* $h \geq 2p-4$. *Furthermore, we may choose the orthonormal frame such that, in addition to* (3.27) *and* (3.28), *we also have*

$$A_\alpha A_3X_1 = X_{\alpha-2}, \quad A_\alpha X_1 = -X_{r+\alpha-2}, \quad p \geq \alpha \geq 4, \tag{3.33}$$

$$Y_i = X_{r+i} = A_3X_i, \quad i = 1,\ldots,r. \tag{3.34}$$

PROOF. As given in the proof of Lemma 3, we decompose the tangent bundle of N into orthogonal decomposition:

$$TN = V \oplus JV, \quad V = V' \oplus V'', \quad JV = JV' \oplus JV''. \tag{3.35}$$

Such a decomposition is given with respect to A_{1*} and A_2. Now, let X_1 be a unit vector in V'. We put $Y_1 = X_{r+1} = A_3X_1$ as before. Then (e) of Lemma 1 implies that $A_3Y_1,\ldots,A_pY_1$ are orthonormal vectors in V' (cf. p. 500 of [4 II]). From this we conclude that $r \geq p-2$ which is equivalent to $h \geq 2p-4$. Now, we put

$$X_i = A_{i+2}A_3X_1 = A_{i+2}Y_1, \quad 2 \leq i \leq 2, \tag{3.36}$$

$$Y_i = X_{r+i} = A_3X_i, \quad \text{for} \quad i = 2,\ldots,p-2,\ldots,r. \tag{3.37}$$

Then, (3.27) holds. Since

$$A_\alpha X_1 = A_\alpha A_3Y_1 = -A_3A_\alpha Y_1 = -A_3X_{\alpha-2} = -Y_{\alpha-2}, \tag{3.38}$$

we also have (3.33). Formulas (3.34) are nothing but (3.37). (Q.E.D.)

From properties (a) and (b) of Lemma 1, we have

$$DZ_\alpha = \sum_{\beta=1}^{p} \theta_{\alpha\beta}Z_\beta, \quad \theta_{\alpha\beta} = -\theta_{\beta\alpha}, \quad \alpha,\beta = 1,\ldots,p. \tag{3.39}$$

for some 1-forms $\theta_{\alpha\beta}$ on N. (3.39) gives

$$DJZ_\alpha = \sum_\beta \theta_{\alpha\beta}JZ_\beta. \tag{3.40}$$

LEMMA 6. *Under the hypothesis and the notations of Lemma* 5, *we have*

$$2\langle\nabla_TX_j,JX_k\rangle = \delta_{jk}\theta_{12}(T), \tag{3.41}$$

$$2\langle\nabla_TY_j,JY_k\rangle = \delta_{jk}\theta_{21}(T), \tag{3.42}$$

$$2\langle\nabla_TX_j,JY_k\rangle = \delta_{jk}\theta_{13}(T) + \sum_{\alpha\geq4} \langle A_\alpha X_j,Y_k\rangle\theta_{1\alpha}(T), \tag{3.43}$$

$$2\langle \nabla_T X_j, Y_k\rangle = \delta_{jk}\theta_{23}(T) + \sum_{\alpha\geq 4} \langle A_\alpha X_j, Y_k\rangle \theta_{2\alpha}(T), \tag{3.44}$$

$$\langle \nabla_T Y_i, Y_k\rangle - \langle \nabla_T X_i, X_k\rangle = \sum_{\alpha\geq 4} \langle A_\alpha X_i, Y_k\rangle \theta_{3\alpha}(T) \tag{3.45}$$

for T *tangent to* N.

PROOF. The proof of this lemma is based mainly on the equation of Codazzi. Let $X_1,\ldots,X_r$, $Y_1,\ldots,Y_r$ be an orthonormal frame of $V' \oplus V'' = V$ with $Y_i = X_{r+i} = A_3 X_i$ as before, then for any vector T tangent to N, Lemma 4 gives

$$\begin{aligned} \sigma(X_i,T) &= \langle JX_i,T\rangle(JZ_2-Z_1) + \langle X_i,T\rangle(Z_2+JZ_1) \\ &\quad + \langle Y_i,T\rangle Z_3 + \langle JY_i,T\rangle JZ_3 + \sum_{\alpha\geq 4} (\langle A_\alpha X_i,T\rangle Z_\alpha + \langle A_{\alpha*}X_i,T\rangle JZ_\alpha), \end{aligned} \tag{3.46}$$

$$\begin{aligned} \sigma(Y_i,T) &= -\langle JY_i,T\rangle(JZ_2+Z_1) - \langle Y_i,T\rangle(Z_2-JZ_1) \\ &\quad + \langle X_i,T\rangle Z_3 + \langle JX_i,T\rangle JZ_3 + \sum_{\alpha\geq 4} (\langle A_\alpha Y_i,T\rangle Z_\alpha + \langle A_{\alpha*}Y_i,T\rangle JZ_\alpha). \end{aligned}$$

From (3.46), (3.47), (2.3) and Lemmas 4 and 5, we obtain

$$\begin{aligned} (\bar\nabla_{X_i}\sigma)(JY_j,JY_k) &= D_{X_i}(\delta_{jk}Z_2-\delta_{jk}JZ_1) - \langle JY_k,\nabla_{X_i}Y_j\rangle(JZ_2+Z_1) \\ &\quad - \langle Y_k,\nabla_{X_i}Y_j\rangle(Z_2-JZ_1) + \langle X_k,\nabla_{X_i}Y_j\rangle Z_3 + \langle JX_k,\nabla_{X_i}Y_j\rangle JZ_3 \\ &\quad + \sum_{\alpha\geq 4} (\langle A_\alpha Y_k,\nabla_{X_i}Y_j\rangle Z_\alpha + \langle A_{\alpha*}Y_k,\nabla_{X_i}Y_j\rangle JZ_\alpha) \\ &\quad - \langle JY_j,\nabla_{X_i}Y_k\rangle(JZ_2+Z_1) - \langle Y_j,\nabla_{X_i}Y_k\rangle(Z_2-JZ_1) \\ &\quad + \langle X_j,\nabla_{X_i}Y_k\rangle Z_3 + \langle JX_j,\nabla_{X_i}Y_k\rangle JZ_3 \\ &\quad + \sum_{\alpha\geq 4} (\langle A_\alpha Y_j,\nabla_{X_i}Y_k\rangle Z_\alpha + \langle A_{\alpha*}Y_j,\nabla_{X_i}Y_k\rangle JZ_\alpha). \end{aligned} \tag{3.48}$$

Moreover, from (3.46), (3.47) and Lemmas 4 and 5, we also obtain

$$\begin{aligned} (\bar\nabla_{JY_j}\sigma)(X_i,JY_k) &= D_{JY_j}(\delta_{ik}JZ_3 + \sum_{\alpha\geq 4} \langle A_\alpha X_i,Y_k\rangle JZ_\alpha) \\ &\quad + \langle Y_k,\nabla_{JY_j}X_i\rangle(JZ_2+Z_1) + \langle Y_k,\nabla_{JY_j}JX_i\rangle(Z_2-JZ_1) \\ &\quad - \langle X_k,\nabla_{JY_j}JX_i\rangle Z_3 - \langle X_k,\nabla_{JY_j}X_i\rangle JZ_3 \\ &\quad - \sum_{\alpha\geq 4} (\langle A_\alpha Y_k,\nabla_{JY_j}JX_i\rangle Z_\alpha + \langle A_{\alpha*}Y_k,\nabla_{JY_j}JX_i\rangle JZ_\alpha) \\ &\quad - \langle X_i,\nabla_{JY_j}Y_k\rangle(JZ_2-Z_1) - \langle X_i,\nabla_{JY_j}JY_k\rangle(Z_2+JZ_1) \\ &\quad - \langle Y_i,\nabla_{JY_j}JY_k\rangle Z_3 - \langle Y_i,\nabla_{JY_j}Y_k\rangle JZ_3 \\ &\quad - \sum_{\alpha\geq 4} (\langle A_\alpha X_i,\nabla_{JY_j}JY_k\rangle Z_\alpha + \langle A_{\alpha*}X_i,\nabla_{JY_j}JY_k\rangle JZ_\alpha). \end{aligned} \tag{3.49}$$

Since the equation of Codazzi gives

$$(\bar\nabla_{X_i}\sigma)(JY_j,JY_k) = (\bar\nabla_{JY_j}\sigma)(X_i,JY_k), \tag{3.50}$$

the Z_1-components of both sides of (3.50) yield

$$2\langle JY_k, \nabla_{X_i} Y_j\rangle = \delta_{jk}\theta_{21}(X_i), \tag{3.51}$$

where we used (3.39), (3.40) and the fact that X_i and Y_k are orthogonal. Similarly, by comparing the JZ_1-, JZ_2-, and JZ_3-components of (3.50), we may also obtain

$$2\langle JY_k, \nabla_{JY_j} X_i\rangle = \delta_{ik}\theta_{13}(JY_j) + \sum_{\alpha\geq 4} \langle A_\alpha X_i, Y_k\rangle\theta_{1\alpha}(JY_j), \tag{3.52}$$

$$2\langle Y_k, \nabla_{JY_j} X_i\rangle = \delta_{ik}\theta_{23}(JY_j) + \sum_{\alpha\geq 4} \langle A_\alpha X_i, Y_k\rangle\theta_{2\alpha}(JY_j), \tag{3.53}$$

$$\begin{aligned} &-\delta_{jk}\theta_{13}(X_i) + \langle JX_k, \nabla_{X_i} Y_j\rangle + \langle JX_j, \nabla_{X_i} Y_k\rangle \\ &= \sum_{\alpha\geq 4} \langle A_\alpha X_i, Y_k\rangle\theta_{\alpha 3}(JY_j) - \langle X_k, \nabla_{JY_j} X_i\rangle - \langle Y_i, \nabla_{JY_j} Y_k\rangle, \end{aligned} \tag{3.54}$$

where we used (3.51) to derive (3.53).
Since $A_\alpha A_3 + A_3 A_\alpha = 0$ for $\alpha \geq 4$, Lemma 5 implies

$$\langle A_\alpha X_i, Y_k\rangle = -\langle A_\alpha X_k, Y_i\rangle. \tag{3.55}$$

Therefore, (3.52) and (3.53) yield

$$\delta_{ik}\theta_{13}(JY_j) = \langle JY_k, \nabla_{JY_j} X_i\rangle + \langle JY_i, \nabla_{JY_j} X_k\rangle, \tag{3.56}$$

$$\delta_{ik}\theta_{23}(JY_j) = \langle Y_k, \nabla_{JY_j} X_i\rangle + \langle Y_i, \nabla_{JY_j} X_k\rangle. \tag{3.57}$$

Furthermore, from (3.55), we see that the left-hand side of (3.54) is symmetric with respect to the indices j and k and the right-hand side is skew-symmetric with respect to j and k, thus we obtain

$$\delta_{jk}\theta_{13}(X_i) = \langle JY_j, \nabla_{X_i} X_k\rangle + \langle JY_k, \nabla_{X_i} X_j\rangle, \tag{3.58}$$

$$\langle \nabla_{JY_j} Y_i, Y_k\rangle - \langle \nabla_{JY_j} X_i, X_k\rangle = \sum_{\alpha\geq 4} \langle A_\alpha X_i, Y_k\rangle\theta_{3\alpha}(JY_j). \tag{3.59}$$

From (3.51) (respectively, (3.52), (3.53) and (3.59)), we obtain (3.42) for T in V' (respectively, (3.43), (3.44), and (3.45) for T in JV''). By using the same method, we may obtain (3.41) - (3.45) for all T in TN. (The computation is long, but straight-forward). (Q.E.D.)

In the following, we denote by R and $R^\perp$ the Riemann curvature tensor and the normal curvature tensor of the leaf N.

LEMMA 7. *Under the hypothesis and the notations of Lemma 5, we have*

$$\begin{aligned} &2R(X_1, Y_1; Y_1, X_1) + 2\langle \nabla_{Y_1} Y_1, \nabla_{X_1} X_1\rangle - 2\langle \nabla_{X_1} Y_1, \nabla_{Y_1} X_1\rangle \\ &= R^\perp(X_1, Y_1; Z_3, Z_2) + \langle D_{Y_1} Z_3, D_{X_1} Z_2\rangle - \langle D_{X_1} Z_3, D_{Y_1} Z_2\rangle. \end{aligned} \tag{3.60}$$

PROOF. From Lemma 5, we have $\langle A_\alpha X_1, Y_1\rangle = \langle A_\alpha X_1, A_3 X_1\rangle = \langle X_1, A_\alpha A_3 A_1\rangle = 0$ for $\alpha \geq 4$. Thus Lemma 6 implies $2\langle \nabla_T Y_1, X_1\rangle = \theta_{32}(T) = \langle D_T Z_3, Z_2\rangle$, from which we obtain (3.60). (Q.E.D.)

4. PROOF OF THEOREM 1.

Under the hypothesis of Theorem 1, if M is non-proper, Lemma 3 implies $p = \dim_{\mathbb{R}} \mathcal{D}^\perp \geq 3$.

If $p \geq 4$, then Lemmas 5 and 6 imply that, for $i \geq 2$, we have

$$2\langle \nabla_T Y_1, X_i\rangle = \sum_{\alpha \geq 4} \langle A_\alpha X_i, Y_1\rangle \theta_{\alpha 2}(T)$$

$$= \sum_{\alpha \geq 4} \langle A_\alpha A_3 X_1, X_i\rangle \theta_{\alpha 2}(T) = \sum_{\alpha \geq 4} \langle X_{\alpha-2}, X_i\rangle \theta_{\alpha 2}(T).$$

Thus, we have $2\langle \nabla_T Y_1, X_i\rangle = \theta_{i+2\,2}(T)$, $i = 2,\dots,r$. Similarly, we have $2\langle \nabla_T X_1, Y_i\rangle = \theta_{i+2\,2}(T)$, $i = 2,\dots,r$. Thus, by applying Lemma 6, we may obtain

$$2\langle \nabla_{Y_1} Y_1, \nabla_{X_1} X_1\rangle - 2\langle \nabla_{X_1} Y_1, \nabla_{Y_1} X_1\rangle$$

$$= \sum_{i=2}^{p-2} \theta_{i+2\,2}(Y_1)[\langle \nabla_{X_1} X_1, X_i\rangle - \langle \nabla_{X_1} Y_1, Y_i\rangle]$$

$$+ \sum_{i=2}^{p-2} \theta_{i+2\,2}(X_1)[\langle \nabla_{Y_1} Y_1, Y_i\rangle - \langle \nabla_{Y_1} X_1, X_i\rangle]$$

$$+ 2\theta_{12}(X_1)\langle \nabla_{Y_1} X_1, JY_1\rangle + 2\theta_{21}(Y_1)\langle \nabla_{X_1} X_1, JY_1\rangle.$$

Therefore, by applying Lemma 6 again, we may find

$$2\langle \nabla_{Y_1} Y_1, \nabla_{X_1} X_1\rangle - 2\langle \nabla_{X_1} Y_1, \nabla_{Y_1} X_1\rangle = \langle D_{X_1} Z_2, D_{Y_1} Z_3\rangle - \langle D_{Y_1} Z_2, D_{X_1} Z_3\rangle. \tag{4.1}$$

Combining (4.1) with (3.60) of Lemma 7, we get

$$2R(X_1, Y_1; Y_1, X_1) = R^{\perp}(X_1, Y_1; Z_3, Z_2). \tag{4.2}$$

From (2.7), (3.46), (3.47), Lemma 5 and the equation of Gauss, we may find

$$R(X_1, Y_1; Y_1, X_1) = -2. \tag{4.3}$$

On the other hand, (2.7), the equation of Ricci, Lemma 1 and Lemma 5 give

$$R^{\perp}(X_1, Y_1; Z_3, Z_2) = 2\langle A_2 X_1, X_1\rangle = 2. \tag{4.4}$$

Equations (4.2), (4.3) and (4.4) give a contradiction. If $p = 3$, then, by (3.27) and the equation of Codazzi, we may obtain (3.41) - (3.45) in such forms that the summation terms in (3.43) - (3.45) were disappeared. By applying these equations, we may obtain a contradiction in a similar way. (Q.E.D.)

<u>**REMARK**</u>. For a CR-submanifold M of a Kaehler manifold, the condition that M is mixed-foliate is equivalent to $AP = -PA$.

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